COMPOSITIO MATHEMATICA

Finitely generated simple sharply 2-transitive groups

Simon André and Vincent Guirardel

Compositio Math. **160** (2024), 1941–1957.

doi:10.1112/S0010437X24007358

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

Compositio Math. 160 (2024) 1941–1957
doi:10.1112/S0010437X24007358

Finitely generated simple sharply 2-transitive groups

Simon André and Vincent Guirardel

Abstract

We construct the first examples of infinite sharply 2-transitive groups which are finitely generated. Moreover, we construct such a group that has Kazhdan property (T), is simple, has exactly four conjugacy classes and we show that this number is as small as possible.

1. Introduction

An action of a group G on a set X containing at least two elements is said to be *sharply 2-transitive* if, for any two couples (x_1, x_2) and (y_1, y_2) of distinct elements of X, there exists a unique element of G mapping (x_1, x_2) to (y_1, y_2). A group G is called *sharply 2-transitive* if there exists a set X on which G acts sharply 2-transitively. For instance, if K is any skew field, the natural action of the affine group $K^* \ltimes K$ on K is sharply 2-transitive. This generalizes to a near-field K where only one of the two distributive rules is required (see, for instance, [DM96, § 7.6]). We say that G is *nearly affine* if $G \simeq K^* \ltimes K$ for some near-field K (this notion is also referred to as *split* in the literature).

Zassenhaus proved that every finite sharply 2-transitive group G is nearly affine, and classified all finite near-fields in [Zas35a, Zas35b]. In the same direction, Tits proved in [Tit52, Tit56] that if G is a locally compact connected group having a continuous sharply 2-transitive action on a locally compact set, then G is nearly affine. In fact, $G \simeq K^* \ltimes K$ where K is either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or a near-field obtained by twisting the multiplication on $\mathbb{H}$. More recently, Glasner and Gulko [GG14] and Glauberman, Mann and Segev [GMS15] proved that if G is any linear group having a sharply 2-transitive action of characteristic distinct from 2 (see § 2.1 for a definition), then again, G is nearly affine.

Until recently, it was an open problem whether every sharply 2-transitive group G is nearly affine. The first counterexamples were constructed in [RST17] and [RT19] by Rips, Segev and Tent. Then, infinite simple sharply 2-transitive groups were constructed in [AT23].

However, none of the examples constructed in [RST17, RT19, AT23] is finitely generated, so one may ask whether Zassenhaus classification extends assuming finite generation: if G is a finitely generated sharply 2-transitive group, does G have to be nearly affine?

In fact, there is an even more basic question: does there exist an infinite, sharply 2-transitive group G which is finitely generated? Indeed, no group of the form $G = K^* \ltimes K$ for a skew field K is known to be finitely generated as the existence of an infinite skew field K whose multiplicative group is finitely generated is a famous open problem (see, for instance, [AM-H00, AKPR19]).

Received 12 April 2023, accepted in final form 31 January 2024, published online 16 September 2024.
2020 Mathematics Subject Classification 20B22, 20F65, 20F67 (primary).

In the present paper, we answer the two questions above: we construct the first examples of infinite sharply 2-transitive groups which are finitely generated, and our examples are not nearly affine.

We also construct such groups having Kazhdan property (T). Recall that this property is, in some sense, opposite to amenability: an amenable discrete group that has property (T) is necessarily finite (see [BHV08] for background). Hence, our examples are in striking contrast with the affine group $K^* \ltimes K$ on a field K, since this group is solvable, in particular amenable. There exist skew fields K such that K^* contains a non-abelian free group, but as mentioned above, no skew field is known for which K^* is finitely generated. In addition, property (T) implies finite generation for discrete groups, so property (T) can be seen as a (strong) reinforcement of finite generation. Property (T) also prevents the existence of non-trivial actions on trees. On the other hand, the previously known exotic sharply 2-transitive groups are constructed as increasing unions of amalgams and Higman–Neumann–Neumann (HNN) extensions, which is an obstruction to having property (T).

Theorem 1.1. *There exists an infinite, finitely generated, sharply 2-transitive group G. Moreover, one can construct such a group G which has property (T), is 2-generated and simple (and, in particular, not nearly affine).*

Remark 1.2. The sharply 2-transitive groups constructed in this paper have characteristic 0, meaning that involutions have a fixed point and that the products of distinct involutions have infinite order (see § 2.1 for more details).

In fact, it is well known (see [Ker74, Chapter II § 6]) that every sharply 2-transitive group is the group of affine transformations of a *near-domain*: this is a structure $(K, +, \cdot)$ similar to a near-field but in which $(K, +)$ is not required to be a commutative group but only a loop with a twisted associativity condition. In this context, the group G of affine transformations $x \mapsto ax + b$ of K acts sharply 2-transitively on K, but it is not a semi-direct product in general. Actually, G is a semi-direct product if and only if K is a near-field, in which case G is nearly affine (see [Ker74, Theorem 7.1]). Thus, examples of exotic sharply 2-transitive groups yield exotic near-domains, and Theorem 1.1 can be reformulated in terms of near-domains as follows.

Corollary 1.3. *There exists an infinite near-domain K which is not a near-field and whose associated group of affine transformations is finitely generated.*

We also consider the problem of minimizing the number of conjugacy classes. First, any sharply 2-transitive group has to have at least three conjugacy classes (except for $\mathbb{Z}/2\mathbb{Z}$), and at least four conjugacy classes in characteristic distinct from 2 (except for the affine group on the field $\mathbb{F}_3$); see Proposition 2.3. Actually, except for the affine groups on $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$, all *finite* sharply 2-transitive groups have at least five conjugacy classes.

On the other hand, Cameron [Cam00] mentions that Cohn's construction [Coh71, Theorem 6.3] can be used to construct an infinite skew field K of characteristic p such that K^* has exactly p conjugacy classes. For $p = 2$, the corresponding affine group $K^* \ltimes K$ would have exactly three conjugacy classes. In characteristic 0, all the previously known sharply 2-transitive groups have infinitely many conjugacy classes.

Also note that in the realm of abstract groups, the first infinite, finitely generated groups with only finitely many conjugacy classes were constructed by Ivanov, and a construction can be found in [Ols91] (see Theorem 41.2). This construction yields for each sufficiently large prime number p a group of exponent p with exactly p conjugacy classes. Then, Osin constructed in

[Osi10] the first infinite finitely generated groups with exactly n conjugacy classes for any natural number $n \geq 2$.

Our construction, relying on Osin's small cancellation techniques [Osi10], provides examples of groups which are sharply 2-transitive in characteristic 0, are finitely generated, and have the smallest possible number of conjugacy classes.

Theorem 1.4. *There exists an infinite simple sharply 2-transitive group which is finitely generated, has property (T) and has exactly four conjugacy classes.*

The proof of our results combines the strategy used in [RT19] and [AT23] with small cancellation theory over relatively hyperbolic groups, as developed notably by Osin in [Osi10] to construct infinite finitely generated groups with exactly two conjugacy classes.

2. Conjugacy classes in a sharply 2-transitive group

2.1 Characteristic of a sharply 2-transitive group

If G acts sharply 2-transitively on a set X (with $|X| \geq 2$), then G has involutions (take any element exchanging two points $x, y \in X$), and all involutions are conjugate (any involution swaps a pair of points, and two involutions can be conjugate to swap the same pair of points).

By analogy with the affine group $K^* \ltimes K$, one says that $G \curvearrowright X$ is of characteristic 2 if involutions have no fixed point.

When the action is not of characteristic 2, every involution has a unique fixed point, and distinct involutions have distinct fixed points (see, for instance, [Ten16, Corollary 3.2]). It follows that there is a natural equivariant bijection between X and the set $\mathcal{I}_G$ of involutions of G, on which G acts by conjugation. In particular, a group has a sharply 2-transitive action of characteristic distinct from 2 on a set X if and only if it acts sharply 2-transitively on its set of involutions. It also follows that all pairs of distinct involutions are conjugate. By analogy with the affine group in characteristic $\neq 2$, we say that an element of G is a *translation* if it is the product of two distinct involutions. Thus, in characteristic $\neq 2$, all translations are conjugate, and one says that $G \curvearrowright X$ has characteristic 0 if translations have infinite order (i.e. if two distinct involutions generate an infinite dihedral group) and characteristic p if translations have order p (i.e. any two distinct involutions generate a dihedral group of order $2p$), and p has to be some prime number (otherwise the subgroup generated by a translation t of order p would contain a translation t' of order $p' < p$, thus not conjugate to t). Note that if K is a field, then the characteristic of K in the usual sense coincides with the characteristic of $K^* \ltimes K$ as defined above. The sharply 2-transitive groups we will construct in this paper will have characteristic 0.

2.2 A lower bound on the number of conjugacy classes

A sharply 2-transitive group with exactly two conjugacy classes is necessarily abelian since all its non-trivial elements have order 2. However, the only abelian sharply 2-transitive group is $\mathbb{Z}/2\mathbb{Z}$ (because the stabilizer of a point in such a group is malnormal), and hence every sharply 2-transitive group of order > 2 has at least three conjugacy classes.

The only finite sharply 2-transitive group with exactly three conjugacy classes is the symmetric group S_3, but there are infinite examples: Cameron [Cam00] mentions that Cohn's construction can be used to construct an infinite skew field K of characteristic 2 such that K^* has exactly 2 conjugacy classes (see Corollary 2 of Theorem 6.3 in [Coh71]); the corresponding affine group $K^* \ltimes K$ has exactly three conjugacy classes.

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

The goal of this section is Proposition 2.3 below showing that any sharply 2-transitive group $G \curvearrowright X$ of characteristic $\neq 2$ with $|X| \geq 4$ has at least 4 conjugacy classes. Moreover, in characteristic 2, the only possibility to have at most 3 conjugacy classes comes from near-fields of characteristic 2. In § 4, we will construct infinite sharply 2-transitive groups of characteristic 0 with exactly 4 conjugacy classes.

Lemma 2.1 [Ten16, Corollary 3.2]. *Let G be a group acting sharply 2-transitively on a set X, with $|X| \geq 2$. Then two distinct involutions cannot have a common fixed point in X.*

Proof. The lemma is obvious if the characteristic is 2, i.e. if involutions do not have fixed points. Otherwise, this is [Ten16, Corollary 3.2]. □

Lemma 2.2. *Let G be a group acting sharply 2-transitively on a set X, with $|X| \geq 2$. The product of two distinct involutions cannot fix a point.*

Proof. Assume towards a contradiction that a product uv of two distinct involutions fixes a point $x \in X$. If v fixes x, so does u, which contradicts the previous lemma. Thus, the involution v exchanges the point x and the point $y = v \cdot x \neq x$. Since $uv \cdot x = x$, u exchanges x and y. By 2-sharpness, $u = v$. □

Proposition 2.3. *Let $G \curvearrowright X$ be a sharply 2-transitive group with $|X| \geq 4$.*

- *If $G \curvearrowright X$ is not of characteristic 2, then G has at least 4 conjugacy classes.*
- *If $G \curvearrowright X$ is of characteristic 2, then G has at least 3 conjugacy classes, and at least 4 conjugacy classes unless $G \simeq K^* \ltimes K$ for some near-field K such that all non-trivial elements of K^* are conjugate.*

Proof. We already saw at the beginning of this section that if $|X| \geq 3$, G has at least 3 conjugacy classes.

Fix a point $x \in X$. By Lemmas 2.1 and 2.2, G_x contains at most one involution and no translation (i.e. a product of two distinct involutions). Moreover, $|X \setminus \{x\}| \geq 3$ and G_x acts transitively on $X \setminus \{x\}$. Thus, G_x contains a non-trivial element g_x which is neither an involution nor a translation.

Assume $G \curvearrowright X$ has characteristic $p \neq 2$, and consider two distinct involutions u, v. Then the translation uv is an element of order p (or infinite order if $p = 0$), hence is not an involution. This shows that the elements $1, u, uv, g_x$ are in four distinct conjugacy classes.

In characteristic 2, the argument above says that if some translation uv is not an involution, then G has at least four conjugacy classes. We may thus assume that the product of any two distinct involutions is an involution, i.e. that the set $\mathcal{I}_G \cup \{1\}$ is a (normal) subgroup of G. By [Ker74, Theorem 7.1], $G = K^* \ltimes K$ for some near-field K (where $K = \mathcal{I}_G \cup \{1\}$ as an additive group). If G has at most 3 conjugacy classes, then K^* has at most 2 conjugacy classes. □

3. The classes of groups $\mathcal{C}$ and $\mathcal{C}'$

In this section that closely follows [AT23, RT19], we introduce two classes of groups $\mathcal{C}$ and $\mathcal{C}'$ that are stable under various types of HNN extensions, amalgams and increasing union. Any sharply 2-transitive group of characteristic 0 is in the class $\mathcal{C}$, but the class $\mathcal{C}'$ is more restrictive.

3.1 The class $\mathcal{C}$

In [AT23] (building on [RT19]), a class of groups $\mathcal{C}$ is introduced that contains all sharply 2-transitive groups of characteristic 0 and which is preserved under various HNN extensions and amalgamated products. We reformulate this definition below (see Definition 3.3).

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

We denote by $D_{\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$ the infinite dihedral group and by $D_{\mathbb{Q}} = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Q}$ the group of isometries of $\mathbb{Q}$.

A subgroup $H \subset G$ is called *malnormal* if the intersection $H \cap gHg^{-1}$ is trivial for every $g \in G \setminus H$, and *quasi-malnormal* if $H \cap gHg^{-1}$ has order at most 2 for $g \in G \setminus H$.

Recall that we denote by $\mathcal{I}_G$ the set of involutions of G, and we denote by $\mathcal{I}_G^{(2)} = \{(u,v) \in \mathcal{I}_G^2 \mid u \neq v\}$ the set of ordered pairs of involutions, on which G acts by conjugation.

DEFINITION 3.1. Let G be a group, and $(u,v) \in \mathcal{I}_G^{(2)}$ be a pair of distinct involutions.

- We say that (u,v) is of type $D_{\mathbb{Z}}$ if $\langle u,v\rangle$ is contained in a (necessarily unique) group $D_{u,v}$ isomorphic to $D_{\mathbb{Z}}$ with $D_{u,v}$ quasi-malnormal in G. If $\langle u,v\rangle = D_{u,v}$ we say that the pair (u,v) is *maximal*.
- We say that (u,v) is of type $D_{\mathbb{Q}}$ if $\langle u,v\rangle$ is contained in a (maybe non-unique, maybe not quasi-malnormal) group isomorphic to $D_{\mathbb{Q}}$.

Remark 3.2.

(1) If (u,v) is of one of the two types, then $\langle u,v\rangle$ is infinite dihedral.
(2) If (u,v) is of type $D_{\mathbb{Z}}$ and (u',v') is of type $D_{\mathbb{Q}}$, then $\langle u,v\rangle \cap \langle u',v'\rangle$ has cardinality at most 2. In particular, the two types are mutually exclusive.
(3) Recall that the *commensurator* $\mathrm{Comm}_G(H)$ of a subgroup $H \subset G$ is the subgroup consisting of elements $g \in G$ such that the intersection of H and gHg^{-1} has finite index in both of them. The pair (u,v) is of type $D_{\mathbb{Z}}$ if and only if the commensurator $\mathrm{Comm}_G(\langle u,v\rangle)$ is an infinite dihedral group. In this case, $D_{u,v} = \mathrm{Comm}_G(\langle u,v\rangle)$ is the unique maximal infinite dihedral group containing $\langle u,v\rangle$.

DEFINITION 3.3. Let G be a group. We say that G belongs to the class $\mathcal{C}$ if:

(1) G acts transitively on $\mathcal{I}_G$, and freely on $\mathcal{I}_G^{(2)}$;
(2) any pair $(u,v) \in \mathcal{I}_G^{(2)}$ of distinct involutions is of type $D_{\mathbb{Z}}$ or $D_{\mathbb{Q}}$;
(3) the set of pairs $(u,v) \in \mathcal{I}_G^{(2)}$ of type $D_{\mathbb{Q}}$ is non-empty and G acts transitively on it.

Remark 3.4. If a group G belongs to $\mathcal{C}$, and if all pairs of involutions of G are of type $D_{\mathbb{Q}}$, then G acts sharply 2-transitively on $\mathcal{I}_G$.

Conversely, any sharply 2-transitive group G of characteristic 0 belongs to the class $\mathcal{C}$, with all pairs of involutions of type $D_{\mathbb{Q}}$. Indeed, G acts sharply 2-transitively on its set of involutions (as recalled in § 2.1). It then suffices to show that any pair of involutions $(u,v) \in \mathcal{I}_G^{(2)}$ is of type $D_{\mathbb{Q}}$. Indeed, given $k \geq 1$, (u,v) is thus conjugate to $(u,(vu)^{k-1}v)$, so consider an inner automorphism $\sigma_k = \mathrm{ad}_{a_k}$ sending (u,v) to $(u,(vu)^{k-1}v)$. Note that $\sigma_k(\langle u,v\rangle)$ is the unique subgroup of index k of $\langle u,v\rangle$. Moreover, for all $k,l \in \mathbb{N} \setminus \{0\}$, $\sigma_k \circ \sigma_l$ agrees with σ_{kl} in restriction to $\langle u,v\rangle$. Since the pair (u,v) has trivial centralizer, it follows that $a_k a_l = a_{kl}$ so σ_k and σ_l commute. Define $D_1 = \langle u,v\rangle$ and $D_n = \sigma_{n!}^{-1}(D_1) = \sigma_n^{-1}(D_{n-1})$. Since $\langle u,v\rangle$ contains $\sigma_n(\langle u,v\rangle)$ with index n, $D_n = \sigma_{n!}^{-1}(\langle u,v\rangle)$ contains $D_{n-1} = \sigma_{n!}^{-1}\sigma_n(\langle u,v\rangle)$ with index n. Thus, $D_1 \subset D_2 \subset \cdots \subset D_n$ with $[D_n : D_{n-1}] = n$. It easily follows that $\bigcup_{n\in\mathbb{N}} D_n$ is an increasing union of infinite dihedral groups, each one having index n in the next one, and whose union is isomorphic to $D_{\mathbb{Q}}$.

Remark 3.5. Assertion (1) implies that the centralizer of each involution is malnormal. Indeed, if $g \notin \mathrm{Cent}_G(i)$, any element $a \in \mathrm{Cent}_G(i)^g \cap \mathrm{Cent}_G(i)$ fixes two distinct involutions, hence is trivial. Assertion (2) implies that any two distinct involutions generate an infinite dihedral group, and thus do not commute.

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

Remark 3.6. Definition 3.3 is equivalent to the definition of the class $\mathcal{C}$ that appears in [AT23]. Since we do not use the equivalence of the two definitions in this paper, we leave the verification of the equivalence to the interested reader.

3.2 Three types of elements

The following terminology will be convenient.

Definition 3.7. Consider a group G and an element $h \in G$ of infinite order. We say that:

(1) h is a *translation* if it is the product of two distinct involutions,
  - it is a *translation of type* $D_{\mathbb{Q}}$ if it is the product of two distinct involutions $h = uv$, with $\langle u, v\rangle$ of type $D_{\mathbb{Q}}$,
  - it is a *translation of type* $D_{\mathbb{Z}}$ if it is the product of two distinct involutions $h = uv$, with (u, v) of type $D_{\mathbb{Z}}$;

(2) h is a *homothety* if h centralizes some involution;

(3) h is *isolated* if h is contained in a malnormal cyclic group $\langle \hat{h}\rangle$. We say that h is *maximal isolated* if $\langle h\rangle$ is malnormal.

Remark 3.8. In any group G, the two types of translations are mutually exclusive (by the second point of Remark 3.2). Moreover, it is clear that an isolated element cannot be a translation or a homothety.

Remark 3.9. The set of translations together with the trivial element does *not* form a subgroup of G in general. This is the case for $G = K^* \ltimes K$ when K is a skew field, and in fact, if and only if $G = K^* \ltimes K$ for some near-field K, where $K = \mathcal{I}_G^2$ as an additive group (see [Ker74, Theorem 7.1]).

Lemma 3.10. *In a group belonging to $\mathcal{C}$, the three possibilities for h (translation, homothety, isolated) are mutually exclusive.*

Proof. In view of the previous remark, we just have to prove that a homothety h cannot be a translation. Let i be an involution centralized by h. Arguing by contradiction, assume that $h = uv$ for two involutions $u \neq v$. We will use several times that two distinct involutions cannot commute in a group of class $\mathcal{C}$. We note that $u, v \neq i$: indeed, if $u = i$, for instance, then v has to commute with i so $v = i$ and $h = 1$ a contradiction. The involution $j = uiu^{-1}$ is therefore distinct from i. The conjugation by u exchanges i and j and so does v since uv commutes with i. By 2-sharpness, $u = v$ a contradiction. □

Remark 3.11. If G is in the class $\mathcal{C}$, then all involutions are conjugate, and all translations of type $D_{\mathbb{Q}}$ are also conjugate (because G acts transitively on pairs of involutions of type $D_{\mathbb{Q}}$).

Definition 3.12. We say that a group G satisfies the *3-type condition* if its elements are of order 1, 2 or ∞, and each element $h \in G$ of infinite order is a translation, a homothety or is isolated.

3.3 The class $\mathcal{C}'$

We will use the following subclass $\mathcal{C}'$ of $\mathcal{C}$.

Definition 3.13 The class $\mathcal{C}'$. We say that a group G belongs to the class $\mathcal{C}'$ if it lies in the class $\mathcal{C}$ and satisfies the 3-type condition.

For example, the affine group $\mathbb{Q}^* \ltimes \mathbb{Q}$ belongs to $\mathcal{C}'$ with no isolated element and no $D_{\mathbb{Z}}$-translation. More generally, if K is a field of characteristic 0, then $K^* \ltimes K$ belongs to $\mathcal{C}'$ if and only if K^* contains no primitive nth root of unity with $n > 2$.

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

The following lemmas give some restrictions for groups in $\mathcal{C}'$ that will be useful.

Lemma 3.14. *If G is a group in which every finite subgroup of G has order at most 2, then every infinite virtually cyclic subgroup of G is isomorphic to $\mathbb{Z}$, $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$ or $D_{\mathbb{Z}}$.*

Proof. If $E \subset G$ is virtually cyclic, it can be written as an extension of a finite normal group $N \subset E$, with E/N isomorphic to $\mathbb{Z}$ or $D_{\mathbb{Z}}$. If N is trivial, we are done. Otherwise, $N \simeq \mathbb{Z}/2\mathbb{Z}$ and since finite subgroups of G have order ≤ 2, E/N has to be torsion-free, so $E/N \simeq \mathbb{Z}$ and $E \simeq \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$. □

Lemma 3.15. *If G belongs to $\mathcal{C}'$, then every non-trivial finite subgroup of G has order 2, and every infinite virtually cyclic subgroup of G is isomorphic to $\mathbb{Z}$, $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$ or $D_{\mathbb{Z}}$.*

Proof. If G belongs to $\mathcal{C}'$, the order of elements of finite order in G is at most 2, so any finite subgroup $F \subset G$ is commutative. Since distinct involutions of G never commute, F is trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The second part of the lemma then follows from Lemma 3.14. □

Lemma 3.16. *Any group G belonging to $\mathcal{C}'$ contains a subgroup isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$.*

Moreover, if $P \subset G$ is almost malnormal (i.e. for every $g \in G$ such that $gPg^{-1} \cap P$ is infinite, then $g \in P$) and contains a pair of involutions of type $D_{\mathbb{Q}}$, then P contains a subgroup isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$.

Proof. The second assertion with $P = G$ implies the first so we prove the second one.

Let (u, v) be a pair of involutions in P of type $D_{\mathbb{Q}}$. Note that there are infinitely many such pairs (u, v'): take for v' any involution in $\langle u, v\rangle \setminus \{u\}$. By Definition 3.3 defining class $\mathcal{C}$, all such pairs are in the same G-orbit, so there exists $h \in G$ that conjugates (u, v) to (u, v'). By almost malnormality of P, $h \in P$. This implies, in particular, that $\mathrm{Cent}_P(u)$ is infinite. Since u is the only involution in $\mathrm{Cent}_G(u)$, $\mathrm{Cent}_P(u)$ contains an element of infinite order which proves the lemma. □

3.4 Increasing unions

Recall from Definition 3.1 that a pair of involutions (u, v) of type $D_{\mathbb{Z}}$ is maximal if $\langle u, v\rangle$ is quasi-malnormal in G.

Definition 3.17. Consider an embedding of groups $G \subset G'$, with G, G' in $\mathcal{C}$.

- We say that the embedding *preserves maximal pairs* if for each maximal pair $(u, v) \in \mathcal{I}_G^{(2)}$ of type $D_{\mathbb{Z}}$, either the pair (u, v) is still a maximal pair of type $D_{\mathbb{Z}}$ in G', or (u, v) is of type $D_{\mathbb{Q}}$ in G'.
- We say that the embedding *preserves maximal isolated elements* if for each malnormal cyclic group $\langle h\rangle$, either $\langle h\rangle$ is still malnormal in G', or h is a translation or a homothety in G'.

We say that the embedding *preserves maximality* if it satisfies both conditions.

Lemma 3.18. *Consider $G = \bigcup_{n\in\mathbb{N}} G_n$ an increasing union of groups in $\mathcal{C}'$. Assume that the embeddings $G_n \subset G_{n+1}$ preserve maximality. Then G is in the class $\mathcal{C}'$ and for each $n \in \mathbb{N}$, the embedding $G_n \subset G$ preserves maximality.*

Proof. First let us check that G is in the class $\mathcal{C}$. Assertion (1) of Definition 3.3 is clear. If (u, v) is of type $D_{\mathbb{Q}}$ in G_n, then it is clearly of type $D_{\mathbb{Q}}$ in G. If $(u, v) \in \mathcal{I}_{G_{n_0}}^{(2)}$ is of type $D_{\mathbb{Z}}$ in every G_n for all $n \geq n_0$, then since embeddings preserve maximality, $D_{u,v}$ does not depend on the group G_n. It follows that $D_{u,v}$ is quasi-malnormal in every G_n hence in G. This proves assertion (2) and that the embedding $G_n \subset G$ preserves maximal pairs. Since any pair of type $D_{\mathbb{Q}}$ in G is of type $D_{\mathbb{Q}}$ in any G_n containing it, assertion (3) is clear, so G belongs to $\mathcal{C}$.

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

To prove that G is in $\mathcal{C}'$, the only non-obvious point to check is that if $h \in G_{n_0}$ is isolated in G_n for every $n \geq n_0$, then it is still isolated in G. The argument is similar to the previous one using that inclusions preserve maximal isolated elements. ☐

3.5 HNN extensions and amalgams

The following proposition unifies several constructions in [AT23] and [RT19].

Proposition 3.19. *Let G be a group in the class $\mathcal{C}$. Consider an HNN extension $G_1 = G*_C$ or an amalgam $G_1 = G *_C H$ and denote by $G_1 \curvearrowright T$ the corresponding Bass–Serre tree. Assume that the following properties hold for some $k \geq 0$.*

(1) *Almost k-acylindricity: the stabilizer of any segment of length $> k$ in T has order ≤ 2.*
(2) *In the case of an amalgam, H has at most one involution (necessarily central in H), and this involution lies in the subgroup C.*

Then G_1 is in the class $\mathcal{C}$.

If, moreover, G and H satisfy the 3-type condition (see Definition 3.12), then G_1 belongs to $\mathcal{C}'$ and the embedding $G \subset G_1$ preserves maximality.

Example 3.20. The proposition implies that if G lies in $\mathcal{C}$ or $\mathcal{C}'$, then so does $G * \mathbb{Z}$.

Remark 3.21. If H has a unique involution, then all elements of infinite order of H are homotheties, and H satisfies the 3-type condition if and only if its elements are of order 1, 2 or ∞.

Proof. Before starting the proof, given two involutions u, v, we say that $\langle u, v\rangle$ is hyperbolic if it does not fix a point in T (otherwise it is elliptic). In this case, we denote by $l_{u,v}$ the unique line of T invariant under $\langle u, v\rangle$. By acylindricity, the pointwise stabilizer of $l_{u,v}$ has order ≤ 2, we claim that it is trivial. Indeed, if $g \in G$ is an involution fixing $l_{u,v}$ pointwise, it is unique and, therefore, has to commute with $\langle u, v\rangle$. Since g has a common fixed point with u, the pair (g, u) is conjugate in G (it cannot be conjugate in H because $g \neq u$). Since no pair of involutions of G commute, we get a contradiction which proves the claim.

We now check that G_1 satisfies assertion (1) of Definition 3.3. The fact that G_1 acts transitively on the set of its involutions is clear: any involution of G_1 is elliptic in T, so it is conjugate in G or H, hence in G by assumption (2). Then, proving that the action on $\mathcal{I}^{(2)}_{G_1}$ is free amounts to checking that the centralizer Z of a pair $(u, v) \in \mathcal{I}^{(2)}_{G_1}$ is trivial. If $\langle u, v\rangle$ is hyperbolic, then any element $g \in Z$ fixes $l_{u,v}$ pointwise so $g = 1$ by the initial claim. If $\langle u, v\rangle$ is elliptic, then the pair (u, v) is conjugate to a pair of involutions of G (it cannot be conjugate in H by assertion (2)). The set $F \subset T$ of fixed points of $\langle u, v\rangle$ is a subtree of diameter $\leq k$. In the case of an amalgam, it is reduced to a point fixed by a conjugate of G by assumption (2). Since Z preserves the bounded tree F, it fixes a point in F. Thus, Z, u, v are contained in a common conjugate of G, so Z is trivial since G belongs to $\mathcal{C}$. This proves that the action of G_1 on $\mathcal{I}^{(2)}_{G_1}$ is free.

Then we prove that G_1 satisfies assertion (2) of Definition 3.3, namely that any pair of distinct involutions is of type $D_\mathbb{Z}$ or $D_\mathbb{Q}$. If $\langle u, v\rangle$ is hyperbolic, then the pair (u, v) is of type $D_\mathbb{Z}$: the global stabilizer $D_{u,v}$ of $l_{u,v}$ of is an infinite dihedral group (it acts faithfully on $l_{u,v}$ by our initial claim), and it is quasi-malnormal; if $gl_{u,v} \neq l_{u,v}$, then $D_{u,v} \cap D^g_{u,v}$ cannot contain any element of infinite order, so $D_{u,v} \cap D^g_{u,v}$ has order at most 2. If $\langle u, v\rangle$ is elliptic, then its commensurator $M = \mathrm{Comm}_{G_1}(\langle u, v\rangle)$ is elliptic because it preserves the union of fixed points of finite index subgroups of $\langle u, v\rangle$, a subtree of diameter $\leq k$. Up to conjugacy, we may thus assume that $\langle u, v\rangle \subset M \subset G$, so $M = \mathrm{Comm}_G(\langle u, v\rangle)$. If (u, v) has type $D_\mathbb{Q}$ in G, it also has type $D_\mathbb{Q}$

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

in G_1. If it has type $D_{\mathbb{Z}}$ in G, then $\mathrm{Comm}_G(\langle u,v\rangle)$ is infinite dihedral, and so is $\mathrm{Comm}_{G_1}(\langle u,v\rangle)$ so (u,v) has type $D_{\mathbb{Z}}$ in G_1. Hence, assertion (2) of Definition 3.3 is proved. Note that we also proved that the embedding $G\subset G_1$ preserves maximal pairs.

Last, since any pair of type $D_{\mathbb{Q}}$ has to be elliptic in T, it is conjugate to a pair in G, so G_1 acts transitively on the set of pairs of type $D_{\mathbb{Q}}$. This shows that G_1 satisfies assertion (3) of Definition 3.3 and concludes the proof that G_1 belongs to $\mathcal{C}$.

We now assume that G and H satisfy the 3-type condition and prove that G_1 satisfies the 3-type condition, hence belongs to the class $\mathcal{C}'$. Clearly, all elements of G_1 have order 1, 2 or ∞. Let $h\in G_1$ be an element of infinite order.

If h is hyperbolic in T, let $l\subset T$ be its axis and let A be the global stabilizer of l. If the pointwise stabilizer of l is non-trivial, it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ by acylindricity and h is a homothety. If the pointwise stabilizer of l is trivial, then A acts faithfully on l so A is either cyclic or infinite dihedral. If dihedral, then h is a translation. If A is cyclic, then it is malnormal because if $gAg^{-1}\cap A\neq 1$, then g must preserve the line l, so $g\in A$. Thus, h is isolated with $\langle\hat{h}\rangle = A$.

Now assume that h is elliptic in T. Consider the subtree F of T consisting of points fixed by some power of h. Here F has diameter at most k and is invariant under the group $M=\mathrm{Comm}_{G_1}(\langle h\rangle)$ so M is elliptic. Thus, we may assume that $\langle h\rangle\subset M\subset H$ or $\langle h\rangle\subset M\subset G$.

If $\langle h\rangle\subset M\subset H$ and H centralizes an involution, then h is a homothety. If $\langle h\rangle\subset M\subset H$ and H is torsion-free, then h cannot be a translation or a homothety in H so h is isolated in H, so $M\simeq\mathbb{Z}$ and h is isolated in G_1.

If $\langle h\rangle\subset M\subset G$, and if h is a translation or a homothety in G, then this is also the case in G_1. If h is isolated in G, then $M\simeq\mathbb{Z}$ and h is isolated in G_1. ☐

The following corollary allows us to turn a pair of involutions of type $D_{\mathbb{Z}}$ into type $D_{\mathbb{Q}}$.

Corollary 3.22 ([AT23, Proposition 4.1] and [RT19, Proposition 1.4]). *Let G be a group in the class $\mathcal{C}'$. Let $(u_0,v_0)\in\mathcal{I}_G^{(2)}$ be a pair of type $D_{\mathbb{Q}}$ and $(u,v)\in\mathcal{I}_G^{(2)}$ be any maximal pair of type $D_{\mathbb{Z}}$. Then the following HNN extension belongs to $\mathcal{C}'$:*

$$G_1=\langle G,t\mid tut^{-1}=u_0,\ tvt^{-1}=v_0\rangle.$$

Moreover, the inclusion $G\subset G_1$ preserves maximality.

Proof. In view of Proposition 3.19, we just have to check that the action on the Bass–Serre tree is almost 2-acylindrical. This easily follows from the fact that $\langle u,v\rangle$ is quasi-malnormal in G, and that $|\langle u_0,v_0\rangle\cap\langle u,v\rangle^g|\leq 2$ for all $g\in G$. ☐

The following corollary allows us to turn an isolated element into a translation or a homothety.

Corollary 3.23 (See also [AT23, Proposition 4.2]). *Let G be a group in $\mathcal{C}'$ and let g,h be two elements of G of infinite order. Suppose that g is a translation or a homothety in G and that h is a maximal isolated element. Then the following HNN extension belongs to $\mathcal{C}'$:*

$$G_1=\langle G,t\mid tgt^{-1}=h\rangle.$$

Moreover, the inclusion $G\subset G_1$ preserves maximality.

Proof. Acylindricity follows from malnormality of $\langle h\rangle$ in G, and from the fact that $\langle h\rangle\cap\langle g\rangle^a=\{1\}$ for all $a\in G$. This last fact holds true because otherwise, some power of h would be conjugate in a group isomorphic to $D_{\mathbb{Z}}$ or to $\mathbb{Z}\times(\mathbb{Z}/2\mathbb{Z})$ contradicting the fact that $\langle h\rangle$ is malnormal in G. ☐

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

The following result allows to make two given homotheties conjugate.

Corollary 3.24. *Let G be a group in the class $\mathcal{C}'$. Let $h_1, h_2 \in G$ be two homotheties centralizing the involutions u_1 and u_2, respectively. Then the following HNN extension belongs to $\mathcal{C}'$:*

$$G_1 = \langle G, t \mid tu_2t^{-1} = u_1,\ th_2t^{-1} = h_1 \rangle.$$

Moreover, the inclusion $G \subset G_1$ preserves maximality.

Proof. This HNN extension is not acylindrical. We rewrite it as an acylindrical amalgam as follows. First, since all involutions of G are conjugate, we may replace h_2 and u_2 by some conjugates and change t accordingly to ensure that $u_2 = u_1$, and we let $u = u_1 = u_2$. Then $G_1 = \langle G, t \mid tut^{-1} = u,\ th_2t^{-1} = h_1 \rangle$. Let $Z = \mathrm{Cent}_G(u)$ be the centralizer of u in G. Then $G_1 = G *_Z H$ where H is the HNN extension

$$H = \langle Z, t \mid tut^{-1} = u,\ th_2t^{-1} = h_1 \rangle.$$

Since no two involutions commute in G, u is the unique involution of Z. Since H is an HNN extension, any involution of H is conjugate to an element of Z, thus conjugate to u. On the other hand, we see that u is central in H, so u is the unique involution of H. Moreover, all elements of H are of order 1, 2 or ∞ because this is the case for $Z \subset G$. As noted in Remark 3.21, this shows that H satisfies the 3-type condition.

Now Z is malnormal in G (see Remark 3.5) and it follows that the Bass–Serre tree of the amalgam $G_1 = G *_Z H$ is 2-acylindrical. Thus, Proposition 3.19 applies. □

4. An infinite simple sharply 2-transitive group with exactly four conjugacy classes

In this section, we use the tools of the previous section to construct an infinite countable simple sharply 2-transitive group of characteristic 0 in $\mathcal{C}'$ with exactly four conjugacy classes (but still not finitely generated). As proved in Proposition 2.3, this number of conjugacy classes is the smallest possible.

Theorem 4.1. *Given a countable group G_0 belonging to $\mathcal{C}'$, there exists a countable group G containing G_0 such that:*

- *G is sharply 2-transitive and belongs to $\mathcal{C}'$;*
- *G has exactly four conjugacy classes, the trivial element, the set of involutions, the set of translations and the set of homotheties.*

Proof. Starting from G_0, we are going to construct inductively an increasing sequence of groups $(G_n)_{n\in\mathbb{N}}$ in the class $\mathcal{C}'$ such that the group $G = \bigcup_{n\in\mathbb{N}} G_n$ is in the class $\mathcal{C}'$, and such that all its pairs of involutions are of type $D_\mathbb{Q}$. As noted in Remark 3.4, this implies that G acts sharply 2-transitively on its set of involutions.

We note that if $g \in G_n$ is a homothety (i.e. it centralizes an involution) in G_n, then it still is a homothety in G_{n+1}. Similarly, if $g \in G_n$ is a translation (i.e. a product of 2 distinct involutions) in G_n, it still is a translation in G_{n+1}, its type may change from $D_\mathbb{Z}$ to $D_\mathbb{Q}$ but not the other way around.

Suppose that the group G_n has already been constructed and let us construct G_{n+1} as an increasing union of groups G_n^m, starting with $G_n^0 = G_n$. We fix (u_0, v_0) a pair of involutions of type $D_\mathbb{Q}$ in G_0. Consider an enumeration $g_1, g_2, \ldots,$ of the maximal isolated elements of G_n, an enumeration $h_1, h_2, \ldots$ of its homotheties, and an enumeration $(u_1, v_1), (u_2, v_2), \ldots$ of the set maximal pairs of involutions of type $D_\mathbb{Z}$. For each homothety h_k, we denote by i_k the unique involution it centralizes (uniqueness follows from 2-sharpness of the action on involutions). Starting with $G_n^0 = G_n$, we define inductively an increasing sequence of groups G_n^m as follows.

(1) Define $G_n^{m+1/3} = \langle G_n^m, t \mid tu_m t^{-1} = u_0,\ tv_m t^{-1} = v_0\rangle$ as in Corollary 3.22 if (u_m, v_m) is maximal of type $D_\mathbb{Z}$ in G_n^m and $G_n^{m+1/3} = G_n^m$ otherwise.
(2) Define $G_n^{m+2/3} = \langle G_n^{m+1/3}, t \mid tg_m t^{-1} = u_0 v_0\rangle$ as in Corollary 3.23 if g_m is a maximal isolated element in $G_n^{m+1/3}$, and $G_n^{m+2/3} = G_n^{m+1/3}$ otherwise.
(3) Define $G_n^{m+1} = \langle G_n^{m+2/3}, t \mid ti_m t^{-1} = i_1,\ th_m t^{-1} = h_1\rangle$ as in Corollary 3.24.

We finally define $G_{n+1} = \bigcup_m G_n^m$. Corollaries 3.22–3.24 show that each group G_n^m is in $\mathcal{C}'$ and that the embedding of each group in the next one preserves maximality. Lemma 3.18 then concludes that G_{n+1} belongs to $\mathcal{C}'$, and that the embedding $G_n \subset G_{n+1}$ preserves maximality. Applying again Lemma 3.18 to $G = \bigcup_n G_n$, we see that G belongs to $\mathcal{C}'$.

We now check that G acts sharply 2-transitively on its set of involutions. Since G belongs to $\mathcal{C}$, it suffices to check that no pair of involutions of G is of type $D_\mathbb{Z}$ (see Remark 3.4). Thus, consider (u, v) a pair of involutions of G, and let $n \in \mathbb{N}$ be such that $u, v \in G_n$. If (u, v) is of type $D_\mathbb{Q}$ in G_n, it is obviously of type $D_\mathbb{Q}$ in G. Otherwise, let $(\tilde{u}, \tilde{v})$ be a maximal pair of type $D_\mathbb{Z}$ in G_n such that $\langle u, v\rangle \subset \langle \tilde{u}, \tilde{v}\rangle$. Since all embeddings preserve maximality, at each step, the pair $(\tilde{u}, \tilde{v})$ remains maximal unless it becomes of type $D_\mathbb{Q}$. Then step 1 of the construction ensures that $(\tilde{u}, \tilde{v})$ and (u, v) become of type $D_\mathbb{Q}$ at some step. This shows that (u, v) is of type $D_\mathbb{Q}$ in G_{n+1}, hence also in G. This shows that G acts sharply 2-transitively on its set of involutions. It also follows that all translations of G are conjugate.

Since G belongs to $\mathcal{C}'$, it remains to show that G has no isolated element and that all homotheties of G are conjugate.

Assume by contradiction that there exists $g \in G$ a maximal isolated element, (equivalently, $\langle g\rangle$ is malnormal in G). Then g is a maximal isolated element in all G_n^m, but step 2 ensures that g is conjugate in $\langle u_0, v_0\rangle$ in G_{n+1}, contradicting that g is isolated in G.

Similarly, step 3 ensures that all homotheties of G_n become conjugate in G_{n+1}. Since any homothety of G is a homothety in some G_n, this shows that all homotheties of G are conjugate. □

Having only four conjugacy classes, the group G above is not far from being simple, but may still fail to be so. The following result shows that we can additionally ensure that G is simple (see also [AT23]).

Theorem 4.2. *Given a countable group G_0 belonging to $\mathcal{C}'$, there exists a countable group G containing G_0 such that:*

- *G is sharply 2-transitive and belongs to $\mathcal{C}'$;*
- *G has exactly four conjugacy classes, the trivial element, the set of involutions, the set of translations and the set of homotheties; and*
- *G is simple.*

Taking, for instance, $G_0 = \mathbb{Q}^* \ltimes \mathbb{Q}$, we get the following result.

Corollary 4.3. *There exists a countable sharply 2-transitive group in the class $\mathcal{C}'$ which is simple and has exactly four conjugacy classes.*

Proof of Theorem 4.2. We claim that one can construct a group G_0' containing G_0 and in the class $\mathcal{C}'$ such that there are four distinct involutions $u_1, u_2, u_3, u_4 \in G_0'$ such that $h = u_1u_2u_3$ and $h' = u_1u_2u_3u_4$ are two homotheties.

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

If the claim holds, then one can take for G the group constructed by applying Theorem 4.1 to G_0'. Indeed, h and h' are still homotheties in G, and let us check that G is simple. Consider N any non-trivial normal subgroup. If N contains an involution or a translation, then it contains all of them so contains h or h' respectively. This shows that in all cases, N contains all homotheties, so N also contains the involution $h'h^{-1}$, hence all involutions, all translations, and $N = G$.

We now prove the claim. The group $G_0 * \langle t\rangle$ belongs to $\mathcal{C}'$ by Example 3.20, and if $u, v, w, z \in G_0$ are 4 distinct involutions, then the element $h = uv(twt^{-1})$ is isolated, and Proposition 3.23 yields a larger group G_1 in which h is a homothety. Similarly, the free product $G_1 * \langle s\rangle$ belongs to $\mathcal{C}'$, the element $h' = h(szs^{-1}) \in G_1 * \langle s\rangle$ is isolated, and we can embed $G_1 * \langle s\rangle$ using Proposition 3.23 into a larger group G_0' to ensure that h' is a homothety. This proves our initial claim. □

5. Getting finite generation

5.1 Small cancellation over relatively hyperbolic groups

Let G be a group hyperbolic relative to a collection of subgroups $\mathcal{H} = \{H_\lambda\}_{\lambda\in\Lambda}$. Recall that an element $g \in H$ is *parabolic* if it is conjugate in some H_λ, and *hyperbolic* if it is not parabolic and of infinite order (we note that in [Osi10], hyperbolic elements are synonymous with non-parabolic, and may have finite order). Given a hyperbolic element $h \in G$, its commensurator $\mathrm{Comm}_G(\langle h\rangle) = \{g \in G \mid gh^ng^{-1} = h^{\pm n} \text{ for some } n \in \mathbb{N}^*\}$ is the unique maximal virtually cyclic subgroup of G containing h. Following [Osi10], we use the following technical definition.

Definition 5.1. Let G be a relatively hyperbolic group. We say that a subgroup $H \subset G$ is *suitable* if it contains two hyperbolic elements $h_1, h_2 \in H$ (of infinite order) such that $\mathrm{Comm}_G(\langle h_1\rangle) \cap \mathrm{Comm}_G(\{h_2\}) = \{1\}$.

We need the following slight refinement of Theorem 2.4 in [Osi10].

Theorem 5.2. *Let G be a group hyperbolic relative to a subgroup P, and $H \subset G$ be a suitable subgroup. Let $t_1, \dots, t_n$ be arbitrary elements of G. Then there exists an epimorphism $\eta : G \twoheadrightarrow Q$ such that:*

(1) *Q is hyperbolic relative to $\eta(P)$;*
(2) *η is injective in restriction to P;*
(3) *$\eta(H)$ is a suitable subgroup of Q;*
(4) *$\eta(t_1), \dots, \eta(t_n)$ belong to $\eta(H)$;*
(5) *for every finite subgroup F of Q, there exists a subgroup $F' \subset G$ isomorphic to F such that $\eta(F') = F$.*

Proof. This is Theorem 2.4 in [Osi10] except for Assertion 5 which is only stated for finite cyclic groups but the argument actually works for all finite groups. One can also refer to [Cou13, Proposition 6.12] where this assertion is proved for small cancellation quotients of hyperbolic groups, or to [DG18, Lemma 4.3] in the context of Dehn fillings. □

5.2 Construction

The main result of this section is the following Theorem from which we will deduce Theorems 1.1 and 1.4.

Theorem 5.3. *Let G_0 be a sharply 2-transitive group belonging to the class $\mathcal{C}'$. Then there exists a 2-generated sharply 2-transitive group G with Kazhdan property (T), such that G contains G_0 and every element of G is conjugate to an element of G_0.*

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

Theorems 1.1 and 1.4 are immediate consequences of the following corollary.

Corollary 5.4. *There exists an infinite sharply 2-transitive group G of characteristic 0 with the following properties:*

- *G is generated by two elements;*
- *G has exactly four conjugacy classes;*
- *G is simple;*
- *G has property (T).*

Proof of the corollary. Start with a countable simple sharply 2-transitive group G_0 of characteristic 0 with four conjugacy classes and belonging to $\mathcal{C}'$, whose existence is proved in Theorem 4.2. Let G be the 2-generated group provided by Theorem 5.3. Then G has at most four conjugacy classes, and at least four by Proposition 2.3. Simplicity of G follows immediately from the simplicity of G_0 since all conjugacy classes of G intersect G_0. □

The following notation will be convenient.

Definition 5.5. Consider a group G with two subgroups $P, H \subset G$. We say that (G, P, H) satisfies $(*)$ if the following hold:

(1) G is a group in the class $\mathcal{C}'$;
(2) G is hyperbolic relative to P;
(3) H is a suitable subgroup of G.

The proof of Theorem 5.3 is based on the following result which will be applied iteratively.

Proposition 5.6. *Consider a group G with two subgroups $P, H \subset G$ such that (G, P, H) satisfies $(*)$. Let $E \subset G$ be a finite or virtually cyclic subgroup of G. Then there exists an epimorphism $\eta : G \twoheadrightarrow \bar{G}$ which is injective on P, such that $(\bar{G}, \eta(P), \eta(H))$ satisfies $(*)$ and such that $\eta(E) \subset \eta(H)$ and $\eta(E)$ is conjugate to a subgroup of $\eta(P)$. Moreover, every involution of $\bar{G}$ is the image of an involution of G.*

We first deduce Theorem 5.3 from Proposition 5.6.

Proof of Theorem 5.3. Let G_0 be a sharply 2-transitive group belonging to $\mathcal{C}'$. Let H_1 be a 2-generated torsion-free hyperbolic group with Kazhdan property (T); the existence of such group is standard: for instance, take a common torsion-free hyperbolic quotient (see [Ols93] and [Cha00, Th. 5.19]) of a free group F_2 and a torsion-free cocompact lattice in $\mathrm{Sp}(2, 1)$. Consider the free product $G_1 = G_0 * H_1$. It is hyperbolic relative to $P_1 := G_0$. Moreover, H_1 is a suitable subgroup of G_1, and it is in the class $\mathcal{C}'$ by Proposition 3.19, so (G_1, P_1, H_1) satisfies $(*)$.

Let $E_1, \ldots, E_n, \ldots$ be an enumeration of all the virtually cyclic subgroups of G_1 (including all cyclic groups of order 2 or ∞). We construct inductively a chain of quotients using Proposition 5.6 as follows. Apply Proposition 5.6 to (G_1, P_1, H_1) and $E_1 \subset G_1$, and denote by $G_2 = \bar{G}_1$ the obtained quotient and by $\eta_1 : G_1 \twoheadrightarrow G_2$ the quotient map. Define $H_2 = \eta(H_1)$ and $P_2 = \eta(P_1) \simeq G_0$ so that the image of E_1 is contained in H_2 and conjugate in P_2. Since (G_2, P_2, H_2) satisfies $(*)$, one can repeat the argument and obtain a chain of quotients

$$G_1 \xrightarrow{\eta_1}\!\!\!\!\!\rightarrow G_2 \xrightarrow{\eta_2}\!\!\!\!\!\rightarrow \cdots \xrightarrow{\eta_{n-1}}\!\!\!\!\!\rightarrow G_n \xrightarrow{\eta_n}\!\!\!\!\!\rightarrow \cdots$$

and subgroups $H_n = \eta_{n-1}(H_{n-1})$, $P_n = \eta_{n-1}(P_{n-1}) \simeq G_0$ of G_n such that (G_n, P_n, H_n) satisfies $(*)$ and such that the image of $E_1, \ldots, E_{n-1}$ in G_n are contained in H_n and conjugate in P_n.

Let G_∞ be the direct limit of this chain and denote the corresponding epimorphism by $\eta_\infty : G_1 \to G_\infty$. Denote $H_\infty = \eta_\infty(H_1)$ and $P_\infty = \eta_\infty(P_1) \simeq G_0$. Any element $g \in G_1$ is contained

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

in some E_j, so its image in G_{j+1} lies in H_{j+1}, hence $\eta_\infty(g) \in H_\infty$. This shows that $\eta_{\infty|H_1}$ is onto, so G_∞, as a quotient of H_1, has property (T) and is 2-generated. Similarly, if $g \in E_j$, then its image in G_{i+1} is conjugate in P_{j+1}, which shows that every element of G_∞ has a conjugate in P_∞.

We finally check that G_∞ acts sharply 2-transitively on its set of involutions. Let $u, v \in G_\infty$ be a pair of involutions, and let us prove that it is conjugate to a pair of involutions of P_∞. There exist n and preimages $u_n, v_n \in G_n$ of u, v such that u_n, v_n are involutions. Since every involution of G_i is the image of an involution of G_{i-1} under η_{i-1}, there exist involutions $u_1, v_1 \in G_1$ that map to $u, v \in G_\infty$, respectively. Since the elementary group $\langle u_1, v_1\rangle = E_j$ for some index j, its image in G_{j+1} is conjugate in P_{j+1} hence $\langle u, v\rangle$ is conjugate in P_∞. Since $P_\infty \simeq G_0$ acts transitively on its pairs of involutions, so does G_∞.

It remains to check that the centralizer of a pair of distinct involutions $u, v \in G_\infty$ is trivial. If $z \in G_\infty$ centralizes $u \neq v$, then there exist n and lifts $u_n, v_n, z_n \in G_n$ of u, v, z such that $u_n \neq v_n$ are involutions and z_n centralizes u_n, v_n. Since G_n belongs to $\mathcal{C}$, it acts freely on its pairs of involutions so $z_n = 1$ and $z = 1$ which concludes the proof. □

It remains to prove Proposition 5.6.

Proof of Proposition 5.6. Consider (G, P, H) satisfying $(*)$. In a first step, we are going to embed G in a group G_1 so that E is conjugate to a subgroup of P in G_1. If E is already parabolic in G, we let $G_1 = G$ so assume otherwise. This implies that E is infinite because, since G belongs to $\mathcal{C}'$, every finite subgroup of G has order at most 2 (Lemma 3.15) and all involutions are conjugate (Definition 3.3).

Since G is relatively hyperbolic, the commensurator $\hat{E}$ of E is virtually cyclic infinite. By Lemma 3.15, $\hat{E}$ is isomorphic to $\mathbb{Z}$, $D_\mathbb{Z}$ or $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$. Note that if $\langle h\rangle$ is a maximal infinite cyclic subgroup of $\hat{E}$, then h is isolated, a translation of type $D_\mathbb{Z}$, or a homothety accordingly.

The following consequence of relative hyperbolicity will be useful: $\bar{P}$ is almost malnormal in $\bar{G}$, i.e. if $\bar{P}^g \cap \bar{P}$ is infinite, then $g \in \bar{P}$ (see, for instance, Lemma 8.3 in [Osi10]). We note that since G lies in the class $\mathcal{C}$, it contains a pair of involutions (u_0, v_0) of type $D_\mathbb{Q}$. Such a pair is conjugate in $\bar{P}$ because a group isomorphic to $D_\mathbb{Q}$ has to be parabolic.

Consider E' a subgroup of P isomorphic to $\hat{E}$: if E is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$, Lemma 3.16 ensures that E' exists, and if $\hat{E} = \langle u, v\rangle \simeq D_\mathbb{Z}$, we choose $E' = \langle u_0, v_0\rangle$. Let $\sigma : \hat{E} \to E'$ be an isomorphism and let G_1 be the HNN extension $G_1 = \langle G, t \ | txt^{-1} = \sigma(x), x \in \hat{E}\rangle$, and let us check that (G_1, P, H) satisfies $(*)$.

The group G_1 belongs to $\mathcal{C}'$:

- if $\hat{E} = \langle h\rangle \simeq \mathbb{Z}$, this follows from Corollary 3.23;
- if $\hat{E} \simeq D_\mathbb{Z}$, this follows from Corollary 3.22;
- if $\hat{E} = \langle h\rangle \times \langle u\rangle \simeq \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$, then this follows from Corollary 3.24.

The group G_1 is hyperbolic relative to P by Dahmani's combination theorem [Dah03] as stated in [Osi10, Theorem 2.5] where we view G as hyperbolic relative to $\{P, \hat{E}\}$ (see, for instance, [Osi10, Theorem 2.1]).

We now check that H is suitable in G_1. By [Osi10, Lemma 2.3], H contains infinitely many non-commensurable hyperbolic elements $h_1, h_2, \ldots \in H$ of infinite order such that $\mathrm{Comm}_G(\langle h_i\rangle) = \langle h_i\rangle$. Up to discarding at most two elements, we may assume that no power of any h_i is G-conjugate in $\hat{E}$ or $\hat{E}'$. To check that $\mathrm{Comm}_{G_1}(\langle h_i\rangle) = \langle h_i\rangle$, consider $G_1 \curvearrowright T$ be the Bass–Serre tree of the HNN extension defining G_1, and $x \in T$ a vertex with stabilizer G.

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

For all $k \geq 1$, the set of fixed points of h_i^k in T is exactly $\{x\}$. It follows that $\mathrm{Comm}_{G_1}(\langle h_i \rangle)$ fixes x so $\mathrm{Comm}_{G_1}(\langle h_i \rangle) = \mathrm{Comm}_G(\langle h_i \rangle) = \langle h_i \rangle$. This shows that H is suitable in G_1.

We thus have embedded G in a group G_1 such that (G_1, P, H) satisfies $(*)$, and such that E is conjugate to a subgroup of P in G_1.

Let $\{t_1, t_2\}$ be a generating set of E, and recall that t denotes the stable letter of the HNN extension defining G_1; in the case where we defined $G = G_1$, we let $t = 1$. Let $\eta_1 : G_1 \twoheadrightarrow \bar{G}$ be the quotient of G_1 given by Theorem 5.2 applied to the elements $t, t_1, t_2 \in G_1$. We denote by $\eta : G \to \bar{G}$ the restriction of η_1. Since G_1 is generated by G and t, and since $\eta_1(t) \in \eta_1(H) \subset \eta_1(G)$, it follows that $\eta : G \to \bar{G}$ is onto. Theorem 5.2 says that η is injective in restriction to P, that the group $\bar{G}$ is hyperbolic relative to $\eta(P)$, that $\eta(H)$ is suitable in $\bar{G}$, and that $\eta(E) = \langle \eta(t_1), \eta(t_2) \rangle$ is contained in $\eta(H)$. Since E is conjugate in P in G_1, $\eta(E)$ is conjugate in $\eta(P)$ in $\bar{G}$.

It also says that any involution $v \in \bar{G}$ is the image of some involution $\tilde{v}$ in G_1. It follows that v is the image of some involution in G: $\tilde{v}$ is conjugate to some involution $\tilde{v}' \in G$, and since η is onto, there exists $g \in G$ such that $\eta(g\tilde{v}'g^{-1}) = v$. We note that since all involutions of G are conjugate to each other, this is also the case in $\bar{G}$.

To check that $(\bar{G}, \eta(P), \eta(H))$ satisfies $(*)$, the only remaining fact to prove is that $\bar{G}$ belongs to $\mathcal{C}'$. In what follows, we use the notation $\bar{P} = \eta(P)$.

We first characterize finite and virtually cyclic subgroups of $\bar{G}$. Since G_1 belongs to $\mathcal{C}'$, every non-trivial finite subgroup of G_1 is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (see Lemma 3.15), and the same holds for $\bar{G}$ by assertion (5) of Theorem 5.2. By Lemma 3.15, it follows that every infinite virtually cyclic subgroup of $\bar{G}$ is isomorphic to $\mathbb{Z}$, $D_\mathbb{Z}$ or $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$.

We now check that $\bar{G}$ belongs to $\mathcal{C}$ (Definition 3.3).

First, for any pair of involutions $(u, v) \in \mathcal{I}_{\bar{G}}^{(2)}$, the group $\langle u, v \rangle$ is infinite dihedral since finite subgroups of $\bar{G}$ have order at most 2. We already noted that $\bar{G}$ acts transitively on its involutions. Let $(u, v) \in \mathcal{I}_{\bar{G}}^{(2)}$ be a pair of involutions, and let us check that its centralizer $Z = Z_{\bar{G}}(\langle u, v \rangle)$ is trivial and that (u, v) is either or type $D_\mathbb{Z}$ or $D_\mathbb{Q}$.

If $\langle u, v \rangle$ is not parabolic, then its commensurator $\mathrm{Comm}_{\bar{G}}(\langle u, v \rangle) \supset \bar{G}(\langle u, v \rangle)$ is virtually cyclic, necessarily isomorphic to $D_\mathbb{Z}$, so (u, v) is of type $D_\mathbb{Z}$ and its centralizer Z is trivial.

If $\langle u, v \rangle$ is parabolic in $\bar{G}$, we may assume that $\langle u, v \rangle \subset \bar{P}$. Then Z is contained in $\bar{P}$ by almost-malnormality of $\bar{P}$, so Z is trivial because $\bar{P}$ is isomorphic to the subgroup P of G, and G belongs to $\mathcal{C}$. Denote by $\tilde{u}, \tilde{v} \in P$ the preimages of u, v by the isomorphism $\eta_{|P} : P \xrightarrow{\sim} \bar{P}$.

If $(\tilde{u}, \tilde{v})$ is of type $D_\mathbb{Q}$ in G, then $\langle \tilde{u}, \tilde{v} \rangle$ is contained in a subgroup $\tilde{D}$ of G isomorphic to $D_\mathbb{Q}$. Such a group $\tilde{D}$ has to be parabolic, hence conjugate in P. It follows that (u, v) is of type $D_\mathbb{Q}$ since $\langle u, v \rangle \subset \eta(\tilde{D}) \simeq D_\mathbb{Q}$. If $(\tilde{u}, \tilde{v})$ is of type $D_\mathbb{Z}$ in G, then $\mathrm{Comm}_P(\langle \tilde{u}, \tilde{v} \rangle) \simeq D_\mathbb{Z}$. Using the isomorphism η, $\mathrm{Comm}_{\bar{P}}(\langle u, v \rangle) \simeq D_\mathbb{Z}$. By almost malnormality of $\bar{P}$, we get that $\mathrm{Comm}_{\bar{G}}(\langle u, v \rangle) = \mathrm{Comm}_{\bar{P}}(\langle u, v \rangle)$ so (u, v) is of type $D_\mathbb{Z}$ in $\bar{G}$.

To prove that $\bar{G}$ belongs to $\mathcal{C}$, it remains to prove that $\bar{G}$ acts transitively on the set of pairs $(u, v) \in \mathcal{I}_{\bar{G}}^{(2)}$ of type $D_\mathbb{Q}$. However, any such pair is conjugate in $\bar{P}$ so it remains to check that $\bar{P}$ acts transitively on its set of pairs $(u, v) \in \mathcal{I}_{\bar{P}}^{(2)}$ of type $D_\mathbb{Q}$. Viewing $\bar{P} \simeq P$ as a subgroup of G, we know that any two pairs in $\mathcal{I}_P^{(2)}$ of type $D_\mathbb{Q}$ are in the same G-orbit. Since P is almost malnormal in G, they are actually in the same P-orbit. This concludes the proof that $\bar{G}$ is in the class $\mathcal{C}$.

To check that $\bar{G}$ belongs to $\mathcal{C}'$, it remains to show that any element $h \in \bar{G}$ of infinite order is a translation, a homothety or is isolated. If $\langle h \rangle$ is not parabolic, this follows from the classification of elementary subgroups (the three cases occurring when $\mathrm{Comm}_{\bar{G}}(\langle h \rangle)$ is isomorphic to $D_\mathbb{Z}$, $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$ or $\mathbb{Z}$, respectively). Thus, we may assume that $h \in \bar{P}$. Let $\tilde{h} \in P$ be its preimage in

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press

P under the isomorphism $\eta : P \xrightarrow{\sim} \bar{P}$. If $\tilde{h}$ is a translation or a homothety in G, it stays so in $\bar{G}$ because η does not kill any involution. If $\tilde{h}$ is isolated in G, it is isolated in P and h is isolated in $\bar{P}$. Its commensurator $\mathrm{Comm}_{\bar{G}}(\langle h\rangle)$ is contained in $\bar{P}$ by almost malnormality. It follows that h is isolated in $\bar{G}$, which proves that $\bar{G}$ belongs to $\mathcal{C}'$ and concludes the proof. □

Acknowledgements

The first named author would like to thank Katrin Tent for stimulating discussions. The second named author benefited from the stimulating atmosphere of the Center Henri Lebesgue, a French government 'Investissements d'Avenir', bearing the reference ANR-11-LABX-0020-01.

Conflicts of interest

None.

Financial support

The first named author acknowledges support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044–390685587, Mathematics Münster: Dynamics-Geometry-Structure and from CRC 1442 Geometry: Deformations and Rigidity. The second named author acknowledges support from the European Research Council (ERC GOAT 101053021) and from the ANR project ANR-22-CE40-0004.

Journal information

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.

References

AM-H00 S. Akbari and M. Mahdavi-Hezavehi, *Normal subgroups of $GL_n(D)$ are not finitely generated*, Proc. Amer. Math. Soc. **128** (2000), 1627–1632.

AT23 S. André and K. Tent, *Simple sharply 2-transitive groups*, Trans. Amer. Math. Soc. **376** (2023), 3965–3993.

AKPR19 A. Atkarskaya, A. Kanel-Belov, E. Plotkin and E. Rips, *Construction of a quotient ring of $\mathbb{Z}_2\mathcal{F}$ in which a binomial $1+w$ is invertible using small cancellation methods*, in *Groups, algebras, and identities*, Contemporary Mathematics (Israel Mathematical Conference Proceedings), vol. 726 (American Mathematical Society, Providence, RI; Bar-Ilan University, Ramat Gan, 2019), 1–76.

BHV08 B. Bekka, P. de la Harpe and A. Valette, *Kazhdan's property*, New Mathematical Monographs, vol. 11 (Cambridge University Press, Cambridge, 2008).

Cam00 P. Cameron, *Classical groups*, Lecture Notes for an M.Sc. course at the University of London (2000), https://webspace.maths.qmul.ac.uk/p.j.cameron/class_gps/.

Cha00 C. Champetier, *L'espace des groupes de type fini*, Topology **39** (2000), 657–680.

Coh71 P. Cohn, *The embedding of firs in skew fields*, Proc. Lond. Math. Soc. (3) **3** (1971), 193–213.

Cou13 R. Coulon, *Small cancellation theory and Burnside problem*, Preprint (2013), arXiv:1302.6933.

Dah03 F. Dahmani, *Combination of convergence groups*, Geom. Topol. **7** (2003), 933–963.

DG18 F. Dahmani and V. Guirardel, *Recognizing a relatively hyperbolic group by its Dehn fillings*, Duke Math. J. **167** (2018), 2189–2241.

DM96 J. Dixon and B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics, vol. 163 (Springer, 1996).

GG14 Y. Glasner and D. Gulko, *Sharply 2-transitive linear groups*, Int. Math. Res. Not. (IMRN) **2014** (2014), 2691–2701.

GMS15 G. Glauberman, A. Mann and Y. Segev, *A note on groups generated by involutions and sharply 2-transitive groups*, Proc. Amer. Math. Soc. **143** (2015), 1925–1932.

Ker74 W. Kerby, *On infinite sharply multiply transitive groups*, Hamburger mathematische Einzelschriften, Heft 6 (Vandenhoeck & Ruprecht, 1974).

Ols91 A. Y. Olshanskii, *Geometry of defining relations in groups*, Mathematics and its Applications (Soviet Series), vol. 70 (Springer, 1991).

Ols93 A. Y. Olshanskii, *On residualing homomorphisms and G-subgroups of hyperbolic groups*, Int. J. Algebra Comput. **3** (1993), 365–409.

Osi10 D. Osin, *Small cancellations over relatively hyperbolic groups and embedding theorems*, Ann. of Math. (2) (2010), 1–39.

RST17 E. Rips, Y. Segev and K. Tent, *A sharply 2-transitive group without a non-trivial abelian normal subgroup*, J. Eur. Math. Soc. (JEMS) **19** (2017), 2895–2910.

RT19 E. Rips and K. Tent, *Sharply 2-transitive groups of characteristic 0*, J. Reine Angew. Math. **2019** (2019), 227–238.

Ten16 K. Tent, *Infinite sharply multiply transitive groups*, Jahresber. Dtsch. Math.-Ver. **118** (2016), 75–85.

Tit52 J. Tits, *Sur les groupes doublement transitifs continus*, Comment. Math. Helv. **26** (1952), 203–224.

Tit56 J. Tits, *Sur les groupes doublement transitifs continus: correction et compléments*, Comment. Math. Helv. **30** (1956), 234–240.

Zas35a H. Zassenhaus, *Kennzeichnung endlicher linearer gruppen als permutationsgruppen*, Abh. Math. Semin. Univ. Hambg. **11** (1935), 17–40.

Zas35b H. Zassenhaus, *Über endliche fastkörper*, Abh. Math. Semin. Univ. Hambg. **11** (1935), 187–220.

Simon André sandre@uni-muenster.de
Institut für Mathematische Logik und Grundlagenforschung, WWU Münster,
Einsteinstraße 62, 48149 Münster, Germany

Vincent Guirardel vincent.guirardel@univ-rennes1.fr
Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

https://doi.org/10.1112/S0010437X24007358 Published online by Cambridge University Press