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POSITIVE SOLUTIONS OF HIGHER-ORDER BOUNDARY-VALUE PROBLEMS

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Abstract We consider a class of even-order boundary-value problems with nonlinear boundary conditions and an eigenvalue parameter λ in the equations. Sufficient conditions are obtained for the existence and non-existence of positive solutions of the problems for different values of λ.

Keywords: existence; non-existence; positive solutions; lower and upper solutions; Schauder fixed-point theorem

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1. Introduction

Consider the nonlinear $2m$th-order boundary-value problem (BVP) consisting of the equation

$$u^{(2m)} = \lambda f(t, u, u'', \dots, u^{(2m-2)}), \quad t \in (0, 1), \tag{1.1}$$

and the boundary condition (BC)

$$u^{(2i)}(0) = g_i(u^{(2i)}(a)), \quad u^{(2i)}(1) = h_i(u^{(2i)}(b)), \quad i = 0, \dots, m-1, \tag{1.2}$$

where $m \geqslant 1$ is an integer, $\lambda > 0$ is a parameter, $a, b \in [0, 1]$ and $f : (0, 1) \times \mathbb{R}^m \to \mathbb{R}$ satisfies the conditions:

(i) for $(x_0, \dots, x_{m-1}) \in \mathbb{R}^m$, $f(\cdot, x_0, \dots, x_{m-1})$ is measurable on $(0, 1)$; and

(ii) for $t \in (0, 1)$ almost everywhere (a.e.), $f(t, \cdot, \dots, \cdot)$ is continuous on $\mathbb{R}^m$;

and $g_i, h_i : \mathbb{R} \to \mathbb{R}$ are continuous for $i = 0, \dots, m-1$. Note that the function f in Equation (1.1) may depend on any or all of the even-order derivatives of the unknown function $u(t)$. By a positive solution of BVP (1.1), (1.2), we mean a function $u \in C^{2m-2}[0, 1]$ such that $u^{(2m-1)}(t)$ is absolutely continuous on $(0, 1)$, $u(t)$ satisfies Equation (1.1) a.e. on $(0, 1)$ and BC (1.2), and $u(t) > 0$ for $t \in (0, 1)$.

For the case

$$f(t, x_0, x_1, \dots, x_{m-1}) \equiv f(t, x_0) \quad \text{and} \quad g_i(x_i) \equiv h_i(x_i) \equiv 0$$

for $i = 0, \dots, m-1$, BVP (1.1), (1.2) reduces to the BVP consisting of the equation

$$u^{(2m)}(t) = \lambda f(t, u), \quad t \in (0, 1), \tag{1.3}$$

and the BC

$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, \dots, m-1. \tag{1.4}$$

There has been a great deal of research work on the existence of positive solutions of BVP (1.3), (1.4) (see, for instance, [**3**, **4**, **13**, **25**] for $m = 1$, [**26**, **27**] for $m = 2$ and [**15**] for $m \geqslant 1$). For comparison, we list several known results on BVP (1.3), (1.4) and its special case, the BVP consisting of the equation

$$u'' = \lambda g(t) h(u), \quad t \in (0, 1), \tag{1.5}$$

and the BC

$$u(0) = u(1) = 0. \tag{1.6}$$

Proposition 1.1 (the main result in [3]). *Let $h(u) = \mathrm{e}^u$ and assume that $g \in C^1((0,1], (-\infty, 0))$ is singular at 0 and is $O(1/t^{2-\delta})$ as $t \to 0^+$, for some $\delta > 0$. There then exists $\lambda^* > 0$ such that BVP (1.5), (1.6) has a positive solution for each $\lambda \in (0, \lambda^*)$, and does not have a positive solution for any $\lambda \in (\lambda^*, \infty)$.*

Proposition 1.2 (Theorem 7 in [25]). *Assume that $g < 0$ is singular at 0 and is $O(1/t^{\alpha})$ as $t \to 0^+$, for some $\alpha \in [0, 1)$, $h \in C(\mathbb{R}, \mathbb{R}^+)$ with $\mathbb{R}^+ = [0, \infty)$ is locally Lipschitz continuous, increasing, $h > 0$ on $\mathbb{R}^+$, and satisfies*

$$\int_0^c \frac{\mathrm{d}u}{\sqrt{H(c) - H(u)}} \leqslant L < \infty \quad \text{for all } c > 0,$$

where $H(u) = \int_0^u h(y)\,\mathrm{d}y$. There then exists $\lambda^* > 0$ such that BVP (1.5), (1.6) has a positive solution for each $\lambda \in (0, \lambda^*)$, and does not have a positive solution for any $\lambda \in (\lambda^*, \infty)$.

Proposition 1.3 (Theorem 1.1 in [4]). *Assume that $g \in C((0, 1), \mathbb{R}^-)$ with $\mathbb{R}^- = (-\infty, 0]$, $g \not\equiv 0$ on $(0, 1)$,*

$$\int_0^1 s^a (1-s)^b |g(s)|\,\mathrm{d}s < \infty$$

for some $a, b \in (0, 1)$, and $h(u)$ is non-decreasing and $h(u) > 0$ for $u \geqslant 0$. Moreover, there exists $c > 0$ such that $h(u) \geqslant cu$ for $u \geqslant 0$. There then exists $\lambda^* > 0$ such that BVP (1.5), (1.6) has a positive solution for each $\lambda \in (0, \lambda^*)$, and does not have a positive solution for any $\lambda \in (\lambda^*, \infty)$.

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Proposition 1.4 (Theorem 2 in [13]). *Assume that $g < 0$ on $(0,1)$ and satisfies*

$$\int_0^1 s|g(s)|\,\mathrm{d}s < \infty,$$

and $h(u) \geqslant \mathrm{e}^u$ for $u \in \mathbb{R}$. There then exists $\lambda^* > 0$ such that BVP (1.5), (1.6) has a positive solution for each $\lambda \in (0, \lambda^*)$, and does not have a positive solution for any $\lambda \in (\lambda^*, \infty)$.

Proposition 1.5 (Theorem 4.3 in [15]). *Assume that $(-1)^m f \in C((0,1)\times\mathbb{R}^+, \mathbb{R}^+)$, $f(t,0) \not\equiv 0$ on $(0,1)$, and $(-1)^m f(t,u)$ is non-decreasing in u. Suppose that, for each $\eta > 0$, there exists $C_\eta > 0$ such that*

$$|f(t,u)| \leqslant C_\eta q(t) \text{ on } (0,1)\times[0,\eta] \quad \text{with } 0 < \int_0^1 s(1-s)q(s)\,\mathrm{d}s < \infty$$

and

$$|f(t,u)| \geqslant l(t)u \text{ on } (0,1)\times\mathbb{R}^+ \qquad \text{with } 0 < \int_0^1 s^2(1-s)^2 l(s)\,\mathrm{d}s < \infty.$$

Let

$$\bar{\lambda} = 30^{m-1}\left(\int_0^1 s^2(1-s)^2 l(s)\,\mathrm{d}s\right)^{-1}.$$

There then exists $\lambda^* \in (0, \bar{\lambda}]$ such that BVP (1.3), (1.4) has a positive solution for each $\lambda \in (0, \lambda^*)$, and does not have a positive solution for any $\lambda \in (\lambda^*, \infty)$.

As in [**15**, Remark 4.1], we observe that Proposition 1.5 improves and generalizes Propositions 1.1–1.4, and more importantly, Proposition 1.5 provides an explicit verifiable range $(\bar{\lambda}, \infty)$ of λ where the BVP has no positive solution. However, none of Propositions 1.1–1.5 provide an explicit verifiable range of λ where the BVP has a positive solution.

The BVP with the equation depending on the derivatives of the unknown function has recently been investigated (see, for example, [**1**, **5**, **6**, **8**–**11**, **16**–**20**] and references therein). In particular, Davis *et al*. [**5**] discussed the existence of at least three positive symmetric concave solutions for the BVP consisting of Equation (1.1) and BC (1.4); Ehme *et al*. [**10**] obtained sufficient conditions for the existence of solutions of fourth-order BVPs based on the existence of a pair of strong lower and upper solutions. BVPs with special nonlinear BCs have also been studied in the literature (see [**8**–**11**, **22**–**24**]. We remark that the BVPs in the general form (1.1), (1.2) are important because of their applications to physical, biological and chemical phenomena (see [**2**, **7**, **21**]). Moreover, they are also interesting in themselves from theoretical perspectives.

Motivated partly by the ideas in [**8**–**10**], in this paper, we study the existence and non-existence of positive solutions of the BVP (1.1), (1.2). Under certain assumptions, we show that there exists $\lambda^* > 0$ such that this BVP has a positive solution for $\lambda \in (0, \lambda^*)$, and has no positive solution for $\lambda \in (\lambda^*, \infty)$. Moreover, we find explicit verifiable ranges of λ where the BVP has and does not have positive solutions, respectively. A comparison theorem plays a key role in the proofs.

The results obtained in this paper generalize and improve many results in the literature, in particular those given by Propositions 1.1–1.5.

This paper is organized as follows. In § 2, we state the main results of this paper and provide an example to show the significance of the results. All the proofs of the main results together with some technical lemmas are given in § 3.

2. Main results

In this paper, for $k = 0, 1, \dots,$ we denote by $C^k[0,1]$ the Banach space of all kth continuously differentiable functions $u(t)$ on $[0,1]$ with the norm

$$\|u\| = \max_{t\in[0,1]}\{|u(t)|, \dots, |u^{(k)}(t)|\},$$

and let $X = C^{2m-2}[0,1]$. Define

$$\mathbb{D} = \begin{cases} \underbrace{\mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}^+ \times \mathbb{R}^- \times \cdots \times \mathbb{R}^+}_{2k-1}, & \text{if } m = 2k-1, \\ \underbrace{\mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}^+ \times \mathbb{R}^- \times \cdots \times \mathbb{R}^-}_{2k}, & \text{if } m = 2k, \end{cases}$$

and, for $\eta > 0$,

$$\mathbb{D}_\eta = \begin{cases} \underbrace{[0,\eta] \times [-\eta,0] \times [0,\eta] \times [-\eta,0] \times \cdots \times [0,\eta]}_{2k-1}, & \text{if } m = 2k-1, \\ \underbrace{[0,\eta] \times [-\eta,0] \times [0,\eta] \times [-\eta,0] \times \cdots \times [-\eta,0]}_{2k}, & \text{if } m = 2k, \end{cases}$$

and

$$(-1)^i\mathbb{R}^+ = \begin{cases} \mathbb{R}^+, & \text{if } i = 2k, \\ \mathbb{R}^-, & \text{if } i = 2k-1. \end{cases}$$

Throughout this paper, we assume that, for $i \geqslant 1$ and $(x_0, \dots, x_{m-1}) \in \mathbb{D}$, $f(t, x_0, x_1, \dots, x_{m-1})$ is non-decreasing in the variables x_{m-2i} on $(-1)^{m-2i}\mathbb{R}^+$ and non-increasing in the variables x_{m-2i-1} on $(-1)^{m-2i-1}\mathbb{R}^+$ (here we do not require that f is monotone in x_{m-1}),

$$(-1)^m f \geqslant 0 \text{ on } (0,1) \times \mathbb{D}, \quad f(t, 0, 0, \dots, 0) \not\equiv 0 \text{ a.e. on } (0,1); \tag{2.1}$$

and for $i = 0, \dots, m-1$, g_i, h_i are non-decreasing on $(-1)^i\mathbb{R}^+$, and

$$(-1)^i g_i \geqslant 0 \quad \text{and} \quad (-1)^i h_i \geqslant 0 \quad \text{on } (-1)^i\mathbb{R}^+. \tag{2.2}$$

In the remainder of the paper we will need the following additional assumptions.

Assumption 2.1. *For any $\eta > 0$, there exists $M_\eta > 0$ such that*

$$(-1)^m f(t, x_0, x_1, \dots, x_{m-1}) \leqslant M_\eta \psi(t) \text{ on } (0,1) \times \mathbb{D}_\eta, \tag{2.3}$$

where $\psi : (0,1) \to \mathbb{R}^+$ satisfies

$$\int_0^1 s(1-s)\psi(s)\,\mathrm{d}s < \infty. \tag{2.4}$$

Assumption 2.2. *There exist $\chi : (0,1) \to \mathbb{R}^+$ and $k \in \{0, \dots, m-1\}$ such that*

$$(-1)^m f(t, x_0, x_1, \dots, x_{m-1}) \geqslant \chi(t)|x_k| \text{ on } (0,1) \times \mathbb{D} \tag{2.5}$$

and

$$0 < \int_0^1 s(1-s)\chi(s)\mu(s)\,\mathrm{d}s < \infty, \tag{2.6}$$

where

$$\mu(t) = \min\{t, 1-t\} \quad \text{for } t \in [0,1]. \tag{2.7}$$

Assumption 2.3. *There exists $r > 0$ such that*

$$\sum_{i=0}^{m-1}(|g_i((-1)^i r)| + |h_i((-1)^i r)|) + \int_0^1 s(1-s)\psi(s)\,\mathrm{d}s \leqslant r, \tag{2.8}$$

where ψ is given in Assumption 2.1.

Remark 2.4. We observe that

(i) Assumption 2.1 holds if $f : [0,1] \times \mathbb{R}^m \to \mathbb{R}$ is continuous or $f(t, x_0, \dots, x_{m-1}) = \psi(t) f_1(x_0, \dots, x_{m-1})$, where $\psi : (0,1) \to \mathbb{R}^+$ satisfies (2.4) and $f_1 \in C(\mathbb{R}^m)$;

(ii) Assumption 2.3 holds if

$$\liminf_{r\to\infty} \frac{1}{r} \sum_{i=0}^{m-1}(|g_i((-1)^i r)| + |h_i((-1)^i r)|) = \rho < 1.$$

Now we state the main results of this paper. The first theorem is a comparison result on the existence of solutions of BVP (1.1), (1.2) among different values of λ.

Theorem 2.5. *Assume that Assumption 2.1 holds, and there exists $\lambda_* > 0$ such that, for $\lambda = \lambda_*$, BVP (1.1), (1.2) has a solution $u_*(t)$ satisfying*

$$(-1)^j u_*^{(2j)}(t) \geqslant 0 \quad \text{for } t \in [0,1] \text{ and } j = 0, \dots, m-1. \tag{2.9}$$

Then, for each $\lambda \in (0, \lambda_*]$, BVP (1.1), (1.2) has a positive solution $u(t)$ satisfying

$$(-1)^j u^{(2j)}(t) \geqslant 0 \quad \text{for } t \in [0,1] \text{ and } j = 0, \dots, m-1. \tag{2.10}$$

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The following provides an explicit interval for λ where BVP (1.1), (1.2) has a positive solution.

Theorem 2.6. *Assume that Assumptions 2.1 and 2.3 hold, and let $\underline{\lambda} = 1/M_r$, where r is given in Assumption 2.3 and M_r defined in Assumption 2.1 with $\eta = r$. Then, for each $\lambda \in (0, \underline{\lambda}]$, BVP (1.1), (1.2) has at least one positive solution satisfying (2.10).*

The next theorem gives explicit values of λ with which BVP (1.1), (1.2) has no solution.

Theorem 2.7. *Assume that Assumption 2.2 holds, and let*

$$\bar{\lambda} = \frac{4 \times 30^{m-k-1}}{\int_0^1 s(1-s)\chi(s)\mu(s)\,\mathrm{d}s},$$

where $\chi(t)$ and k are given in Assumption 2.2, and $\mu(t)$ is defined by (2.7). Then, for any $\lambda \in (\bar{\lambda}, \infty)$, BVP (1.1), (1.2) has no solution satisfying (2.10).

Combining Theorems 2.5–2.7, we obtain the following result.

Theorem 2.8. *Assume that Assumptions 2.1–2.3 hold, and let $\underline{\lambda}$ and $\bar{\lambda}$ be defined in Theorems 2.6 and 2.7, respectively. There then exists $\lambda^* \in [\underline{\lambda}, \bar{\lambda}]$ such that, for each $\lambda \in (0, \lambda^*)$, BVP (1.1), (1.2) has at least one positive solution satisfying (2.10) and, for any $\lambda \in (\lambda^*, \infty)$, it does not have a solution satisfying (2.10).*

Remark 2.9. Theorems 2.5–2.8 are generalizations and improvements of Propositions 1.1–1.5 because

(i) the results are given for more general BVP (1.1), (1.2) where f may depend on higher-order derivatives of the unknown function and the BC may be nonlinear with multiple points involved;

(ii) weaker assumptions are imposed, in particular, the monotonicity of f in x_{m-1} is no longer required;

(iii) stronger conclusions are reached; in fact, in addition to the existence of λ^*, explicit intervals are found in Theorems 2.6 and 2.7, where the BVP has or does not have positive solutions, respectively.

Remark 2.10. The existence of positive solutions of BVP (1.1), (1.2) with $\lambda = \lambda^*$ is not given in Theorem 2.8. However, with further assumptions, we can show that BVP (1.1), (1.2) has a positive solution satisfying (2.10) when $\lambda = \lambda^*$. We omit the details.

Similar results to Theorems 2.5–2.8 also hold for the BVP consisting of Equation (1.1) and the more general form of BCs

$$\left.\begin{aligned} u^{(2i)}(0) &= \hat{g}_i(u^{(2i)}(a_1), u^{(2i)}(a_2), \dots, u^{(2i)}(a_l)),\\ u^{(2i)}(1) &= \hat{h}_i(u^{(2i)}(b_1), u^{(2i)}(b_2), \dots, u^{(2i)}(b_l)), \end{aligned}\right\} \quad i = 0, \dots, m-1, \tag{2.11}$$

where $l \geqslant 1$ is an integer, $a_1, \dots, a_l, b_1, \dots, b_l \in [0,1]$, and $\hat{g}_i, \hat{h}_i : \mathbb{R}^l \to \mathbb{R}$, $i = 0, \dots, m-1$, are continuous, non-decreasing in all their arguments on $((-1)^i\mathbb{R}^+)^l$, and

$(-1)^i \hat{g}_i, (-1)^i \hat{h}_i \geqslant 0$ on $((-1)^i \mathbb{R}^+)^l$. Note that, for $m = 1$, BC (2.11) includes the linear multipoint BC

$$u(0) = \sum_{i=1}^{n} c_i u(t_i), \qquad u(1) = \sum_{i=1}^{n} d_i u(t_i),$$

where $n \geqslant 1$ is an integer, $t_i \in (0, 1)$, and $c_i, d_i \geqslant 0$ for $i = 1, \dots, n$, which has been extensively studied in the literature (see, for example, [**12**, **16**, **18**] and the references therein).

We will use the following assumption.

Assumption 2.11. *There exists $r > 0$ such that*

$$\sum_{i=0}^{m-1} (|\hat{g}_i((-1)^i r, \dots, (-1)^i r)| + |\hat{h}_i((-1)^i r, \dots, (-1)^i r)|) + \int_0^1 s(1-s)\psi(s)\,\mathrm{d}s \leqslant r,$$

where ψ is given in Assumption 2.1.

Now we state the parallel results for BVP (1.1), (2.11) to Theorems 2.5–2.8 for the BVP (1.1), (1.2). The first theorem is a comparison result on the existence of solutions of BVP (1.1), (2.11) among different values of λ.

Theorem 2.12. *Assume that Assumption 2.1 holds, and there exists $\lambda_* > 0$ such that, for $\lambda = \lambda_*$, BVP (1.1), (2.11) has a solution $u_*(t)$ satisfying (2.9). Then, for each $\lambda \in (0, \lambda_*]$, BVP (1.1), (1.2) has a positive solution $u(t)$ satisfying (2.10).*

The following provides an explicit interval for λ where BVP (1.1), (2.11) has a positive solution.

Theorem 2.13. *Assume that Assumptions 2.1 and 2.11 hold, and let $\underline{\lambda}$ be defined as in Theorem 2.6, where r is given in Assumption 2.11. Then, for each $\lambda \in (0, \underline{\lambda}]$, the BVP (1.1), (2.11) has at least one positive solution satisfying (2.10).*

The next theorem gives explicit values of λ with which the BVP (1.1), (2.11) has no solution.

Theorem 2.14. *Assume that Assumption 2.2 holds, and let $\bar{\lambda}$ be defined as in Theorem 2.7. Then, for each $\lambda \in (\bar{\lambda}, \infty)$, the BVP (1.1), (2.11) has no solution satisfying (2.10).*

Combining Theorems 2.12–2.14, we obtain the following.

Theorem 2.15. *Assume that Assumptions 2.1, 2.2 and 2.11 hold, and let $\underline{\lambda}$ and $\bar{\lambda}$ be defined in Theorems 2.13 and 2.14, respectively. There then exists $\lambda^* \in [\underline{\lambda}, \bar{\lambda}]$ such that, for each $\lambda \in (0, \lambda^*)$, the BVP (1.1), (2.11) has at least one positive solution satisfying (2.10) and, for any $\lambda \in (\lambda^*, \infty)$, it does not have a solution satisfying (2.10).*

In the next section, we only prove Theorems 2.5–2.8. With minor modification of the arguments, one can prove Theorems 2.12–2.15. We omit the details.

In the rest of this section, we give an example to illustrate our results.

Example 2.16. Consider the BVP consisting of the equation

$$u^{(4)} = \lambda(u - \tfrac{1}{6}t^{-1/2}u'' + 1), \quad t \in (0,1), \tag{2.12}$$

and the BC

$$\left.\begin{aligned} u(0) &= \tfrac{1}{10}u^{1/2}(\tfrac{1}{3}), & u(1) &= \tfrac{1}{5}u^{2/3}(\tfrac{1}{2}), \\ u''(0) &= \tfrac{1}{5}u''(\tfrac{1}{3}), & u''(1) &= \tfrac{1}{10}u''(\tfrac{1}{2}), \end{aligned}\right\} \tag{2.13}$$

where $\lambda > 0$ is a parameter.

With $m = 2$, $\mathbb{D} = \mathbb{R}^+ \times \mathbb{R}^-$ and $f(t, x_0, x_1) = x_0 - t^{-1/2}x_{1/6} + 1$, we see that f is non-decreasing in x_0 on $\mathbb{R}^+$ and (2.1) holds. With $g_0(x_0) = x_0^{1/2}/10$, $g_1(x_1) = x_1/5$, $h_0(x_0) = x_0^{2/3}/5$, and $h_1(x_1) = x_1/10$, we see that g_i, h_i are non-decreasing on $(-1)^i\mathbb{R}^+$ for $i = 0, 1$, and (2.2) holds.

Let $\psi(t) = t^{-1/2}/6 + 2$ for $t \in (0,1)$. Then $\psi(t)$ satisfies (2.4). For any $\eta > 0$, let $M_\eta = \max\{1, \eta\}$ and $\mathbb{D}_\eta = [0, \eta] \times [-\eta, 0]$. Then, for $(t, x_0, x_1) \in (0,1) \times \mathbb{D}_\eta$,

$$f(t, x_0, x_1) = x_0 - \tfrac{1}{6}t^{-1/2}x_1 + 1 \leqslant M_\eta(\tfrac{1}{6}t^{-1/2} + 2) = M_\eta\psi(t),$$

i.e. (2.3) is satisfied. Hence Assumption 2.1 holds.

Let $\chi(t) \equiv 1$ for $t \in (0,1)$. Then $\chi(t)$ satisfies (2.6), and for $(t, x_0, x_1) \in (0,1) \times \mathbb{D}$

$$f(t, x_0, x_1) = x_0 - t^{-1/2}x_1 + 1 > \chi(t)x_0 = \chi(t)|x_0|,$$

i.e. (2.5) is satisfied for $k = 0$. Hence Assumption 2.2 holds.

Let $r = 1$. Then

$$\sum_{i=0}^{1}(|g_i((-1)^i r)| + |h_i((-1)^i r)|) = \tfrac{3}{5}.$$

From

$$\int_0^1 s(1-s)\psi(s)\,\mathrm{d}s = \int_0^1 s(1-s)(\tfrac{1}{6}s^{-1/2} + 2)\,\mathrm{d}s = \tfrac{17}{45}$$

we have that

$$\sum_{i=0}^{1}(|g_i((-1)^i r)| + |h_i((-1)^i r)|) + \int_0^1 s(1-s)\psi(s)\,\mathrm{d}s = \tfrac{3}{5} + \tfrac{17}{45} < 1 = r,$$

i.e. (2.8) is satisfied. Thus Assumption 2.3 holds.

Note that $m = 2$, $M_r = 1$, $k = 0$, and $\chi(t) \equiv 1$ on $(0,1)$. For $\underline{\lambda}$ and $\bar{\lambda}$ defined in Theorems 2.6 and 2.7, respectively, we have that $\underline{\lambda} = 1$ and $\bar{\lambda} = 2304$. Thus, from Theorem 2.8, there exists $\lambda^* \in [1, 2304]$ such that, for each $\lambda \in (0, \lambda^*)$, the BVP (2.12), (2.13) has at least one positive solution satisfying (2.10) and, for any $\lambda \in (\lambda^*, \infty)$, it does not have a solution that satisfies (2.10).

3. Proofs

It is well known that the Green function for the BVP

$$u''(t) = 0 \text{ on } (0,1) \text{ with } u(0) = u(1) = 0$$

is given by

$$G(t,s) = \begin{cases} t(s-1), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(t-1), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{3.1}$$

Let $G_1(t,s) = G(t,s)$ and recursively define

$$G_j(t,s) = \int_0^1 G(t,\tau) G_{j-1}(\tau,s) \,\mathrm{d}\tau, \quad j = 2,\dots,m. \tag{3.2}$$

Then $G_j(t,s)$ is the Green function for the BVP

$$u^{(2j)}(t) = 0, \quad t \in (0,1), \qquad u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0,\dots,j-1,$$

for $j = 1,\dots,m$. Clearly, (3.1) implies that

$$0 \leqslant -G(t,s) \leqslant s(1-s) \quad \text{for } (t,s) \in [0,1] \times [0,1]. \tag{3.3}$$

By (3.2), (3.3), and by induction, it is easy to see that, for $j = 1,\dots,m$,

$$0 \leqslant (-1)^j G_j(t,s) \leqslant s(1-s) \quad \text{for } (t,s) \in [0,1] \times [0,1]. \tag{3.4}$$

For any $u \in X$ and $f : (0,1) \times X \to \mathbb{R}$, if

$$\int_0^1 s(1-s)|f(s,u(\cdot))| \,\mathrm{d}s < \infty, \tag{3.5}$$

then from (3.1), (3.2) and (3.4), we see that, for $j = 0,\dots,m-1$,

$$\begin{aligned} \left| \left(\int_0^1 G_m(t,s) f(s,u(\cdot)) \,\mathrm{d}s \right)^{(2j)} \right| &= \left| \int_0^1 G_{m-j}(t,s) f(s,u(\cdot)) \,\mathrm{d}s \right| \\ &\leqslant \int_0^1 s(1-s)|f(s,u(\cdot))| \,\mathrm{d}s, \end{aligned} \tag{3.6}$$

and, for $j = 1,\dots,m-1$,

$$\begin{aligned} &\left| \left(\int_0^1 G_m(t,s) f(s,u(\cdot)) \,\mathrm{d}s \right)^{(2j-1)} \right| \\ &\quad = \left| \int_0^t \int_0^1 \tau G_{m-j}(\tau,s) f(s,u(\cdot)) \,\mathrm{d}s \,\mathrm{d}\tau + \int_t^1 \int_0^1 (\tau-1) G_{m-j}(\tau,s) f(s,u(\cdot)) \,\mathrm{d}s \,\mathrm{d}\tau \right| \\ &\quad \leqslant \int_0^t \int_0^1 s(1-s)|f(s,u(\cdot))| \,\mathrm{d}s \,\mathrm{d}\tau + \int_t^1 \int_0^1 s(1-s)|f(s,u(\cdot))| \,\mathrm{d}s \,\mathrm{d}\tau \\ &\quad = \int_0^1 s(1-s)|f(s,u(\cdot))| \,\mathrm{d}s. \end{aligned} \tag{3.7}$$

Moreover, from [**1**, Lemma 2.1] or [**15**, Lemma 3.1], we have that, for $j = 1, \dots, m$,

$$|G_j(t,s)| \geqslant \frac{1}{30^{j-1}} t(1-t)s(1-s) \quad \text{on } [0,1]\times[0,1]. \tag{3.8}$$

We refer the reader to [**1**, **15**] for related discussions about Green's functions.

The following lemmas will be used in the proofs of our main results. The first one is an analogue to Lemma 2.1 in [**8**, **9**], and can be proved in the same way.

Lemma 3.1. *$x(t)$ is a solution of the BVP (1.1), (1.2) if and only if $x(t)$ is a solution of the integral equation*

$$x(t) = \sum_{i=0}^{m-1} [g_i(x^{(2i)}(a))p_i(t) + h_i(x^{(2i)}(b))q_i(t)] \\ + \lambda \int_0^1 G_m(t,s) f(s, x(s), x''(s), \dots, x^{(2m-2)}(s))\,\mathrm{d}s,$$

where $G_m(t,s)$ is defined by (3.2) with $j = m$, and p_i and q_i are, respectively, the unique solutions of the BVPs

$$p_i^{(2m)}(t) = 0 \text{ on } (0,1), \quad p_i^{(2j)}(0) = \delta_{ij}, \quad p_i^{(2j)}(1) = 0, \qquad i,j = 0,\dots,m-1,$$
$$q_i^{(2m)}(t) = 0 \text{ on } (0,1), \quad q_i^{(2j)}(0) = 0, \quad q_i^{(2j)}(1) = \delta_{ij}, \quad i,j = 0,\dots,m-1,$$

with

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

In fact, p_i and q_i, $i = 0, \dots, m-1$, are polynomials of degree less than $2m$.

Lemma 3.2 shows some properties of the functions p_i and q_i given in Lemma 3.1.

Lemma 3.2. *Let p_i, q_i, $i = 0, \dots, m-1$, be defined as in Lemma 3.1. Then*

(i) *$\|p_i\| \leqslant 1$ and $\|q_i\| \leqslant 1$ for $i = 0, \dots, m-1$;*

(ii) *for $t \in [0,1]$ and $i = 0, \dots, m-1$,*

$$p_i^{(2j)}(t) = 0, \quad q_i^{(2j)}(t) = 0 \quad \text{for } j \in \{i+1, \dots, m\},$$

and

$$(-1)^{i-j} p_i^{(2j)}(t) \geqslant 0, \quad (-1)^{i-j} q_i^{(2j)}(t) \geqslant 0 \quad \text{for } j \in \{0, \dots, i\}.$$

Proof. The proof of part (i) was shown in [**8**, Lemma 2.3]. In the following, we prove part (ii).

For $t \in [0,1]$, $i = 0, \dots, m-1$, and $j = 0, \dots, m$, let $w_{i,j}(t) = p_i^{(2j)}(t)$. For $i = m-1$, from the definitions of p_{m-1} and q_{m-1}, $p_{m-1}^{(2m)}(t) = q_{m-1}^{(2m)}(t) = 0$ for $t \in [0,1]$. Then $w''_{m-1,m-1}(t) = 0$ on $(0,1)$, $w_{m-1,m-1}(0) = 1$ and $w_{m-1,m-1}(1) = 0$.

So $w_{m-1,m-1}(t) = 1 - t \geqslant 0$ for $t \in [0,1]$. Note that $w''_{m-1,m-2}(t) = w_{m-1,m-1}(t) \geqslant 0$ on $[0,1]$, $w_{m-1,m-2}(0) = w_{m-1,m-2}(1) = 0$ and we conclude that $w_{m-1,m-2}(t) \leqslant 0$ for $t \in [0,1]$. By induction, $(-1)^{m-1-j} w_{m-1,j}(t) \geqslant 0$ for $t \in [0,1]$ and $j \in \{0,\dots,m\}$, i.e. $(-1)^{m-1-j} p^{(2j)}_{m-1}(t) \geqslant 0$ on $[0,1]$ for $j \in \{0,\dots,m\}$. By the same argument, we can show that $(-1)^{m-1-j} q^{(2j)}_{m-1}(t) \geqslant 0$ on $[0,1]$ for $j \in \{0,\dots,m\}$. Thus, we have proved that part (ii) holds if $i = m-1$.

For $i \in \{0,\dots,m-2\}$, $w''_{i,m-1}(t) = 0$ on $[0,1]$ and $w_{i,m-1}(0) = w_{i,m-1}(1) = 0$. Thus, $w_{i,m-1}(t) = 0$ for $t \in [0,1]$. By the definition of p_i and induction, it is easy to see that $w_{i,j}(t) = 0$ for $t \in [0,1]$ and $j \in \{i+1,\dots,m\}$, i.e. $p^{(2j)}_i(t) = 0$ on $[0,1]$ for $j \in \{i+1,\dots,m\}$. Note now that $w''_{i,i}(t) = w_{i,i+1}(t) = 0$ on $[0,1]$, $w_{i,i}(0) = 1$ and $w_{i,i}(1) = 0$, and we have that $w_{i,i}(t) = 1 - t \geqslant 0$ for $t \in [0,1]$. Similarly to the case where $i = m-1$, it is easy to prove by induction that $(-1)^{i-j} p^{(2j)}_i(t) \geqslant 0$ for $t \in [0,1]$ and $j \in \{0,\dots,i\}$. The same reasoning can be used to show that $q^{(2j)}_i(t) = 0$ on $[0,1]$ for $j \in \{i+1,\dots,m\}$, and $(-1)^{i-j} q^{(2j)}_i(t) \geqslant 0$ on $[0,1]$ for $j \in \{0,\dots,i\}$. This completes the proof. □

The following is a generalized version of the Arzela–Ascoli theorem from $C[a,b]$ to $C^k[a,b]$ (see [**19**]).

Lemma 3.3. *Let k be a non-negative integer. Assume that $\{u_n(t)\}_{n=1}^{\infty}$ is a sequence in $C^k[0,1]$ such that $\{u_n(t)\}_{n=1}^{\infty}$ is uniformly bounded and $\{u_n^{(i)}(t)\}_{n=1}^{\infty}$, $i = 0,\dots,k$, are equicontinuous. Then $\{u_n(t)\}_{n=1}^{\infty}$ has a subsequence which converges uniformly to a function $u(t)$ in $C^k[0,1]$.*

The next result is about a property of functions given by Lemma 3.4 in [**14**].

Lemma 3.4. *Assume that $v \in C[0,1] \bigcap C^2(0,1)$ with $v(t) \geqslant 0$ and $v''(t) \leqslant 0$ on $(0,1)$. Then*

$$v(t) \geqslant \mu(t) \max_{\tau \in [0,1]} v(\tau) \quad \text{for } t \in [0,1],$$

where $\mu(t)$ is defined by (2.7).

3.1. Proof of Theorem 2.5

To prove Theorem 2.5, we need to introduce the definition of lower and upper solutions and present several related lemmas.

Definition 3.5. Let $\alpha \in X$ such that $\alpha^{(2m-1)}$ is absolutely continuous on $(0,1)$. Then $\alpha(t)$ is said to be a lower solution of BVP (1.1), (1.2) if

$$\alpha^{(2m)}(t) \geqslant \lambda f(t, \alpha(t), \alpha''(t), \dots, \alpha^{(2m-2)}(t)) \text{ a.e. on } (0,1),$$

$$\left.\begin{aligned} (-1)^{m-i+1}(\alpha^{(2i)}(0) - g_i(\alpha^{(2i)}(a))) \leqslant 0, \\ (-1)^{m-i+1}(\alpha^{(2i)}(1) - h_i(\alpha^{(2i)}(b))) \leqslant 0, \end{aligned}\right\} \quad i = 0,\dots,m-1.$$

Let $\beta \in X$ such that $\beta^{(2m-1)}$ is absolutely continuous on $(0,1)$. Then $\beta(t)$ is said to be an upper solution of the BVP (1.1), (1.2) if

$$\beta^{(2m)}(t) \leqslant \lambda f(t, \beta(t), \beta''(t), \ldots, \beta^{(2m-2)}(t)) \text{ a.e. on } (0,1),$$

$$\left.\begin{aligned} (-1)^{m-i+1}(\beta^{(2i)}(0) - g_i(\beta^{(2i)}(a))) \geqslant 0, \\ (-1)^{m-i+1}(\beta^{(2i)}(1) - h_i(\beta^{(2i)}(b))) \geqslant 0, \end{aligned}\right\} \quad i = 0, \ldots, m-1.$$

If $\alpha, \beta \in X$ satisfy the condition that, for $t \in [0,1]$,

$$(-1)^{m-i+1}\alpha^{(2i)}(t) \leqslant (-1)^{m-i+1}\beta^{(2i)}(t), \quad i = 0, \ldots, m-1. \tag{3.9}$$

then, for $i = 0, \ldots, m-1$, we define γ_i and δ_i by

$$\gamma_i = \min_{t\in[0,1]} \min\{\alpha^{(2i)}(t), \beta^{(2i)}(t)\} \tag{3.10}$$

and

$$\delta_i = \max_{t\in[0,1]} \max\{\alpha^{(2i)}(t), \beta^{(2i)}(t)\}, \tag{3.11}$$

and, for $u \in X$ and $i = 0, \ldots, m-1$, we define $\tilde{u}^{[2i]} : [0,1] \to \mathbb{R}$ by

$$\begin{aligned}&(-1)^{m-i+1}\tilde{u}^{[2i]}(t) \\ &\qquad = \max\{(-1)^{m-i+1}\alpha^{(2i)}(t), \min\{(-1)^{m-i+1}u^{(2i)}(t), (-1)^{m-i+1}\beta^{(2i)}(t)\}\}.\end{aligned} \tag{3.12}$$

Then, for $i = 0, \ldots, m-1$, $\gamma_i \leqslant \delta_i$, $\tilde{u}^{[2i]}(t)$ is continuous on $[0,1]$, and

$$\left.\begin{aligned} \tilde{\alpha}^{[2i]}(t) = \alpha^{(2i)}(t), \qquad \tilde{\beta}^{[2i]}(t) = \beta^{(2i)}(t), \\ (-1)^{m-i+1}\alpha^{(2i)}(t) \leqslant (-1)^{m-i+1}\tilde{u}^{[2i]}(t) \leqslant (-1)^{m-i+1}\beta^{(2i)}(t). \end{aligned}\right\} \tag{3.13}$$

Define a functional $\tilde{f} : (0,1) \times X \to \mathbb{R}$ by

$$\tilde{f}(t, u(\cdot)) = \begin{cases} f(t, \tilde{u}^{[0]}(t), \tilde{u}^{[2]}(t), \ldots, \tilde{u}^{[2m-4]}(t), \alpha^{(2m-2)}(t)) + \dfrac{u^{(2m-2)}(t) - \alpha^{(2m-2)}(t)}{1 + (u^{(2m-2)}(t))^2}, \\ \qquad\qquad \text{if } u^{(2m-2)}(t) < \alpha^{(2m-2)}(t), \\ f(t, \tilde{u}^{[0]}(t), \tilde{u}^{[2]}(t), \ldots, \tilde{u}^{[2m-4]}(t), u^{(2m-2)}(t)), \\ \qquad\qquad \text{if } \alpha^{(2m-2)}(t) \leqslant u^{(2m-2)}(t) \leqslant \beta^{(2m-2)}(t), \\ f(t, \tilde{u}^{[0]}(t), \tilde{u}^{[2]}(t), \ldots, \tilde{u}^{[2m-4]}(t), \beta^{(2m-2)}(t)) + \dfrac{u^{(2m-2)}(t) - \beta^{(2m-2)}(t)}{1 + (u^{(2m-2)}(t))^2}, \\ \qquad\qquad \text{if } u^{(2m-2)}(t) > \beta^{(2m-2)}(t). \end{cases} \tag{3.14}$$

Then, for $t \in (0,1)$, $\tilde{f}(t, u(\cdot))$ is continuous in u for $u \in X$. Consider the BVP consisting of the equation

$$u^{(2m)} = \lambda \tilde{f}(t, u(\cdot)), \quad t \in (0,1), \tag{3.15}$$

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and the BC

$$u^{(2i)}(0) = g_i(\tilde{u}^{[2i]}(a)), \quad u^{(2i)}(1) = h_i(\tilde{u}^{[2i]}(b)), \quad i = 0, \dots, m-1. \tag{3.16}$$

Lemma 3.6. *Let $\lambda > 0$ be fixed. Assume that the BVP (1.1), (1.2) has a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ satisfying (3.9), and $[\gamma_i, \delta_i] \subseteq (-1)^i \mathbb{R}^+$ for $i = 0, \dots, m-1$, where γ_i and δ_i are defined by (3.10) and (3.11), respectively. If $u(t)$ is a solution of the BVP (3.15), (3.16), then $u(t)$ satisfies the condition that, for $t \in [0,1]$ and $i = 0, \dots, m-1$,*

$$(-1)^{m-i+1}\alpha^{(2i)}(t) \leqslant (-1)^{m-i+1}u^{(2i)}(t) \leqslant (-1)^{m-i+1}\beta^{(2i)}(t). \tag{3.17}$$

Consequently, $u(t)$ is a solution of the BVP (1.1), (1.2).

Proof. We first prove that

$$(-1)^{m-i+1}u^{(2i)}(t) \leqslant (-1)^{m-i+1}\beta^{(2i)}(t) \quad \text{for } t \in [0,1] \text{ and } i = 0, \dots, m-1.$$

Suppose by contradiction that there exists $t_0 \in [0,1]$ such that $u^{(2m-2)}(t_0) > \beta^{(2m-2)}(t_0)$. Without loss of generality, assume that $u^{(2m-2)}(t) - \beta^{(2m-2)}(t)$ is maximized at t_0. If $t_0 = 0$, then, from (3.13), (3.16), the monotonicity of g_{m-1}, and the fact that $\beta(t)$ is an upper solution of the BVP (1.1), (1.2), we see that

$$u^{(2m-2)}(0) = g_{m-1}(\tilde{u}^{[2m-2]}(a)) \leqslant g_{m-1}(\beta^{(2m-2)}(a)) \leqslant \beta^{(2m-2)}(0), \tag{3.18}$$

which is a contradiction. A similar contradiction occurs at $t_0 = 1$. If $t_0 \in (0,1)$, then there exists $\hat{t}$ in a neighbourhood of t_0 such that $u^{(2m-2)}(\hat{t}) > \beta^{(2m-2)}(\hat{t})$ and $u^{(2m)}(\hat{t}) \leqslant \beta^{(2m)}(\hat{t})$. For otherwise, there exists a small neighbourhood $\mathcal{N}$ of t_0 such that $u^{(2m)}(t) > \beta^{(2m)}(t)$ almost everywhere in $\mathcal{N}$. This implies that $u^{(2m-2)}(t) - \beta^{(2m-2)}(t)$ is strictly concave-up in $\mathcal{N}$, contradicting the assumption that $u^{(2m-2)}(t) - \beta^{(2m-2)}(t)$ is maximized at t_0. Since $\beta(t)$ is an upper solution of the BVP (1.1), (1.2), from (3.13) and the monotonicity of f, we have that

$$\begin{aligned} 0 &\geqslant u^{(2m)}(\hat{t}) - \beta^{(2m)}(\hat{t}) \\ &\geqslant \lambda f(\hat{t}, \tilde{u}^{[0]}(\hat{t}), \tilde{u}^{[2]}(\hat{t}), \dots, \beta^{(2m-2)}(\hat{t})) + \frac{u^{(2m-2)}(\hat{t}) - \beta^{(2m-2)}(\hat{t})}{1 + (u^{(2m-2)}(\hat{t}))^2} \\ &\qquad - \lambda f(\hat{t}, \beta(\hat{t}), \beta''(\hat{t}), \dots, \beta^{(2m-2)}(\hat{t})) \\ &\geqslant \frac{u^{(2m-2)}(\hat{t}) - \beta^{(2m-2)}(\hat{t})}{1 + (u^{(2m-2)}(\hat{t}))^2} \\ &> 0. \end{aligned}$$

We again reach a contradiction. Thus $u^{(2m-2)}(t) \leqslant \beta^{(2m-2)}(t)$ for $t \in [0,1]$.

Using the monotonicity of g_{m-2}, similar to that in (3.18), we can show that

$$u^{(2m-4)}(0) - \beta^{(2m-4)}(0) \geqslant 0 \quad \text{and} \quad u^{(2m-4)}(1) - \beta^{(2m-4)}(1) \geqslant 0. \tag{3.19}$$

By Lemma 2.2 in [**8**, **9**], we have that, for $t \in [0,1]$,

$$\begin{aligned} u^{(2m-4)}(t) - \beta^{(2m-4)}(t) = [u^{(2m-4)}(0) - \beta^{(2m-4)}(0)](1-t) \\ + [u^{(2m-4)}(1) - \beta^{(2m-4)}(1)]t \\ + \int_0^1 G(t,s)[u^{(2m-2)}(s) - \beta^{(2m-2)}(s)]\,\mathrm{d}s. \end{aligned} \tag{3.20}$$

From (3.3), $0 \geqslant G(t,s) \geqslant -s(1-s)$ for $(t,s) \in [0,1] \times [0,1]$. Note that $u^{(2m-2)}(t) - \beta^{(2m-2)}(t) \leqslant 0$ on $[0,1]$; we get from (3.19) and (3.20) that $u^{(2m-4)}(t) - \beta^{(2m-4)}(t) \geqslant 0$ on $[0,1]$. Repeated application of the above argument yields that

$$(-1)^{m-i+1} u^{(2i)}(t) \leqslant (-1)^{m-i+1} \beta^{(2i)}(t) \quad \text{for } t \in [0,1] \text{ and } i = 0, \dots, m-1.$$

In the same way, we can show that

$$(-1)^{m-i+1} \alpha^{(2i)}(t) \leqslant (-1)^{m-i+1} u^{(2i)}(t) \quad \text{for } t \in [0,1] \text{ and } i = 0, \dots, m-1.$$

Hence (3.17) holds for $t \in [0,1]$ and $i = 0, \dots, m-1$. From (3.12), $\tilde{u}^{[2i]}(t) \equiv u^{(2i)}(t)$ on $[0,1]$ for $i = 0, \dots, m-1$, and then $\tilde{f}(t, u(\cdot)) \equiv f(t, u(t), u''(t), u^{(2m-2)}(t))$ on $[0,1]$. Therefore, $u(t)$ is a solution of the BVP (1.1), (1.2). This completes the proof. □

Lemma 3.7. *Let $\lambda > 0$ be fixed and let Assumption 2.1 hold. Assume that the BVP (1.1), (1.2) has a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ satisfying (3.9), and $[\gamma_i, \delta_i] \subseteq (-1)^i \mathbb{R}^+$ for $i = 0, \dots, m-1$, where γ_i and δ_i are defined by (3.10) and (3.11), respectively. Then the BVP (1.1), (1.2) has at least one solution $u(t)$ satisfying (3.17) for $t \in [0,1]$ and $i = 0, \dots, m-1$.*

Proof. Let $\tilde{u}^{[2i]}$, $i = 0, \dots, m-1$, and $\tilde{f}$ be defined by (3.12) and (3.14), respectively. Let $\eta = \max\{\|\alpha\|, \|\beta\|\}$. Then from Assumption 2.1 and (3.14) we see that, for $u \in X$ and $t \in [0,1]$,

$$|\tilde{f}(t, u(\cdot))| \leqslant M_\eta \psi(t) + \|\alpha\| + \|\beta\| + 1, \tag{3.21}$$

where ψ is given in Assumption 2.1. Define an operator $\tilde{T} : X \to X$ by

$$(\tilde{T}u)(t) = \sum_{i=0}^{m-1} [g_i(\tilde{u}^{[2i]}(a)) p_i(t) + h_i(\tilde{u}^{[2i]}(b)) q_i(t)] + \lambda \int_0^1 G_m(t,s) \tilde{f}(s, u(\cdot))\,\mathrm{d}s, \tag{3.22}$$

where $G_m(t,s)$ is given by (3.2) with $j = m$. In a manner similar to the proof of Lemma 3.1, we see that $u(t)$ is a solution of the BVP (3.15), (3.16) if and only if u is a fixed point of $\tilde{T}$. Clearly, $\tilde{T} : X \to X$ is continuous. In the following, we show that $\tilde{T}(X)$ is compact. In view of (3.13), there exists $d > 0$ such that, for all $u \in X$ and $i = 0, \dots, m-1$,

$$|g_i(\tilde{u}^{[2i]}(t))| \leqslant d \quad \text{and} \quad |h_i(\tilde{u}^{[2i]}(t))| \leqslant d \quad \text{on } [0,1]. \tag{3.23}$$

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From (2.4) and (3.21), we see that (3.5) holds with f replaced by $\tilde{f}$. Thus, from Lemma 3.2 (i), (3.6), (3.7), (3.21)–(3.23) and (2.4), we have that, for $u \in X$, $t \in [0,1]$ and $j = 0, \dots, 2m-2$,

$$
\begin{aligned}
|(\tilde{T}u)^{(j)}(t)| &\leqslant \sum_{i=0}^{m-1} [|g_i(\tilde{u}^{(2i)}(a))| + |h_i(\tilde{u}^{(2i)}(b))|] + \lambda \int_0^1 s(1-s)|\tilde{f}(s,u(\cdot))| \,\mathrm{d}s \\
&\leqslant 2md + \lambda \int_0^1 s(1-s)(M_\eta \psi(s) + \|\alpha\| + \|\beta\| + 1) \,\mathrm{d}s \\
&< \infty.
\end{aligned}
$$

This means that $\tilde{T}$ is uniformly bounded on X, and $(\tilde{T}u)^{(j)}(t)$ is equicontinuous on $[0,1]$ for $j = 0, \dots, 2m-3$. Now we show that $(\tilde{T}u)^{(2m-2)}(t)$ is equicontinuous on $[0,1]$. From (3.22),

$$
\begin{aligned}
(\tilde{T}u)^{(2m-2)}(t) = \sum_{i=0}^{m-1} [g_i(\tilde{u}^{[2i]}(a)) p_i^{(2m-2)}(t) + h_i(\tilde{u}^{[2i]}(b)) q_i^{(2m-2)}(t)] \\
+ \lambda \int_0^1 G(t,s)\tilde{f}(s,u(\cdot)) \,\mathrm{d}s.
\end{aligned}
$$

Hence, it suffices to show that the operator $A : X \to X$ defined by

$$
(Au)(t) = \int_0^1 G(t,s)\tilde{f}(s,u(\cdot)) \,\mathrm{d}s
$$

is equicontinuous on $[0,1]$. From (2.4), we see that, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$
\int_0^\delta s(1-s)(M_\eta \psi(s) + \|\alpha\| + \|\beta\| + 1) \,\mathrm{d}s \leqslant \tfrac{1}{6}\varepsilon \tag{3.24}
$$

and

$$
\int_{1-\delta}^1 s(1-s)(M_\eta \psi(s) + \|\alpha\| + \|\beta\| + 1) \,\mathrm{d}s \leqslant \tfrac{1}{6}\varepsilon. \tag{3.25}
$$

Since $G(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$, for the above ε, there exists $\zeta = \zeta(\varepsilon) > 0$ such that

$$
|G(t_1,s) - G(t_2,s)| \leqslant \frac{\varepsilon}{3\int_\delta^{1-\delta}(M_\eta \psi(\tau) + \|\alpha\| + \|\beta\| + 1) \,\mathrm{d}\tau} \tag{3.26}
$$

for $s \in [0,1]$ and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| \leqslant \zeta$. Combining (3.24)–(3.26) and considering (3.3) and (3.21), we have that, for any $u \in X$ and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| \leqslant \zeta$,

$$
|(Au)(t_1) - (Au)(t_2)| = \left| \left(\int_0^\delta + \int_{1-\delta}^1 + \int_\delta^{1-\delta} \right) (G(t_1,s) - G(t_2,s))\tilde{f}(s,u(\cdot)) \,\mathrm{d}s \right|
$$

$$\begin{aligned}
&\leqslant 2\biggl(\int_0^{\delta}+\int_{1-\delta}^{1}\biggr)s(1-s)(M_\eta\psi(s)+\|\alpha\|+\|\beta\|+1)\,\mathrm{d}s\\
&\qquad+\int_{\delta}^{1-\delta}|(G(t_1,s)-G(t_2,s))|(M_\eta\psi(s)+\|\alpha\|+\|\beta\|+1)\,\mathrm{d}s\\
&\leqslant \tfrac{2}{3}\varepsilon+\tfrac{1}{3}\varepsilon\\
&=\varepsilon.
\end{aligned}$$

This implies that A is equicontinuous on $[0,1]$, and so is $(\tilde{T}u)^{(2m-2)}(t)$. By Lemma 3.3, $\tilde{T}(X)$ is compact. Using the Schauder fixed-point theorem, we see that there exists a fixed point u of $\tilde{T}$ in X. Hence, $u(t)$ is a solution of the BVP (3.15), (3.16). Therefore, from Lemma 3.6, $u(t)$ satisfies (3.17) for $t\in[0,1]$ and $i=0,\dots,m-1$, and is consequently a solution of the BVP (1.1), (1.2). This completes the proof. □

Proof of Theorem 2.5. We consider two cases when m is even and odd, respectively.

(1) Assume m is even. In this case, from (2.1), $f\geqslant 0$ on $(0,1)\times\mathbb{D}$. Let $\alpha(t)\equiv u_*(t)$ and $\beta(t)\equiv 0$ for $t\in[0,1]$. Then, from (2.9), $\alpha(t)$ and $\beta(t)$ satisfy (3.9), $(\alpha(t),\dots,\alpha^{(2m-2)}(t))\in\mathbb{D}$, and $(\beta(t),\dots,\beta^{(2m-2)}(t))\in\mathbb{D}$. Since for $\lambda\in(0,\lambda_*)$,

$$\begin{aligned}
\alpha^{(2m)}(t)&=\lambda_* f(t,\alpha(t),\alpha''(t),\dots,\alpha^{(2m-2)}(t))\\
&\geqslant \lambda f(t,\alpha(t),\alpha''(t),\dots,\alpha^{(2m-2)}(t))\quad\text{on }(0,1),
\end{aligned}$$

$\alpha^{(2i)}(0)=g_i(\alpha^{(2i)}(a))$ and $\alpha^{(2i)}(1)=g_i(\alpha^{(2i)}(b))$ for $i=0,\dots,m-1$, $\alpha(t)$ is a lower solution of the BVP (1.1), (1.2) for $\lambda\in(0,\lambda_*)$. On the other hand, from (2.2) we see that, for $\lambda\in(0,\infty)$,

$$\beta^{(2m)}(t)\equiv 0\leqslant \lambda f(t,\beta(t),\beta''(t),\dots,\beta^{(2m-2)}(t)),\quad t\in(0,1),$$

$$\left.\begin{aligned}
(-1)^{m-i+1}(\beta^{(2i)}(0)-g_i(\beta^{(2i)}(a)))\geqslant 0,\\
(-1)^{m-i+1}(\beta^{(2i)}(1)-g_i(\beta^{(2i)}(b)))\geqslant 0,
\end{aligned}\right\}\quad i=0,\dots,m-1.$$

Thus $\beta(t)$ is an upper solution of the BVP (1.1), (1.2) for $\lambda\in(0,\infty)$. Moreover, $[\gamma_i,\delta_i]\subseteq(-1)^i\mathbb{R}^+$ for $i=0,\dots,m-1$. Thus, Lemma 3.7 implies that, for each $\lambda\in(0,\lambda_*)$, the BVP (1.1), (1.2) has at least one solution $u(t)$ satisfying (3.17) for $t\in[0,1]$ and $i=0,\dots,m-1$. Clearly, $u(t)$ satisfies (2.10). Now we show that $u(t)$ is a positive solution. Note from (2.1) that $u(t)\not\equiv 0$ on $[0,1]$. From (2.10) with $j=1$ we see that $u''(t)\leqslant 0$ on $(0,1)$. Thus, by Lemma 3.4, $u(t)>0$ for $t\in(0,1)$, i.e. $u(t)$ is a positive solution of the BVP (1.1), (1.2).

(2) Assume m is odd. In this case, from (2.1), $f\leqslant 0$ on $(0,1)\times\mathbb{D}$. Let $\alpha(t)\equiv 0$ and $\beta(t)\equiv u_*(t)$ for $t\in[0,1]$. Then as in case (1), we see that $\alpha(t)$ and $\beta(t)$ satisfy (3.9), $(\alpha(t),\dots,\alpha^{(2m-2)}(t))\in\mathbb{D}$, $(\beta(t),\dots,\beta^{(2m-2)}(t))\in\mathbb{D}$, $\alpha(t)$ is a lower solution of the BVP (1.1), (1.2) for $\lambda\in(0,\infty)$, and $\beta(t)$ is an upper solution of the BVP (1.1), (1.2) for $\lambda\in(0,\lambda_*)$. The rest of the proof is similar to case (1) and hence is omitted. □

3.2. Proof of Theorems 2.6–2.8

Proof of Theorem 2.6. Define a set K in X by

$$K = \{u \in X \mid (-1)^j u^{(2j)}(t) \geqslant 0 \text{ for } t \in [0,1] \text{ and } j = 0, \dots, m-1\},$$

and an operator $T : K \to X$ by

$$(Tu)(t) = \sum_{i=0}^{m-1} [g_i(u^{(2i)}(a))p_i(t) + h_i(u^{(2i)}(b))q_i(t)] \\ + \lambda \int_0^1 G_m(t,s) f(s, u(s), u''(s), \dots, u^{(2m-2)}(s)) \, ds, \quad (3.27)$$

where $\lambda > 0$. By Lemma 3.1, $u(t)$ is a solution of the BVP (1.1), (1.2) if and only if u is a fixed point of the operator T. For $u(t) \in K$, $t \in [0,1]$ and $j = 0, \dots, m-1$,

$$(Tu)^{(2j)}(t) = \sum_{i=0}^{m-1} [g_i(u^{(2i)}(a))p_i^{(2j)}(t) + h_i(u^{(2i)}(b))q_i^{(2j)}(t)] \\ + \lambda \int_0^1 G_{m-j}(t,s) f(s, u(s), u''(s), \dots, u^{(2m-2)}(s)) \, ds. \quad (3.28)$$

We observe from (2.2) and Lemma 3.2 (ii) that, for $u(t) \in K$, $(t,s) \in [0,1] \times [0,1]$ and $i, j = 0, \dots, m-1$,

$$(-1)^j g_i(u^{(2i)}(a))p_i^{(2j)}(t) \geqslant 0 \quad \text{and} \quad (-1)^j h_i(u^{(2i)}(b))q_i^{(2j)}(t) \geqslant 0, \quad (3.29)$$

and, from (2.1) and (3.4),

$$(-1)^j G_{m-j}(t,s) f(t, u(s), u''(s), \dots, u^{(2m-2)}(s)) \geqslant 0. \quad (3.30)$$

Combining (3.28)–(3.30), we obtain the result that

$$(-1)^j (Tu)^{(2j)}(t) \geqslant 0 \quad \text{for } t \in [0,1] \text{ and } j = 0, \dots, m-1. \quad (3.31)$$

Thus, $T : K \to K$. Let r be as given in Assumption 2.3 and define $K_r = \{u \in K \mid \|u\| \leqslant r\}$. From Assumption 2.3, (2.8) holds. Thus, in view of (2.2) and from the monotonicity of g_i, h_i, we see that, for $u \in K_r$,

$$\sum_{i=0}^{m-1} (|g_i(u^{(2i)}(a))| + |h_i(u^{(2i)}(b))|) + \int_0^1 s(1-s)\psi(s) \, ds \leqslant r.$$

Note from Assumption 2.1 that f satisfies (3.5) with $u \in K_r$. Thus, for $\lambda \in (0, \underline{\lambda}]$, $u \in K_r$, $t \in [0,1]$ and $j = 0, \dots, 2m-2$, by Lemma 3.2 (i), Assumption 2.1 with $\eta = r$, (3.6) and (3.7), we have that

$$\begin{aligned} |(Tu)^{(j)}(t)| &\leqslant \sum_{i=0}^{m-1} [|g_i(u^{(2i)}(a))| + |h_i(u^{(2i)}(b))|] + \lambda M_r \int_0^1 s(1-s)\psi(s) \, ds \\ &\leqslant \sum_{i=0}^{m-1} (|g_i(u^{(2i)}(a))| + |h_i(u^{(2i)}(b))|) + \int_0^1 s(1-s)\psi(s) \, ds \\ &\leqslant r, \end{aligned}$$

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i.e. $\|Tu\| \leqslant r$. Thus $T : K_r \to K_r$ for $\lambda \in (0, \underline{\lambda}]$. As in the proof of Lemma 3.7, we can show that $T(K_r)$ is compact. By the Schauder fixed-point theorem, there exists a fixed point u of T in K_r for each $\lambda \in (0, \underline{\lambda}]$. Hence, for each $\lambda \in (0, \underline{\lambda}]$, the BVP (1.1), (1.2) has a solution $u(t)$. Since $u(t) \in K$, $u(t)$ satisfies (2.10). Using the same argument as in the proof of Theorem 2.5, we see that $u(t)$ is positive. This completes the proof. □

Proof of Theorem 2.7. From (2.6), we see that $\bar{\lambda} \in (0, \infty)$. Suppose by contradiction that there exists $\lambda \in (\bar{\lambda}, \infty)$ such that the BVP (1.1), (1.2) has a solution $u(t)$ satisfying (2.10). For the k given in Assumption 2.2, we now claim that $u^{(2k)}(t) \not\equiv 0$ on $[0, 1]$. Otherwise, $u^{(2j)}(t) \equiv 0$ on $(0, 1)$ for $j = k, \dots, m$. By the monotonicity of f, it is easy to see that

$$|f(t, u(t), \dots, u^{(2k-2)}(t), 0, \dots, 0)| \geqslant |f(t, 0, \dots, 0)| \not\equiv 0 \text{ a.e. on } (0, 1).$$

But this contradicts the assumption that $u(t)$ is a solution of Equation (1.1).

From (2.10) we see that

$$(-1)^k u^{(2k)}(t) \geqslant 0 \quad \text{and} \quad (-1)^k u^{(2k+2)}(t) \leqslant 0 \quad \text{on } (0, 1).$$

Then Lemma 3.4 implies that

$$(-1)^k u^{(2k)}(t) \geqslant \mu(t) \max_{\tau \in [0,1]} \{(-1)^k u^{(2k)}(\tau)\}. \tag{3.32}$$

Thus $(-1)^k u^{(2k)}(t) > 0$ for $t \in (0, 1)$. Note from Lemma 3.1 that u is a fixed point of the operator T defined by (3.27), i.e. $(Tu)(t) \equiv u(t)$ on $[0, 1]$. Hence, from (3.28)–(3.30) with $j = k$, we have that, for $t \in [0, 1]$,

$$(-1)^k u^{(2k)}(t) \geqslant \lambda \int_0^1 |G_{m-k}(t, s)|\, |f(s, u(s), u''(s), \dots, u^{(2m-2)}(s))| \, \mathrm{d}s.$$

Then, from (2.5), (3.8) and (3.32),

$$\begin{aligned}
(-1)^k u^{(2k)}(\tfrac{1}{2}) &\geqslant \frac{1}{4 \times 30^{m-k-1}} \lambda \int_0^1 s(1-s)\chi(s)(-1)^k u^{(2k)}(s) \, \mathrm{d}s \\
&\geqslant \frac{1}{4 \times 30^{m-k-1}} \lambda \max_{\tau \in [0,1]} \{(-1)^k u^{(2k)}(\tau)\} \int_0^1 s(1-s)\chi(s)\mu(s) \, \mathrm{d}s \\
&\geqslant \frac{1}{4 \times 30^{m-k-1}} \lambda (-1)^k u^{(2k)}(\tfrac{1}{2}) \int_0^1 s(1-s)\chi(s)\mu(s) \, \mathrm{d}s.
\end{aligned}$$

Hence,

$$1 \geqslant \frac{1}{4 \times 30^{m-k-1}} \lambda \int_0^1 s(1-s)\chi(s)\mu(s) \, \mathrm{d}s,$$

and so

$$\lambda \leqslant \frac{4 \times 30^{m-k-1}}{\int_0^1 s(1-s)\chi(s)\mu(s) \, \mathrm{d}s} = \bar{\lambda},$$

which contradicts the assumption that $\lambda \in (\bar{\lambda}, \infty)$. Thus, for any $\lambda \in (\bar{\lambda}, \infty)$, the BVP (1.1), (1.2) has no solution that satisfies (2.10). This completes the proof. □

Proof of Theorem 2.8. Theorem 2.8 readily follows from Theorems 2.5–2.7. □

References

1. R. P. Agarwal, D. O'Regan and S. Staněk, Singular Lidstone boundary value problem with given maximal values for solutions, *Nonlin. Analysis* **55** (2003), 859–881.
2. C. F. Beards, *Vibration analysis with applications to control systems* (Edward Arnold, London, 1995).
3. Y. S. Choi, A singular boundary value problem arising from near-ignition analysis of flame structure, *Diff. Integ. Eqns* **4** (1991), 891–895.
4. R. Dalmasso, Positive solutions of singular boundary value problems, *Nonlin. Analysis* **27** (1996), 645–652.
5. J. M. Davis, P. W. Eloe and J. Henderson, Triple positive solutions and dependence on higher order derivatives, *J. Math. Analysis Applic.* **237** (1999), 710–720.
6. J. M. Davis, J. Henderson and P. J. Y. Wong, General Lidstone problems: multiplicity and symmetry of solutions, *J. Math. Analysis Applic.* **251** (2000), 527–548.
7. E. Dulácska, *Soil settlement effects on building*, Developments in Geotechnical Engineering, vol. 69 (Elsevier, 1992).
8. J. Ehme and J. Henderson, Existence and local uniqueness for nonlinear Lidstone boundary value problems, *J. Inequal. Pure. Appl. Math.* **1**(1) (2000), Article 8, pp. 1–9 (available at www.emis.famaf.unc.edu.ar/journals/JIPAM/v1n1/021_99.html).
9. J. Ehme, P. W. Eloe and J. Henderson, Existence of solutions for $2n$th order nonlinear generalized Sturm–Liouville boundary value problems, *Math. Inequal. Appl.* **4** (2001), 247–255.
10. J. Ehme, P. W. Eloe and J. Henderson, Upper and lower solution methods for fully nonlinear boundary value problems, *J. Diff. Eqns* **180** (2002), 51–64.
11. D. Franco, D. O'Regan and J. Perán, Fourth-order problems with nonlinear boundary conditions, *J. Computat. Appl. Math.* **174** (2005), 315–327.
12. C. P. Gupta and S. Trofimchuk, Solvability of a multi-point boundary value problem and related *a priori* estimates, *Can. Appl. Math. Q.* **6** (1998), 45–60.
13. K. S. Ha and Y. H. Lee, Existence of multiple positive solutions of singular boundary value problems, *Nonlin. Analysis* **28** (1997), 1429–1938.
14. L. Kong and Q. Kong, Positive solutions of nonlinear m-point boundary value problems on a measure chain, *J. Diff. Eqns Applic.* **9** (2003), 615–627.
15. L. Kong and Q. Kong, Even order nonlinear eigenvalue problems on a measure chain, *Nonlin. Analysis* **52** (2003), 1891–1909.
16. L. Kong and Q. Kong, Multi-point boundary value problems of second-order differential equations, I, *Nonlin. Analysis* **58** (2004), 909–931.
17. L. Kong and Q. Kong, Second-order boundary value problems with nonhomogeneous boundary conditions, I, *Math. Nachr.* **278** (2005), 173–193.
18. L. Kong and Q. Kong, Multi-point boundary value problems of second-order differential equations, II, preprint.
19. L. Kong and Q. Kong, Second-order boundary value problems with nonhomogeneous boundary conditions, II, preprint.
20. B. Lou, Positive solutions of nonlinear focal boundary value problems, *Rocky Mt. J. Math.* **33** (2003), 1417–1435.
21. L. Meirovitch, *Dynamics and control of structures* (Wiley, New York, 1990).
22. I. Rachůnková and J. Tomeček, Impulsive BVPs with nonlinear boundary conditions for the second order differential equations without growth restrictions, *J. Math. Analysis Applic.* **292** (2004), 525–539.

23. H. B. Thompson, Second order ordinary differential equations with fully nonlinear two point boundary conditions, *Pac. J. Math.* **172** (1996), 255–276.
24. H. B. Thompson, Second order ordinary differential equations with fully nonlinear two point boundary conditions, II, *Pac. J. Math.* **172** (1996), 279–297.
25. F. H. Wong, Existence of positive solutions of singular boundary value problems, *Nonlin. Analysis* **21** (1993), 397–406.
26. Q. L. Yao and Z. B. Bai, Existence of positive solutions for boundary value problems of $u^{(4)}(t) - \lambda h(t)f(u(t)) = 0$, *Chin. Ann. Math.* A **20** (1999), 575–578.
27. B. G. Zhang and L. Kong, Existence of positive solutions of fourth order singular boundary value problems, *Chin. J. Contemp. Math.* **22** (2001), 207–214.

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