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On the monodromy at infinity of a polynomial map, II

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Abstract. To a polynomial map $f \colon \mathbf{C}^{n+1} \to \mathbf{C}$ one can attach a monodromy transformation on the complex cohomology of its generic fiber that reflects its asymptotic behaviour. In this paper this transformation is determined for a class of generic polynomials in terms of data attached to projective hypersurfaces with isolated singularities.

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1. Introduction

In the last years a lot of work has been concentrated on the study of the behaviour at infinity of polynomial maps (see for example [27], [28], [3], [4], [29], [13], [15]–[17], [30], [22], [31], [9], among others). This behaviour can be very complicated, therefore one of the main ideas was to find special classes of polynomial maps which have, in some sense, nice properties at infinity. In this paper, we completely determine the complex algebraic monodromy at infinity for a special class of polynomial maps (which is complicated enough to show the nature of the general problem).

Next, we give the precise definitions: Let $f \colon \mathbf{C}^{n+1} \to \mathbf{C}$ be a map given by a polynomial with complex coefficients (which will be also denoted by f). Then there exists a finite set $\Gamma \subset \mathbf{C}$ such that the map

$$f|_{\mathbf{C}^{n+1} - f^{-1}(\Gamma)} \colon \mathbf{C}^{n+1} - f^{-1}(\Gamma) \to \mathbf{C} - \Gamma$$

is a locally trivial $\mathcal{C}^{\infty}$-fibration ([18]). We denote by Γ_f the smallest subset of the complex plane with this property. Γ_f contains the set Σ_f of critical values of f, but in general it is bigger. Fix $t_0 \in \mathbf{C}$ such that $|t_0| > \max\{|t| : t \in \Gamma_f\}$. The complex

algebraic monodromy associated with the path $s \mapsto t_0\, e^{2\pi i s}$, $s \in [0, 1]$, is denoted by

$$(T_f^\infty)^* \colon H^*(f^{-1}(t_0), \mathbf{C}) \to H^*(f^{-1}(t_0), \mathbf{C}).$$

This isomorphism is called the monodromy at infinity of f. As we will see later, $(T_f^\infty)^*$ is a very delicate invariant of f.

On studying topological properties of polynomial maps, one usually imposes some condition which insures the absence of vanishing cycles 'at infinity' for a suitable compactification of the map f (tameness, Malgrange condition, ..., cf. e.g. [3] and [30]). From this point of view, a class of polynomial maps which looks natural to study is the following.

DEFINITION. A polynomial $f \in \mathbf{C}[X_1, \ldots, X_{n+1}]$ will be called a $(*)$-polynomial if it verifies the following condition

$$(*) \quad \begin{cases} \text{For } t \in \mathbf{C} - \Sigma_f, \text{ the closure in } \mathbf{P}^{n+1} \text{ of the affine} \\ \text{hypersurface } \{f = t\} \text{ is non-singular.} \end{cases}$$

The goal of this article is the computation of $(T_f^\infty)^*$ for $(*)$-polynomials.

We will assume that $n \geqslant 2$. The case $n = 1$ is completely clarified in [9].

If $d = \deg(f)$ and $f = f_d + f_{d-1} + \cdots$ is the decomposition of f into homogeneous components, condition $(*)$ is equivalent to

$$\{x \in \mathbf{C}^{n+1} \mid \operatorname{grad} f_d(x) = 0, \ f_{d-1}(x) = 0\} = \{0\},$$

where grad denotes the gradient vector. The local analogue of $(*)$-polynomials are the superisolated singularities, introduced by Luengo in [14]. The local algebraic monodromy of superisolated surface singularities was determined by E. Artal in [1].

In the first part [9] of this sequence of papers, the following results are given (besides others):

(a) A $(*)$-polynomial f satisfies $\Gamma_f = \Sigma_f$ and any fiber of f has the homotopy type of a bouquet of n-dimensional spheres (cf. [7]). In particular, the only interesting monodromy transformation is $(T_f^\infty)^n$, which in the sequel will be denoted simply by T_f^∞.

(b) The hypersurface $X^\infty \subset \mathbf{P}^n$ given by $f_d = 0$ has only isolated singularities, and the monodromy at infinity (actually, the whole topology at infinity) depends only on the hypersurface X^∞. Thus the study of the monodromy at infinity of $(*)$-polynomials is very strongly linked with the study of projective hypersurfaces with isolated singularities. This link gives results in both directions (see below the Main Theorem, Corollary 3 and the comments after it).

(c) The characteristic polynomial of T_f^∞ is computable in terms of the characteristic polynomials of the local monodromies of the isolated singularities of X^∞ (cf. Corollary 2).

(d) On the other hand, the nilpotent part of X^∞ cannot be determined only from local data attached to the isolated singularities of X^∞, it depends essentially on the *position* of these singular points.

Part of the *global information* about the position of the singular points of X^∞ is already encoded in its Betti numbers. More subtle invariants are hidden in the complement $\mathbf{P}^n - X^\infty$ of X^∞, or in the cyclic coverings of $\mathbf{P}^n$ branched along X^∞. For algebraic surfaces, O. Zariski related this kind of invariants with the defect (or superabundance) of some linear systems, respectively with some Betti numbers of cyclic coverings. In the sequel we give the numerical invariants of X^∞ which will provide our description of T_f^∞.

For X a quasi-projective variety, denote by $b_q(X)$ (respectively, $p_q(X)$) the dimension of $H^q(X, \mathbf{C})$ (respectively, the dimension of the qth primitive cohomology of X). The numbers $p_n(X^\infty) = b_{n-1}(\mathbf{P}^n - X^\infty)$ and $p_{n-1}(X^\infty) = b_n(\mathbf{P}^n - X^\infty)$ are in general global invariants of X^∞ (Here, if $n = 2$, we define $p_2(X^\infty) = b_2(X^\infty) - 1$). We define a map $h\colon \pi_1(\mathbf{P}^n - X^\infty) \to \mathbf{Z}/d\mathbf{Z}$ as follows. If $n > 2$ then h is just the Hurewicz map (in fact, isomorphism)

$$\pi_1(\mathbf{P}^n - X^\infty) \to H_1(\mathbf{P}^n - X^\infty, \mathbf{Z}) = \mathbf{Z}/d\mathbf{Z}.$$

If $n = 2$, let r denote the number of irreducible components of X^∞, of degrees $d_1, \ldots, d_r$. Then h is defined as the composition

$$\pi_1(\mathbf{P}^n - X^\infty) \to H_1(\mathbf{P}^n - X^\infty, \mathbf{Z}) = \frac{\mathbf{Z}^r}{(d_1, \ldots, d_r)} \xrightarrow{\alpha} \mathbf{Z}/d\mathbf{Z},$$

where α is defined by $\alpha[(a_1, \ldots, a_r)] = \overline{\Sigma a_i}$. The composition of h with the characters $\rho_s\colon \mathbf{Z}/d\mathbf{Z} \to \mathbf{C}^*$ defined by $\rho_s(1) = \mathrm{e}^{2\pi i s/d}$ (for $1 \leqslant s \leqslant d - 1$) provide one-dimensional flat bundles $\mathbf{V}_s$ over $\mathbf{P}^{n+1} - X^\infty$ with monodromy representation $\rho_s \circ h$.

Let $j\colon \mathbf{P}^n - X^\infty \hookrightarrow \mathbf{P}^n$ denote the inclusion map. It is not difficult to see that the direct image sheaf $j_* \mathbf{V}_s$ $(= R^0 j_* \mathbf{V}_s)$ coincides with the extension by zero $j_! \mathbf{V}_s$, $s = 1, \ldots, d - 1$. We define the 'equivariant defect' by

$$\beta_s = \begin{cases} p_n(X^\infty) = b_{n-1}(\mathbf{P}^n - X^\infty) & \text{if } s = 0, \\ b_{n+1}(\mathbf{P}^n, j_* \mathbf{V}_s) & \text{if } s = 1, \ldots, d - 1. \end{cases}$$

Here, $b_{n+1}(\mathbf{P}^n, j_* \mathbf{V}_s)$ is the dimension of the sheaf cohomology $H^{n+1}(\mathbf{P}^n, j_* \mathbf{V}_s)$. This vector space is the $\mathrm{e}^{2\pi i s/d}$-eigenspace of $H^{n+1}(X_0')$, where X_0' is the dth cyclic covering of $\mathbf{P}^n$ branched along X^∞ and the action is induced by the natural Galois action (cf. (2.12), see also Section 2, VIII).

Set $\mathrm{Sing}(X^\infty) = \{p_1, \ldots, p_k\}$, and let F_i, μ_i, T_i be respectively the local Milnor fiber, the Milnor number and the local algebraic monodromy $H^{n-1}(F_i, \mathbf{C}) \to H^{n-1}(F_i, \mathbf{C})$ of the isolated hypersurface singularity (X^∞, p_i). We will call an invariant *local* if it can be expressed in terms of the local operators $\{T_i\}_{i=1}^k$ and the numbers n and d. We define the following local numerical invariants

$$\chi_0 = -\sum_{i=1}^k \mu_i + \frac{(-1)^n + (d-1)^{n+1}}{d} + (-1)^{n+1}$$

$$= (-1)^n (\chi(\mathbf{P}^n - X^\infty) - 1)$$

$$\chi_s = \chi_0 + (-1)^n \text{ for } s = 1, \ldots, d-1.$$

Now we are ready to formulate our main result. If T is an operator, let T_α denote its restriction to its generalized α-eigenspace and let $\#_l T_\alpha$ be the number of Jordan blocks of T_α of size l. Set $\#T_\alpha = \Sigma_{l \geqslant 1} \#_l T_\alpha$. With this notations one has.

MAIN THEOREM

(I) *If $\alpha = \mathrm{e}^{2\pi i s/d}$, $s = 0, \ldots, d-1$, then:*

(a) $\#_1(T_f^\infty)_\alpha = \chi_s + 2\beta_s - \Sigma_{i=1}^k \#(T_i)_\alpha$.
(b) $\#_2(T_f^\infty)_\alpha = -\beta_s + \Sigma_{i=1}^k \#_1(T_i)_\alpha$.
(c) $\#_{l+1}(T_f^\infty)_\alpha = \Sigma_{i=1}^k \#_l(T_i)_\alpha$ for $l \geqslant 2$.

(II) *If $\alpha^d \neq 1$, then $(T_f^\infty)_\alpha = \oplus_{i=1}^k \alpha^d \cdot (T_i)_{\alpha^{1-d}}$, i.e.* $\#_l(T_f^\infty)_\alpha = \Sigma_{i=1}^k \#_l(T_i)_{\alpha^{1-d}}$ *for all $l \geqslant 1$.*

COROLLARY 1.

(a) $\#_l(T_f^\infty)_\alpha = 0$ *for* $l \geqslant n+2$.
(b) $\#_{n+1}(T_f^\infty)_\alpha = 0$ *if* $\alpha^d \neq 1$ *or* $\alpha = 1$.

COROLLARY 2 [9, (3.3)]. *The characteristic polynomial of T_f^∞ is given by the following local formula.*

$$\det(\lambda \cdot \mathrm{Id} - T_f^\infty) =$$

$$= (\lambda - 1)^{(-1)^{n+1}} \cdot (\lambda^d - 1)^{((d-1)^{n+1} + (-1)^n)/d} \cdot \prod_{i=1}^k \frac{\det(\lambda^{d-1} \cdot \mathrm{Id} - T_i)}{(\lambda^d - 1)^{\mu_i}}.$$

Another byproduct of the main theorem is the following.

COROLLARY 3. *If $\alpha = \mathrm{e}^{2\pi i s/d}$, $s = 0, \ldots, d-1$, then*

$$\frac{\Sigma_{i=1}^k \#(T_i)_\alpha - \chi_s}{2} \leqslant \beta_s \leqslant \sum_{i=1}^k \#_1(T_i)_\alpha.$$

If $\Sigma_{i=1}^{k} \#(T_i)_\alpha \leqslant \chi_s$, then the lower bound given by Corollary 3 is useless, but in some cases it gives even the right value of β_0. For example, if $n = 2$ and $l_1, \ldots, l_d \in \mathbf{C}[X, Y, Z]$ are linear forms defining an arrangement of lines in $\mathbf{P}^2$ such that no more than two of them meet at a point, then $\beta_0 = d - 1$, and this is exactly the bound given by Corollary 3 above. From the discussion in [6, p. 161] follows that if (X^∞, p_i) are all nondegenerate singularities (i.e., $\Sigma_{i=1}^{k} \#(T_i)_1 = 0$), then the defect $\beta_0 = 0$. Also, in Zariski's book [26] we can find similar criteria for $n = 2$. Notice that our bound gives a sharper criterion: If $\Sigma_{i=1}^{k} \#_1(T_i)_1 = 0$ then $\beta_0 = 0$ (cf. also Remark 2.30 below). Indeed, it is not difficult to construct examples of isolated hypersurface singularities whose monodromy admits 1 as eigenvalue but it has no Jordan block of size one corresponding to eigenvalue 1. For example, the complex algebraic monodromy of the singularity given by $(x^p + y^q)(x^q + y^p) + z^p = 0$, where $2 \leqslant p < q$ and $\gcd(p, q) = 1$, has $p - 1$ Jordan blocks of size two for eigenvalue 1 but no blocks of size one for 1. Then the procedure described in [2, (1.1)] gives projective hypersurfaces to which the criterion above applies. For another corollary of the Main Theorem, see (2.16).

2. Proof of the main theorem

I. *The main construction and two exact sequences*

Let $f \colon \mathbf{C}^{n+1} \to \mathbf{C}$ be a polynomial map which satisfies the condition $(*)$. By [9, (2.6)], we can assume that f is of the form $f_d + x_{n+1}^{d-1}$, where f_d is homogeneous of degree d (and no singularity of X^∞ is on the hyperplane $x_{n+1} = 0$). Set (cf. [9, Sect. 5])

$$\mathcal{X} = \{([x], t) \in \mathbf{P}^{n+1} \times D : t(f_d(x_1, \ldots, x_{n+1}) + x_0 x_{n+1}^{d-1}) = x_0^d\},$$

where D denotes a disk of sufficiently small radius in the complex plane with center at the origin. Then the map $\pi \colon \mathcal{X} \to D$ given by $\pi([x], t) = t$ induces a locally trivial $\mathcal{C}^\infty$-fibration over $D - \{0\}$ with projective fibers, these are exactly the projective closures of the fibers of f. Moreover, if we denote by T the algebraic monodromy $H^n(\overline{f^{-1}(t_0)}) \to H^n(\overline{f^{-1}(t_0)})$ of the projective closure $\overline{f^{-1}(t_0)}$ associated with the path $s \mapsto t_0 \, e^{2\pi i s}$ ($s \in [0, 1]$, $|t_0|$ sufficiently large), then the monodromy of π over ∂D (with its natural orientation) is exactly T^{-1}. Theorem (4.6) in [9] says basically that the knowledge of T is equivalent to that of T_f^∞.

(2.1) THEOREM [9, (4.6)].

(a) For any $\alpha \neq 1$, $(T_f^\infty)_\alpha = T_\alpha$.

(b) For $\alpha = 1$ one has

 (i) $\#_1(T_f^\infty)_1 = b_n(X^\infty) + p_{n-1}(X^\infty) - \#T_1$.

 (ii) $\#_2(T_f^\infty)_1 = \#_1 T_1 - b_n(X^\infty)$.

(iii) $\#_{l+1}(T_f^\infty)_1 = \#_l T_1$ *for* $l \geqslant 2$.

The big disadvantage of the map π is that its central fiber $\pi^{-1}(0)$ is non-reduced. For this reason we consider the following construction (semi-stable normalization, cf. e.g. [24, (2.1)]): Let D' again be a disc of small radius and consider $\delta \colon D' \to D$ given by $\delta(t) = t^d$. Then the normalization $\mathcal{X}'$ of $\mathcal{X} \times_\delta D'$ can be identified with

$$\mathcal{X}' = \{([x], t) \in \mathbf{P}^{n+1} \times D' \mid f_d(x_1, \ldots, x_{n+1}) + t x_0 x_{n+1}^{d-1} = x_0^d\}.$$

Now $\pi' \colon \mathcal{X}' \to D'$ ($\pi([x], t) = t$) induces a locally trivial $\mathcal{C}^\infty$-fibration over $D' - \{0\}$ with algebraic monodromy $T^{-d} \colon H^n((\pi')^{-1}(t_0)) \to H^n((\pi')^{-1}(t_0))$.

Notice that now both $\mathcal{X}'$ and the central fiber $X_0' = (\pi')^{-1}(0)$ have only isolated singularities: $\mathrm{Sing}(\mathcal{X}') = \mathrm{Sing}(X^\infty) \times \{0\}$. In fact, the central fiber is the d-fold cyclic covering of $\mathbf{P}^n$ branched along X^∞, in particular if we set $\mathrm{Sing}(X_0') = \{p_1', \ldots, p_k'\}$, then the isolated singularities (X_0', p_i') are the dth suspensions of the singularities $(X^\infty, p_i)_{i=1}^k$ and the map π' provides smoothings of them. Let F_i' (respectively, T_i') be the Milnor fiber of (X_0', p_i') (respectively, the monodromy $H^n(F_i') \to H^n(F_i')$ corresponding to the smoothing given by π'), $1 \leqslant i \leqslant k$. Then the exact sequence of vanishing cycles is

$$0 \to H^n(X_0') \to H^n(X_t') \to \bigoplus_{i=1}^{k} H^n(F_i') \to P^{n+1}(X_0') \to 0, \qquad (2.2)$$

where $X_t' = (\pi')^{-1}(t)$ (for some fixed $t \neq 0$) and $P^{n+1}(X_0')$ is the primitive cohomology $\mathrm{Ker}[H^{n+1}(X_0') \to H^{n+1}(X_t')]$ (for details see [9, (10), (5.3)]).

Our second exact sequence is given by the generalized invariant cycle theorem proved in the Appendix of [9]

$$0 \to H^n(X_0') \to \mathrm{Ker}((T^{-d})_1 - \mathrm{Id}) \to \bigoplus_{i=1}^{k} H_{\{p_i'\}}^{n+1}(\mathcal{X}') \to 0, \qquad (2.3)$$

(we recall that $H_{\{p_i'\}}^{n+1}(\mathcal{X}')$ denotes cohomology with supports in $\{p_i'\}$, i.e. $H_{\{p_i'\}}^{n+1}(\mathcal{X}') = H^{n+1}(\mathcal{X}', \mathcal{X}' - \{p_i'\})$). Both sequences are exact sequences of mixed Hodge structures and there is a natural monodromy action on them, which at the level of $H^n(X_t')$ is T^{-d}. The main point of the paper is the construction of an action on these exact sequences which at the level of $H^n(X_t')$ is T^{-1}. More precisely: we would like to understand the monodromy of π, but this map has a non-reduced central fiber, which makes the study difficult. Then we go to the normalization of the d-fold covering π, which is π', and we lift the monodromy of π to the level of π'.

First, notice that $\pi' \colon \mathcal{X}' \to D'$ has a natural Galois action of the cyclic group $\mathbf{Z}/d\mathbf{Z}$ over $\pi \colon \mathcal{X} \to D$, that is, we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{\quad G \quad} & \mathcal{X}' \\
\big\downarrow{\scriptstyle\pi'} \quad \searrow \; \mathcal{X} \; \swarrow \quad & & \big\downarrow{\scriptstyle\pi'} \\
D' & \xrightarrow[\quad \pi \quad]{} & D' \\
\searrow \quad D \quad \swarrow & &
\end{array}
\tag{2.4}
$$

where if we set $\xi = \mathrm{e}^{2\pi i/d}$, the horizontal map $D' \to D'$ is given by $t \mapsto t\xi^{-1}$ and G is given by $G([x_0 : \cdots : x_{n+1}], t) = ([\xi x_0 : \cdots : x_{n+1}], t\xi^{-1})$.

Now we lift the geometric monodromy of π (over $D - \{0\}$) to the level of $\pi' : \mathcal{X}' \to D'$. Fix a point $t_0 \in D' - \{0\}$, consider the circle $S^1_{t_0} = \{z \in D' : |z| = t_0\}$, and take $\mathcal{E}' = (\pi')^{-1}(S^1_{t_0})$. The fibration $\mathcal{E}' \to S^1_{t_0}$ is still denoted by π', its monodromy transformation is T^{-d}. Take a local trivialization over the positive arc $[t_0, t_0\xi]$ i.e., a diffeomorphism h such that the following diagram commutes

$$
\begin{array}{ccc}
[0, 1/d] \times X'_{t_0} & \xrightarrow{\quad h \quad} & (\pi')^{-1}(\mathrm{arc}[t_0, t_0\xi]) \\
& \simeq & \\
{\scriptstyle (s,x) \mapsto t_0\, \mathrm{e}^{2\pi i s}} \searrow & & \swarrow {\scriptstyle \pi'} \\
& S^1_{t_0}. &
\end{array}
\tag{2.5}
$$

Then the geometric monodromy of π can be identified at the level of π' with the composition

$$
X'_{t_0} \xrightarrow{\; h(1/d,\cdot) \;} X'_{t_0\xi} \xrightarrow{\; G \;} X'_{t_0}.
\tag{2.6}
$$

This lifting construction can be extended over D' as follows: Since $X'_0 = (\pi')^{-1}(0)$ has only isolated singularities, it is possible to construct a flow

$$
\begin{array}{ccc}
[0, 1] \times \mathcal{X}' & \xrightarrow{\quad \phi \quad} & \mathcal{X}' \\
\big\downarrow{\scriptstyle \mathrm{id} \times \pi'} & & \big\downarrow{\scriptstyle \pi'} \\
[0, 1] \times D' & \xrightarrow{\quad \varphi \quad} & D',
\end{array}
$$

such that the above diagram is commutative, $\varphi(s, t) = t\, \mathrm{e}^{2\pi i s}$, and $\phi(s, x) = x$ for any $x \in X'_0$ (see, for example, [5]). Now consider the composition $G \circ \phi(1/d, \cdot)$

over D'

$$
\begin{array}{ccccc}
\mathcal{X}' & \xrightarrow{\ \phi(1/d,\cdot)\ } & \mathcal{X}' & \xrightarrow{\ G\ } & \mathcal{X}' \\
 & {}_{\pi'}\searrow & \downarrow & \swarrow_{\pi'} & \\
 & & D'. & &
\end{array}
\qquad (2.7)
$$

This will be called the 'lifted geometric monodromy'. In the next subsections we will determine the isomorphisms induced by it on the vector spaces which appear in the exact sequences (2.2) and (2.3). Obviously, on $H^n(X'_t)$ the induced 'lifted geometric monodromy' is exactly T^{-1}.

The action on the spaces $H^q(X'_0)$ can be determined as follows. Since $\phi(s, x) = x$ for any $x \in X'_0$, the isomorphism $\phi(1/d, \cdot)$ restricted to X'_0 is the identity. Therefore, the action on $H^q(X'_0)$ is induced by the Galois action $G \colon X'_0 \to X'_0$, $G([x_0 : \cdots : x_{n+1}]) = [\xi x_0, \ldots, x_{n+1}]$. This action will be denoted by G^q.

II. *The action on* $\oplus_{i=1}^k H^n(F'_i)$

If $\varphi \colon H \to H$ is a linear map, we will denote by $c_l(\varphi) \colon H^{\oplus l} \to H^{\oplus l}$ the linear map defined by $c_l(\varphi)(x_1, \ldots, x_l) = (\varphi(x_l), x_1, \ldots, x_{l-1})$. Then we have:

(2.8) THEOREM.

(a) *If $S(F_i)$ denotes the suspension of F_i then we have a homotopy equivalence $F'_i \sim \bigvee_{d-1} S(F_i)$, therefore an isomorphism $H^n(F'_i) \simeq H^{n-1}(F_i)^{\oplus(d-1)}$*
(b) *Under the isomorphism above, the 'lifted monodromy action' on $H^n(F'_i)$ is $c_{d-1}(T_i)$.*

Proof. Part (a) was proved already in [9, (5.3)], but it follows also from our discussion here. For (b) we use a similar construction as in [loc. cit.]. The map-germ $(\mathcal{X}', p'_i) \xrightarrow{\pi'} (D', 0)$ can be identified with

$$
\mathcal{Y}_i := \{(y_0, y, t) \colon g_i(y) + t y_0 = y_0^d\} \xrightarrow{\pi'_i} (D', 0)
$$

where y_0 and $y = (y_1, \ldots, y_n)$ are local affine coordinates ($y_i = x_i/x_{n+1}$, $i = 0, \ldots, n$), g_i is the local equation of $(X^\infty, p_i) \subset (\mathbf{P}^n, p_i)$ and π'_i is given by $\pi'_i(y_0, y, t) = t$. The Galois action on $\mathcal{Y}_i$ is $G(y_0, y, t) = (y_0 \xi, y, t\xi^{-1})$. As in the global situation (2.5), consider the (local) locally trivial fibrations induced by π'_i, $\mathcal{E}'_i = (\pi'_i)^{-1}(S^1_{t_0}) \cap \mathcal{Y}_i \xrightarrow{\pi'_i} S^1_{t_0}$, with fiber F'_i. Consider the local trivializations over the positive arc $[t_0, t_0\xi]$

$$[0, 1/d] \times F_i' \xrightarrow[\simeq]{h_i} (\pi_i')^{-1}(\text{arc}[t_0, t_0\xi])$$

with downward maps $(s,x) \mapsto t_0\, e^{2\pi i s}$ and π_i' to $S^1_{t_0}$.

Then the 'lifted geometric action' on $F_i' := (\pi_i')^{-1}(t_0)$ is the composition

$$(\pi_i')^{-1}(t_0) \xrightarrow{h_i(1/d,\cdot)} (\pi_i')^{-1}(t_0\xi) \xrightarrow{G} (\pi_i')^{-1}(t_0), \qquad (2.9)$$

which will be denoted h_i'. We will prove that this geometric action induces $c_{d-1}(T_i)$ at the cohomology level.

Remark. It is not difficult to see that $(h_i')^d$ is the monodromy of π_i'. In [9] this is identified with $c_{d-1}(T_i^d) \approx [c_{d-1}(T_i)]^d$.

As in [9, (5.3)], consider the isolated complete intersection singularity given by $\mathcal{Y}_i' \xrightarrow{\varphi} D' \times D'$, $\varphi(y_0, y, t) = (t, y_0)$. The discriminant of φ is $\Delta = \{ty_0 = y_0^d\}$ and φ is a locally trivial fibration over $D' \times D' - \Delta$ with fiber F_i.

For $t_0\, e^{2\pi i\beta} \in S^1_{t_0}$, the intersection points of the line $\{t = t_0\, e^{2\pi i\beta}\}$ with the discriminant $\{ty_0 = y_0^d\}$ of φ are

$$q_0(\beta) = (t_0\, e^{2\pi i\beta}, 0) \quad \text{and} \quad q_j(\beta) = (t_0\, e^{2\pi i\beta}, e^{2\pi i(j+\beta)/d-1} \cdot \sqrt[d-1]{t_0}),$$

for $j = 1, \ldots, d-1$. We will use the following notations

$$\mathrm{I}_j(\beta) = \text{segment } [q_0(\beta), q_j(\beta)] \quad (\text{in } \{t_0\, e^{2\pi i\beta}\} \times D'),$$

$$\mathrm{I}(\beta) = \bigcup_{j=1}^{d-1} \mathrm{I}_j(\beta),$$

$$B = \bigcup_{\beta \in [0,1]} \mathrm{I}(\beta) \subset D' \times D',$$

$$r_j(\beta) = \text{middlepoint of } \mathrm{I}_j(\beta) = (t_0\, e^{2\pi i\beta}, \tfrac{1}{2} \sqrt[d-1]{t_0}\, e^{2\pi i(j+\beta)/d-1}).$$

It is obvious that, for all $1 \leqslant j \leqslant d-1$, $\varphi^{-1}(q_j(\beta))$ is contractible and $\varphi^{-1}(r_j(\beta))$ is exactly F_i. Therefore $\varphi^{-1}(\mathrm{I}_j(\beta))$ can be identified with the suspension $S(F_i)$ and $\varphi^{-1}(\mathrm{I}(\beta)) \sim \bigvee_{d-1} S(F_i)$.

The inclusion $B \subset S^1_{t_0} \times D'$ admits a strong deformation retract which can be lifted. Consider the torus $\mathbf{T} = S^1_{t_0} \times S^1_{(1/2)\sqrt[d-1]{t_0}}$ which contains the points

$r_j(\beta)$. By the identification of $F_i' = (\pi_i')^{-1}(t_0) = \varphi^{-1}(\{t_0\} \times D')$ with $\varphi^{-1}(\mathbf{I}(0)) \sim \bigvee_{d-1} S(F_i)$, the homology of F_i' is generated by a wedge of suspensions of cycles which lie above the points $r_j(0)$, $1 \leqslant j \leqslant d-1$. When we move t on the positive arc $[t_0, t_0\xi]$, then these points move on the path $[0, 1/d] \to \mathbf{T}$ given by $\beta \mapsto r_j(\beta)$. We denote these paths by γ_j, with endpoints $r_j(0)$ and $r_j(1/d)$, i.e.
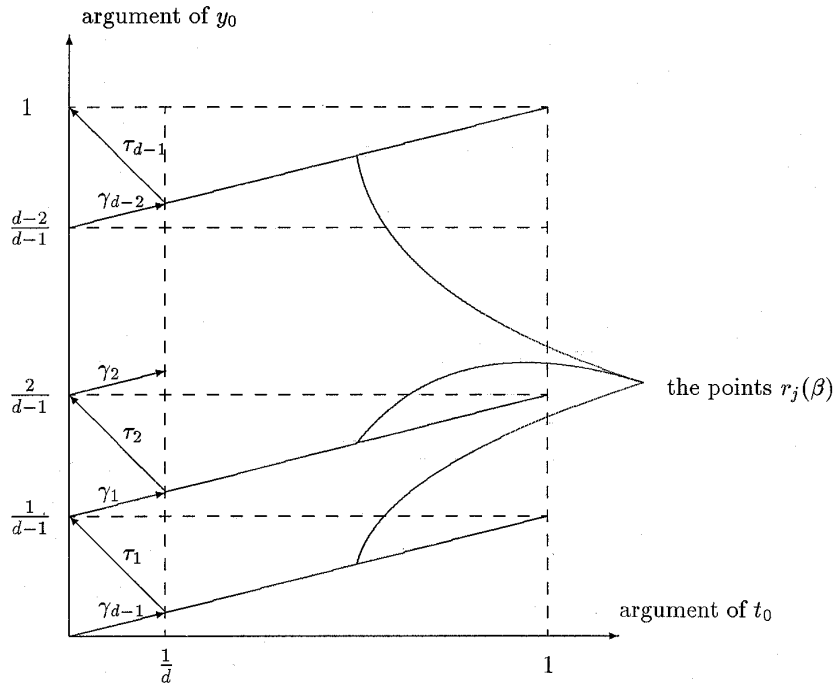
$$\gamma_j(s) = (t_0\, \mathrm{e}^{2\pi i s}, \tfrac{1}{2}\, \sqrt[d-1]{t_0}\, \mathrm{e}^{2\pi i (j+s)/d-1}), \quad s \in \left[0, \frac{1}{d}\right].$$

The local trivialization over $\cup_j \gamma_j$ corresponds to $h_i(1/d, \cdot)$ in (2.9) (we will explain this identification more precisely later). Next, we identify the Galois action with some local trivialization over some paths.

Consider the paths $\tau_j \colon [0, 1/d] \to \mathbf{T}$ defined by

$$\tau_j(s) = (t_0\, \mathrm{e}^{2\pi i((1/d)-s)}, \tfrac{1}{2}\, \sqrt[d-1]{t_0}\, \mathrm{e}^{2\pi i((j-1/d-1)+(1/d(d-1))+s)}),$$

which connects $r_{j-1}(1/d)$ and $r_j(0)$.

Notice that the Galois action $(y_0, y, t) \mapsto (y_0\xi, y, t\xi^{-1})$ induces

$$G \colon \varphi^{-1}\left(r_{j-1}\left(\frac{1}{d}\right)\right) \xrightarrow{\ \sim\ } \varphi^{-1}(r_{j-1}(0)).$$

Consider the isomorphism (up to isotopy) given by the local trivialization of φ above the oriented path τ_j

$$\mathrm{Tr}_j \colon \varphi^{-1}\left(r_{j-1}\left(\frac{1}{d}\right)\right) \xrightarrow{\sim} \varphi^{-1}(r_{j-1}(0)).$$

Fact. The composition

$$\varphi^{-1}(r_{j-1}(0)) \xrightarrow{G^{-1}} \varphi^{-1}\left(r_{j-1}\left(\frac{1}{d}\right)\right) \xrightarrow{\mathrm{Tr}_j} \varphi^{-1}(r_{j-1}(0)) \tag{2.10}$$

is isotopic to the identity.

   Proof of the fact. Consider the map $\delta \colon D' \times D' \to D'$ given by $\delta(t, y_0) = y_0^d - t y_0$ (one has $\delta^{-1}(0) =$ the discriminant of φ). First notice that $\delta(r_{j-1}(1/d)) = \delta(r_j(0))$. Therefore, the composition (2.10) can be identified with

$$g_i^{-1}(\delta(r_j(0))) \xrightarrow{\mathrm{Id}} g_i^{-1}\left(\delta\left(r_{j-1}\left(\frac{1}{d}\right)\right)\right) \xrightarrow{\widetilde{\mathrm{Tr}_j}} g_i^{-1}(\delta(r_j(0))),$$

where the first map is the identity $y \mapsto y$ (the second component of G) and $\widetilde{\mathrm{Tr}_j}$ is the trivialization of g_i above the loop $s \mapsto \delta(\tau_k(s))$, ($s \in [0, 1/d]$). Now, it is easy to verify that $\delta(\tau_k(s))$ can be written in the form $B \cdot \mathrm{e}^{2\pi i(a+ds)} + A$, where $|A| > |B|$. Therefore the loop $s \mapsto \delta(\tau_k(s))$ is isotopic to zero in $D' - \{0\}$. So, $\widetilde{\mathrm{Tr}_j}$ (and hence $\mathrm{Tr}_j \circ G^{-1}$ too) is isotopic to the identity. $\qquad\square$

The above fact shows that the Galois action G can be replaced by the local trivialization above the paths $\{\tau_j\}_j$.

   Since $h(1/d, \cdot)$ in (2.9) corresponds to the local trivialization above $\{\gamma_j\}_j$ and the Galois action to the local trivialization above $\{\tau_j\}_j$, then the composed map in (2.9) corresponds to the trivialization above $\{\tau_{j+1} \circ \gamma_j\}_j$. Now we identify the fibers of φ above the points

$$r_{d-1}(0), r_{d-1}\left(\frac{1}{d}\right), r_1(0), r_1\left(\frac{1}{d}\right), \ldots, r_{d-1}(0), r_{d-2}\left(\frac{1}{d}\right),$$

via the paths

$$\gamma_{d-1}, \tau_1, \gamma_1, \tau_2, \gamma_2, \tau_3, \ldots, \gamma_{d-2}. \tag{2.11}$$

The fiber F_i' is

$$S\varphi^{-1}(r_{d-1}(0)) \vee S\varphi^{-1}(r_1(0)) \vee \cdots \vee S\varphi^{-1}(r_{d-2}(0))$$

and the 'lifted monodromy action' is induced by

$$S(\tau_1 \circ \gamma_{d-1})_* \vee S(\tau_2 \circ \gamma_1)_* \vee \cdots \vee S(\tau_{d-1} \circ \gamma_{d-2})_*.$$

But this (because of the identification of fibers via the paths in (2.11)) is exactly the isomorphism $c_{d-1}(Q)$, where Q is the monodromy of φ above the loop

$$l = \gamma_{d-1} \circ \tau_1 \circ \gamma_1 \circ \tau_2 \circ \cdots \circ \gamma_{d-2} \circ \tau_{d-1}.$$

The loop l in the complement of the discriminant of φ is homotopic to $s \mapsto (t_0, \mathrm{e}^{2\pi i s}(\sqrt[d-1]{t_0}/2))$, $s \in [0,1]$. The linking number of this (second) loop with $\{y_0 = 0\}$ is one, and with $\{y_0^{d-1} = t_0\}$ is zero. Therefore $Q = T_i$, in particular the map h_i' induced by $G \circ h_i(1/d, \cdot)$ is $c_{d-1}(T_i)$.   □

III. *The action on $H^{n+1}_{\{p_i'\}}(\mathcal{X}')$*

In this subsection we prove that the 'lifted action' on $H^{n+1}_{\{p_i'\}}(\mathcal{X}')$ is trivial. Let $\mathcal{K}_i'$ be the link of $(\mathcal{X}', p_i') = \{g_i(y) + ty_0 - y_0^d = 0\}$ (we use the same notations as in II). The map

$$\pi' \colon \{g_i(y) + ty_0 - y_0^d = 0\} \to D', \qquad (y_0, y, t) \mapsto t$$

gives an open book decomposition of $\mathcal{K}_i'$. Let $K_i = \{t = 0\} \subseteq \mathcal{K}_i'$ be the link of t, then $\arg = \arg(t) \colon \mathcal{K}_i' - K_i \to S^1$ is a $\mathcal{C}^\infty$-locally trivial fibration. Consider the flow $\phi \colon [0,1] \times \mathcal{K}_i' \to \mathcal{K}_i'$ such that

 (a) If $x \in \{t = 0\} = K_i$, then $\phi(s, x) = x$ for any s.
 (b) If $x \notin K_i$, then $\arg(\phi(s, x)) = \mathrm{e}^{2\pi i s} \arg(s)$.

The wanted geometric action is the composed map

$$\mathcal{K}_i' \xrightarrow{\phi(1/d, \cdot)} \mathcal{K}_i' \xrightarrow{G} \mathcal{K}_i'.$$

Now $\phi(1/d, \cdot)$ is isotopic to the identity via the flow $\phi(s, \cdot)$, $s \in [0, 1/d]$. The Galois action is isotopic to the identity as well. To see this consider the isotopy

$$(s, (y_0, y, t)) \mapsto (y_0(s), y(s), t(s))$$
$$= (y_0\, \mathrm{e}^{2\pi i s/d}, y, (t - y_0^{d-1})\, \mathrm{e}^{-2\pi i s/d} + y_0^{d-1}\mathrm{e}^{2\pi i s(d-1)/d}).$$

If $s = 0$, then $(y_0(0), y(0), t(0)) = (y_0, y, t)$, if $s = 1$ then $(y_0(0), y(0), t(0)) = (y_0\xi, y, t\xi^{-1}) = G(y_0, y, t)$.

IV. *The exact sequences revisited*

We summarize the results of the subsections I–III: One has the following two exact sequences, with the 'lifted monodromy action':

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^n(X'_0) & \longrightarrow & H^n(X'_t) & \longrightarrow & \bigoplus_{i=1}^{k} H^n(F'_i) & \longrightarrow & P^{n+1}(X'_0) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle G^n} & & \downarrow{\scriptstyle T^{-1}} & & \downarrow{\scriptstyle \oplus\, c_{d-1}(T_i)} & & \downarrow{\scriptstyle G^{n+1}} & & \\
0 & \longrightarrow & H^n(X'_0) & \longrightarrow & H^n(X'_t) & \longrightarrow & \bigoplus_{i=1}^{k} H^n(F'_i) & \longrightarrow & P^{n+1}(X'_0) & \longrightarrow & 0
\end{array}
$$

and (E.2):

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H^n(X'_0) & \longrightarrow & \mathrm{Ker}\,(T^{-d} - \mathrm{Id}) & \longrightarrow & \bigoplus_{i=1}^{k} H^{n+1}_{\{p_i\}}(\mathcal{X}') & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle G^n} & & \downarrow{\scriptstyle T^{-1}} & & \downarrow{\scriptstyle \text{Identity}} & & \\
0 & \longrightarrow & H^n(X'_0) & \longrightarrow & \mathrm{Ker}\,(T^{-d} - \mathrm{Id}) & \longrightarrow & \bigoplus_{i=1}^{k} H^{n+1}_{\{p'_i\}}(\mathcal{X}') & \longrightarrow & 0
\end{array}
$$

The Main Theorem will follow from these exact sequences and from some mixed Hodge-theoretical arguments.

We end this subsection with some facts about the Galois action $G^*\colon H^*(X'_0) \to H^*(X'_0)$, where $p\colon X'_0 \to \mathbf{P}^n$ is the dth cyclic covering branched along X^∞ (cf. [8]).

One has that $H^q(X'_0, \mathbf{C}) = \mathbf{H}^q(\mathbf{P}^n, \mathbf{R}p_*\mathbf{C}_{X'_0}) = H^q(\mathbf{P}^n, p_*\mathbf{C}_{X'_0})$ and the restriction of $p_*\mathbf{C}_{X'_0}$ to the complement of X^∞ is a flat bundle. Its corresponding monodromy representation is given by the composed map $\pi_1(\mathbf{P}^n - X^\infty) \xrightarrow{h} \mathbf{Z}/d\mathbf{Z} \xrightarrow{r} \mathrm{Aut}(\mathbf{Z}^d)$ (see Section 1 for the definition of h), where $r(1) := \sigma\colon \mathbf{Z}/d\mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$ is the permutation $\sigma(x_1, \ldots, x_d) = (x_d, x_1, \ldots, x_{d-1})$.

But also the Galois action is induced by σ. So, we have a direct sum decomposition $p_*\mathbf{C}_{X'_0} = \mathbf{C}_{\mathbf{P}^n} \oplus \oplus_{s=1}^{d-1} j_*\mathbf{V}_s$ such that $G|_{\mathbf{C}_{\mathbf{P}^n}}$ is the identity and $G|_{j_*\mathbf{V}_s}$ is the multiplication by $\xi^s = \mathrm{e}^{2\pi i s/d}$. Therefore

$$
(H^q(X'_0); G^q) = \left(H^q(\mathbf{P}^n) \oplus \left(\bigoplus_{s=1}^{d-1} H^q(\mathbf{P}^n, j_*\mathbf{V}_s) \right); \bigoplus_{s=0}^{d-1} \xi^s \right). \tag{2.12}
$$

V. *The proof of the main Theorem, case $\alpha = 1$*

Consider the exact sequence (E.2) with its actions. Since the (generalized) 1-eigenspace of T^{-1} on $\mathrm{Ker}(T^{-d} - \mathrm{Id})$ is exactly $\mathrm{Ker}(T^{-1} - \mathrm{Id})$, the decomposition (2.12) provides the following exact sequence

$$
0 \to H^n(\mathbf{P}^n) \to \mathrm{Ker}(T^{-1} - \mathrm{Id}) \to \bigoplus_{i=1}^{k} H^{n+1}_{\{p'_i\}}(\mathcal{X}') \to 0. \tag{2.13}
$$

This is again an exact sequence of mixed Hodge structures. Now, it is on the one hand clear that the weight of $H^n(\mathbf{P}^n)$ is n and, on the other hand

$$\dim \mathrm{Gr}^W_{n-l+1} H^{n+1}_{\{p'_i\}}(\mathcal{X}') = \#_l(T_i)_1 \quad (\text{for } l \in \mathbf{Z}),$$

(see [9, (5.5)]). Since the weight filtration on $H^n(X^l_t)$ is the monodromy weight filtration of T^{-1} centered at n, one has $\dim \mathrm{Gr}^W_{n-l+1} \mathrm{Ker}(T^{-1} - \mathrm{Id}) = \#_l(T^{-1})$. This shows that

$$(T^{-1})_1 = \bigoplus_{i=1}^{k} (T_i)_1 \oplus \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathrm{id}_\mathbf{C} & \text{if } n \text{ is even.} \end{cases} \tag{2.14}$$

Now notice that

$$p_{n-1}(X^\infty) - p_n(X^\infty) = \chi_0, \tag{2.15}$$

(see, for example (2.29), or [6, Chapter 5, (3.7) and (4.4.ii)]). Hence the result follows from (2.1), (2.14) and (2.15). $\square$

(2.16) *Remark*. Using the exact sequence of vanishing cycles of $X^\infty \subset \mathbf{P}^n$, by standard mixed Hodge theoretical arguments, one can prove that the dimensions $\dim \mathrm{Gr}^W_{n-l} P^{n-1}(X^\infty)$ $(l \geqslant 1)$ are equal to the numbers on the right-hand side of the equalities in the Main Theorem, case $\alpha = 1$. Hence, the Main Theorem gives

$$\#_l(T^\infty_f)_1 = \dim \mathrm{Gr}^W_{n-l} P^{n-1}(X^\infty) \quad \text{for } l \in \mathbf{Z},$$

where W denotes the weight filtration.

VI. *The proof of the main Theorem, case $\alpha^d = 1, \alpha \neq 1$*

First notice that the Galois action $G: X^l_0 \to X^l_0$ is an algebraic map, therefore $G^q: H^q(X^l_0) \to H^q(X^l_0)$ preserves the weight filtration. Since $H^n(\mathbf{P}^n, j_*\mathbf{V}_s)$ is the ξ^s-eigenspace of G^q (cf. 2.12), it has a natural induced weight filtration (actually, it has a natural mixed Hodge structure). Now let $\alpha = \mathrm{e}^{2\pi i s/d} = \xi^s$ for $1 \leqslant s \leqslant d-1$. Then by (E.2) one has

$$H^n(\mathbf{P}^n, j_*\mathbf{V}_s) = \mathrm{Ker}(T^{-1} - \alpha). \tag{2.17}$$

In particular, for $l \geqslant 1$ one has

$$\#_l(T^{-1})_\alpha = \dim \mathrm{Gr}^W_{n-l+1} H^n(\mathbf{P}^n, j_*\mathbf{V}_s). \tag{2.18}$$

Actually, (2.18) together with (2.1) already determine $(T^\infty_f)_\alpha$, but this is not exactly the assertion of the Main Theorem, we want some more information about the right-hand side of (2.18).

Consider the exact sequence (E.1). Using (2.17) one has

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(T^{-1}-\alpha) & \longrightarrow & H^n(X'_t)_\alpha & \longrightarrow & [\oplus H^n(F'_i)]_\alpha & \longrightarrow & H^{n+1}(\boldsymbol{P}^n, j_*\boldsymbol{V}_s) & \longrightarrow & 0 \\
& & \downarrow{\cdot\alpha} & & \downarrow{(T^{-1})_\alpha} & & \downarrow{[\oplus c_{d-1}(T_i)]_\alpha} & & \downarrow{\cdot\alpha} & & \\
0 & \longrightarrow & \mathrm{Ker}(T^{-1}-\alpha) & \longrightarrow & H^n(X'_t)_\alpha & \longrightarrow & [\oplus H^n(F'_i)]_\alpha & \longrightarrow & H^{n+1}(\boldsymbol{P}^n, j_*\boldsymbol{V}_s) & \longrightarrow & 0.
\end{array}
$$

By [19], the weight filtration of $H^n(X'_t)_\alpha$ is the monodromy weight filtration of $(T^{-1})_\alpha$ centered at n, thus the quotient $H^n(X'_t)/\mathrm{Ker}(T^{-1}-\alpha)$ has a (polarized) mixed Hodge structure with weight filtration equal to the monodromy weight filtration of the class $[T_\alpha^{-1}]$ of $(T^{-1})_\alpha$ centered at $n+1$. On the other hand, $H^{n+1}(X'_0)$ is pure of weight $n+1$ ([24]) and G^{n+1} preserves the weight filtration, hence $H^{n+1}(\boldsymbol{P}^n, j_*\boldsymbol{V}_s)$ is pure of weight $n+1$ (cf. (2.12)). These two facts show that the weight filtration of $[\oplus H^n(F'_i)]_\alpha$ is the monodromy weight filtration of $(\oplus c_{d-1}(T_i))_\alpha$. Now comparing the dimensions of the primitive cohomologies of $[T_\alpha^{-1}]$ and $(c_{d-1}(T_i))_\alpha$ one has

$$
\begin{cases}
\#_2(T^{-1})_\alpha = -\dim H^{n+1}(\boldsymbol{P}^n, j_*\boldsymbol{V}_s) + \sum_{i=1}^k \#_1(c_{d-1}(T_i))_\alpha, \\[2mm]
\#_{l+1}(T^{-1})_\alpha = \sum_{i=1}^k \#_l(c_{d-1}(T_i))_\alpha \quad \text{for } l \geqslant 2.
\end{cases}
\tag{2.19}
$$

Since $\alpha^d = 1$ and $\alpha \neq 1$, $(c_{d-1}(T_i))_\alpha = (T_i)_{\alpha^{-1}}$. Therefore

$$
\begin{cases}
\#_2(T^{-1})_\alpha = -\beta_s + \sum_{i=1}^k \#_1(T_i)_{\alpha^{-1}}, \\[2mm]
\#_{l+1}(T^{-1})_\alpha = \sum_{i=1}^k \#_l(T_i)_{\alpha^{-1}} \quad \text{for } l \geqslant 2.
\end{cases}
\tag{2.20}
$$

Since $\beta_{d-s} = \beta_s$ $(s = 1, \ldots, d-1)$, parts (Ib,Ic) follow from (2.20) and (2.1.a). In order to prove (Ia) (i.e., to compute $\#_1(T_f^\infty)_\alpha = \#_1 T_\alpha$) notice that from (2.18) one has

$$
\#(T^{-1})_\alpha = \dim H^n(\boldsymbol{P}^n, j_*\boldsymbol{V}_s)
\tag{2.21}
$$

and from (2.20)

$$
\sum_{l \geqslant 2} \#_l(T^{-1})_\alpha = -\beta_s + \sum_{i=1}^k \#(T_i)_{\alpha^{-1}}.
\tag{2.22}
$$

Therefore

$$\#_1(T^{-1})_\alpha = \dim H^n(\mathbf{P}^n, j_*\mathbf{V}_s)$$

$$+\dim H^{n+1}(\mathbf{P}^n, j_*\mathbf{V}_s) - \sum_{i=1}^{k} \#(T_i)_{\alpha^{-1}}.$$

Now (Ia) follows from this identity and the following result:

(2.23) PROPOSITION. *For $s = 1, \dots, d-1$ one has:*

 (a) $\dim H^q(\mathbf{P}^n, j_*\mathbf{V}_s) = 0$ *for* $q \neq n, n+1$.
 (b) $\dim H^n(\mathbf{P}^n, j_*\mathbf{V}_s) - \dim H^{n+1}(\mathbf{P}^n, j_*\mathbf{V}_s) = \chi_s$.

(*For the definition of χ_s, see the introduction*). *In particular, the Euler characteristic of $(\mathbf{P}^n, j_*\mathbf{V}_s)$ is local.*

   Proof. The first part follows from (2.12), because $P^q(X_0') = 0$ if $q \neq n, n+1$. The second part will be proved (together with some other relations) in subsection VIII, (2.29).

VII. *The proof of the main theorem, case $\alpha^d \neq 1$*

Consider the exact sequence (E.1). Since $(G^*)^d =$ Id, the generalized α-eigenspaces are

$$(H^n(X_t')_\alpha, (T^{-1})_\alpha) \simeq \left(\left(\bigoplus_{i=1}^{k} H^n(F_i') \right)_\alpha, (\oplus c_{d-1}(T_i))_\alpha \right). \tag{2.23}$$

By (2.1), $T_\alpha = (T_f^\infty)_\alpha$. Also, if $\Pi_j(\lambda - \xi_j)$ is the characteristic polynomial of T_i, then $\Pi_j(\lambda^{d-1} - \xi_j)$ is the characteristic polynomial of $c_{d-1}(T_i)$. Therefore, if α is an eigenvalue of $c_{d-1}(T_i)$, then $\alpha^{d-1} = \xi_j$ for some ξ_j and the unipotent (or nilpotent) part of $c_{d-1}(T_i)_\alpha$ and $(T_i)_{\xi_j}$ can be identified. This ends the proof of the Main Theorem.

VIII. *The relation with the Milnor fiber of $f_d: \mathbf{C}^{n+1} \to \mathbf{C}$*

Consider the homogeneous singularity $f_d: \mathbf{C}^{n+1} \to \mathbf{C}$ with one-dimensional singular locus, let F be its Milnor fiber. It is well-known that its (reduced) homology is concentrated in $H_n(F)$ and $H_{n-1}(F)$. Let $h_q: H_q(F) \to H_q(F)$ be the algebraic monodromy of f_d, where $q = n-1, n$. In this subsection we identify our local and global invariants with numerical invariants given by the transformations h_{n-1}, h_n. We recall that in the case $n = 2$, the Milnor fibre of a homogeneous singularity with one-dimensional singular locus was studied in [8].
   Recall that we denoted $p: X_0' \to \mathbf{P}^n$ the dth cyclic covering branched along X^∞. Then $p^{-1}(X^\infty)$ can be identified with X^∞ and $X_0' - p^{-1}(X^\infty)$ with F. This gives a cyclic (unramified) covering $F \to \mathbf{P}^n - X^\infty$ with fiber $\mathbf{Z}/d\mathbf{Z}$. By duality

(see, e.g., [23, Thm 19, p. 297]), $H_q(F) = H^{2n-q}(X'_0, p^{-1}(X^\infty))$, therefore the exact sequence of the pair $(X'_0, p^{-1}(X^\infty))$ reads as follows

$$0 \longrightarrow P^{n-1}(X^\infty) \longrightarrow H_n(F) \longrightarrow P^n(X'_0) \longrightarrow P^n(X^\infty) \longrightarrow H_{n-1}(F) \longrightarrow P^{n+1}(X'_0) \to 0$$

$$\downarrow Id \qquad \downarrow h_n^{-1} \qquad \downarrow G^n \qquad \downarrow Id \qquad \downarrow h_{n-1}^{-1} \qquad \downarrow G^{n+1}$$

$$0 \longrightarrow P^{n-1}(X^\infty) \longrightarrow H_n(F) \longrightarrow P^n(X'_0) \longrightarrow P^n(X^\infty) \longrightarrow H_{n-1}(F) \longrightarrow P^{n+1}(X'_0) \to 0$$

In the above diagram, we have also inserted the corresponding Galois actions. The Galois action on $X'_0 = \{f_d(x_1, \ldots, x_{n+1}) = x_0^d\}$ is $[x_0 : \cdots : x_{n+1}] \mapsto [\xi x_0 : \cdots : x_{n+1}]$. If on $X'_0 - p^{-1}(X^\infty)$ we take affine coordinates $y_i = x_i/x_0$, $1 \leqslant i \leqslant n + 1$, then the induced action is $(y_1, \ldots, y_{n+1}) \mapsto \xi^{-1}(y_1, \ldots, y_{n+1})$. This is the inverse of the geometric monodromy of the Milnor fiber F.

Now, if we consider the generalized eigenspaces of the Galois action in (2.26), one has the following identifications

$$\begin{cases} (H_n(F), h_n^{-1})_{\neq 1} = (P^n(X'_0), G^n)_{\neq 1} \\ (H_{n-1}(F), h_{n-1}^{-1})_{\neq 1} = (P^{n+1}(X'_0), G^{n+1})_{\neq 1} \\ \dim H_n(F)_1 = p_{n-1}(X^\infty) \quad \text{and} \quad \dim H_{n-1}(F)_1 = p_n(X^\infty). \end{cases} \qquad (2.27)$$

We recall (cf. (2.12)) that $(P^q(X'_0), G^q)_{\neq 1} = (\oplus_{s=1}^{d-1} H^q(\mathbf{P}^n, j_* \mathbf{V}_s), \oplus_{s=1}^{d-1} \xi^s)$. In particular, all our global invariants are equivalent to the characteristic polynomial of h_{n-1}, i.e. $\beta_s = \operatorname{rank} H_{n-1}(F)_\alpha$ $(0 \leqslant s \leqslant d - 1)$. Now let us consider the zeta function of $f_d : \mathbf{C}^{n+1} \to \mathbf{C}$. This is basically given in [21]

$$\frac{\det(\lambda \cdot \operatorname{Id} - h_n)}{\det(\lambda \cdot \operatorname{Id} - h_{n-1})} = (\lambda - 1)^{(-1)^{n+1}} \cdot (\lambda^d - 1)^{((d-1)^{n+1} + (-1)^n)/d}$$

$$\times \prod_{i=1}^k (\lambda^d - 1)^{-\mu_i} = \prod_{s=0}^{d-1} (\lambda - e^{2\pi i s/d})^{\chi_s}. \qquad (2.28)$$

Now, (2.28) and (2.27) give

$$\begin{cases} p_{n-1}(X^\infty) - p_n(X^\infty) = \chi_0, \\ \dim H^n(\mathbf{P}^n, j_* \mathbf{V}_s) - \dim H^{n+1}(\mathbf{P}^n, j_* \mathbf{V}_s) = \chi_s \quad (s = 1, \ldots, d - 1). \end{cases} \qquad (2.29)$$

This proves (2.15) and (2.23.b).

(2.30) *Remark*. By (2.27) and Corollary 3 one has (for $s = 0, \ldots, d - 1$)

$$\dim H_{n-1}(F)_{e^{2\pi i s/d}} = \beta_s \leqslant \sum_{i=1}^k \#_1(T_i)_{e^{2\pi i s/d}}.$$

Similar restrictions can be found in [20, (5.4) and Sect. 9].

(2.31) *Remark* (The relation with $\pi_{n-1}(\mathbf{P}^n - X^\infty)$). Here we present the connection between the present paper and [12], more precisely, between the defects β_s $(0 \leqslant s \leqslant d - 1)$ and $\pi_{n-1}(\mathbf{P}^n - X^\infty)$.

First, assume that $n = 2$. Denote $G = \pi_1(\mathbf{P}^n - X^\infty)$, $G' = [G, G]$, and $G'' = [G', G']$. Then $0 \to \pi_1(F) \to G \to \mathbf{Z}/d\mathbf{Z} \to 0$ is an exact sequence, actually $\pi_1(F) = G'$. Therefore, $H_1(F) = G'/G''$, and it has a natural action of $\mathbf{Z}/d\mathbf{Z}$. By (2.27) one has:

$$\beta_s = \text{rank}\left(\left(\frac{G'}{G''} \right) \otimes \mathbf{Q} \right)_\alpha; \qquad \alpha = \text{e}^{2\pi i s/d}; \quad s = 0, \ldots, d - 1.$$

Now, assume that $n > 2$. From the covering $F \to \mathbf{P}^n - X^\infty$ and Hurewicz Theorem one has: $\pi_1(\mathbf{P}^n - X^\infty) = \mathbf{Z}/d\mathbf{Z}$, $\pi_q(\mathbf{P}^n - X^\infty) = 0$ if $1 < q < n - 1$, and $\pi_{n-1}(\mathbf{P}^n - X^\infty) = \pi_{n-1}(F) = H_{n-1}(F)$. Hence by (2.27)

$$\beta_s = \text{rank}(\pi_{n-1}(\mathbf{P}^n - X^\infty) \otimes \mathbf{Q})_\alpha; \qquad \alpha = e^{2\pi i s/d}; \quad s = 0, \ldots, d - 1,$$

where the action of $\pi_1(\mathbf{P}^n - X^\infty)$ on $\pi_{n-1}(\mathbf{P}^n - X^\infty)$ is the natural one.

3. Examples

I. *Zariski's plane sextics*

Set $d = 6$ and let $f_6 \in \mathbf{C}[X; Y; Z]$ be a form defining a plane sextic in $\mathbf{P}^2$ with six cusps and no other singularities. Then $\chi_0 = 8$ and $\chi_s = 9$ for $s = 1, \ldots, 5$. Moreover (since the characteristic polynomial of the local monodromy of a cusp singularity is $t^2 - t + 1$), $\beta_s = 0$ if $s = 0, 2, 3, 4$. Our main theorem gives:

(a) If $\alpha^d \neq 1$, then $(T_f^\infty)_\alpha$ has only one-dimensional Jordan blocks, and $(T_f^\infty)_\alpha = \text{Id}_{\mathbf{C}^6}$ if $\alpha = \text{e}^{\pi i \varphi}$, $\varphi \in \{ \frac{1}{15}, \frac{7}{15}, \frac{11}{15}, \frac{13}{15}, \frac{17}{15}, \frac{19}{15}, \frac{23}{15}, \frac{29}{15} \}$ (i.e., if $\alpha^5 = \text{e}^{2\pi i s/6}$ for $s = 1$ or $s = 5$ and $\alpha^6 \neq 1$). Otherwise $(T_f^\infty)_\alpha = 0$.

(b) $(T_f^\infty)_{\text{e}^{2\pi i s/6}}$ has only one-dimensional blocks if $s = 0, 2, 3, 4$. The number of them is 8 if $s = 0$ and 9 if $s = 2, 3, 4$.

(c) $(T_f^\infty)_{\text{e}^{2\pi i s/6}}$ has only one and two-dimensional blocks if $s = 1$ or $s = 5$. The number of one-dimensional blocks is $3 + 2\beta_s$, and the number of two-dimensional blocks is $6 - \beta_s$.

Now, by the identification (2.27) and [6, Thm 2.9] one has $\beta_s = 1$, $(s = 1, 5)$ if the cusps are on a conic and $\beta_s = 0$ otherwise.

II. *Nodal hypersurfaces*

Assume that f_d defines a hypersurface in $\mathbf{P}^n$ with only nodal (i.e. A_1) singularities, let k denote the number of nodes. It follows from the Main Theorem that the

maximal size of a Jordan block of T_f^∞ is two. The numbers β_s can be computed using [6, VI, Thm (4.5)] and (2.27).

If dn is even, set $S = \mathbf{C}[X_1, \ldots, X_{n+1}]$, $q = (dn/2) - n - 1$, and let S_q denote the homogeneous component of degree q of S. If $\Sigma \subset X^\infty$ denotes the set of nodes of $\{f_d = 0\}$, let $S_q(\Sigma) = \{h \in S_q \mid h_{|\Sigma} = 0\}$, and $\mathrm{defect}(S_q(\Sigma)) := k - \mathrm{codim}_{S_q}(S_q(\Sigma))$.

From the Main Theorem we get the following possibilities for T_f^∞:

(a) n is odd, d is odd.

　　Here $\beta_s = 0$ for all s. Thus T_f^∞ has no Jordan blocks of size two, i.e. it is of finite order.

(b) n is odd, d is even.

　　In this case $\beta_s = 0$ for $s \neq d/2$ and T_f^∞ can have Jordan blocks of size two only for eigenvalue -1, the number of them is $\#_2(T_f^\infty)_{-1} = k - \beta_{d/2} = k - \mathrm{defect}(S_q(\Sigma)) = \mathrm{codim}_{S_q}(S_q(\Sigma))$.

(c) n is even.

　　In this case, $\beta_s = 0$ for $s \neq 0$ and $\beta_0 = p_n(X^\infty)$. It follows that T_f^∞ can have Jordan blocks of size two only for eigenvalue 1, and $\#_2(T_f^\infty)_1 = k - p_n(X^\infty) = k - \mathrm{defect}(S_q(\Sigma)) = \mathrm{codim}_{S_q}(S_q(\Sigma))$. This number, in general, is not zero. For example, the defect $(S_5(\Sigma)) = \beta_0$ of the recently constructed quintic hypersurface in $\mathbf{P}^4$ with $k = 130$ nodes [25] is 29 [loc. cit., p. 864]. The quintic constructed by Hirzebruch [11] has 126 nodes and defect $\beta_0 = 25$. Actually, there are quintics in $\mathbf{P}^4$ with 118 nodes and defect $18 \leqslant \beta_0 \leqslant 19$ [10].

We recall that, as proved by Artal in [1], the Jordan block structure of the local monodromy of a superisolated singularity given by $f_d + f_{d+1} + \cdots \in \mathbf{C}[X, Y, Z]$ depends also on the position of the singularities of the projective curve given by $f_d = 0$.

Acknowledgements

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     After the submission of this paper, the following preprints and papers related to the study of polynomial maps have appeared as well.

27. Artal Bartolo, E., Cassou-Noguès, P. and Dimca, A.: Sur la topologie des polynômes complexes, Preprint.

28. Artal Bartolo, E., Luengo Velasco, I. and Melle Hernandez, A.: On the topology of a generic fibre of a polynomial map, Preprint.

29. Dimca, A.: Monodromy at infinity for polynomials in two variables, Preprint.

30. Parusiński, A.: A note on singularities at infinity of complex polynomials. In Symplectic singularities and geometry of gauge fields, Banach Center Publications, 39 (1997), 131–141.

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