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ON UNIFORMLY STRONGLY PRIME GAMMA RINGS

G.L. BOOTH AND N.J. GROENWALD

The concept of uniformly strongly prime (usp) is introduced for Γ-ring, and a usp radical $\tau(M)$ is defined for a Γ-ring M. If M has left and right unities, then $\tau(L)^+ = \tau(M) = \tau(R)^*$, where L and R denote, respectively, the left and right operator rings of M, and $\tau(\cdot)$ denotes the usp radical of a ring. If m, n are positive integers, then $\tau(M_{mn}) = (\tau(M))_{mn}$, where M_{mn} is the matrix Γ_{nm}-ring. τ is shown to be a special radical in the variety of Γ-rings. τ_1 is the upper radical determined by the class of usp Γ-rings of bound 1. $\tau \subseteq \tau_1$, but the reverse inclusion does not hold in general. The place of τ and τ_1 in the hierarchy of radicals for Γ-rings is shown.

1. BASIC CONCEPTS

Let M and Γ be additive abelian groups. If, for all $x, y, z \in M$, $\gamma, \mu \in \Gamma$, we have

(i) $x\gamma y \in M$;

(ii) $x\gamma(y\mu z) = (x\gamma y)\mu z$;

(iii) $x\gamma(y+z) = x\gamma y + x\gamma z$; $x(\gamma+\mu)y = x\gamma y + x\mu y$; $(x+y)\gamma z = x\gamma z + y\gamma z$

then M is called a *Γ-ring*. If U and V are subsets of M and ϕ is a subset of Γ, then we define

$$U\phi V = \{u\gamma v : u \in U, \quad \gamma \in \phi, \quad v \in V\}.$$

If A is a subgroup of M^+, and $A\Gamma M \subseteq A$, $M\Gamma A \subseteq A$, then A is an *ideal* of M, denoted by $A \lhd M$. Similar notation will be used for ideals of rings. If $A \lhd M$, the factor Γ-ring M/A is defined in the natural way. If $P \lhd M$, and $U, V \lhd M$, $U\Gamma V \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$, then P is called a *prime ideal* of M. M is a *prime Γ-ring* if the zero ideal of M is prime. The following result is proved along the same lines as the corresponding one for rings.

PROPOSITION 1.1. *Let M be a Γ-ring and let $P \lhd M$. Then the following are equivalent:*

(a) *P is a prime ideal of M;*

(b) *For all $x, y \in P$, $x\Gamma M\Gamma y \subseteq P$ implies $x \in P$ or $y \in P$.*

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If $A \lhd M$, the *left annihilator* of M is the set

$$l(A) = \{x \in M : x\Gamma A = 0\}.$$

Similarly,

$$r(A) = \{x \in M : A\Gamma x = 0\}.$$

If $A \lhd M$, and $0 \neq I \lhd M$ implies $I \cap A \neq 0$, then A is called an *essential ideal* of M, denoted by $A \lhd \cdot M$. If M and M' are Γ-rings, and there exists a group isomorphism $f\colon M \to M'$ satisfying $f(x\gamma y) = f(x)\gamma f(y)$, for all $x, y \in M$, $\gamma \in \Gamma$, then M and M' are said to be isomorphic, denoted by $M \cong M'$.

Let $x \in M$, $\gamma \in \Gamma$. Define $[x, \gamma]\colon M \to M$ by $[x, \gamma]y = x\gamma y$ for all $y \in M$. The subring L of $\operatorname{end}(M)$ consisting of all sums $\sum_i [x_i, \gamma_i]$, $x_i \in M$, $\gamma_i \in \Gamma$, is called the *left operator ring* of M. A right operator ring R of M is defined similarly, and consists of all sums of the form $\sum_i [\gamma_i, x_i]$, $\gamma_i \in \Gamma$, $x_i \in M$.

If $A \subseteq L$, $A^+ = \{x \in M : [x, \gamma] \in A \text{ for all } \gamma \in \Gamma\}$.
If $B \subseteq R$, $B^* = \{x \in M : [\gamma, x] \in B \text{ for all } \gamma \in \Gamma\}$.
If $C \subseteq L$, $C^{+'} = \{l \in L : lM \subseteq C\}$ and $C^{*'} = \{r \in R : Mr \subseteq C\}$.

It is easily seen that all of these mappings take ideals to ideals, and preserve intersections.

If L contains an element d such that $dx = x$ for all $x \in M$, then d is called a *left unity* for M. It is easily seen that d is the (2-sided) unity of L in this case. Similarly, if there exists $e \in R$ such that $xe = x$ for all $x \in M$, then e is called a *right unity* for M, and e is the unity of R in this case.

For further details of Γ-rings and their operator rings, see the references.

2. Uniformly strongly prime Γ-rings

Following Olson [**11**], a ring R is called *uniformly strongly prime* (usp), if R contains a finite subset F such that $xFy = 0$ implies that $x = 0$ or $y = 0$, for all $x, y \in R$. F is called an *insulator* for R. If $P \lhd R$, then P is called a *usp ideal* of R if there exists a finite subset F of R such that $xFy \subseteq P$ implies that $x \in P$ or $y \in P$ for all $x, y \in R$. It is clear that $P \lhd R$ is usp if and only if the factor ring R/P is usp. Moreover, usp implies strongly prime and hence prime.

In [**4**] the concept of a strongly prime Γ-ring is introduced. Several characterisations of this concept are available. In ([**4**], Proposition 2.2(c)), it is shown that a Γ-ring M is strongly prime if and only if, for all $x \in M$, there exist finite subsets F of M and ϕ and Λ of Γ such that $x\phi F\Lambda y = 0$ implies $y = 0$, for all $y \in M$. In view of Proposition 1.1, strongly prime implies prime.

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A Γ-ring M will be called *uniformly strongly prime (usp)* if there exist finite subsets F and Δ of M and Γ, respectively, such that $x\Delta F\Delta y = 0$ implies $x = 0$ or $y = 0$, for all $x, y \in M$.

The pair (F, Δ) will be called an *insulator* for M. It is obvious that usp implies strongly prime for Γ-rings. If $P \lhd N$, then P is called usp if M and Γ contains finite subsets F and Δ, respectively, such that $x\Delta F\Delta y \subseteq P$ implies $x \in P$ or $y \in P$, for all $x, y \in M$. It is clear that, if $P \lhd M$, then P is usp if and only if M/P is a usp Γ-ring.

The following characterisation of usp Γ-rings will be of use later.

LEMMA 2.1. *Let M be a Γ-ring. Then the following are equivalent:*

(a) *M is usp;*

(b) *there exist finite subsets F of M and ϕ and Λ of Γ such that $x\phi F\Lambda y = 0$ implies $x = 0$ or $y = 0$, for all $x, y \in M$.*

PROOF: (a) $\Rightarrow$ (b) is obvious. So suppose the conditions of (b) hold. Then let $\Delta = \phi \cup \Lambda$. Since $x\phi F\Lambda y \subseteq x\Delta F\Delta y$ for all $x, y \in M$, (F, Δ) is the required insulator of M. ∎

THEOREM 2.2. *Let M be a Γ-ring with left operator ring L. Then:*

(a) *If M has a left unity $\sum_{i=1}^{m}[d_i, \delta_i]$ and M is usp, then L is usp;*

(b) *If M has a right unity $\sum_{i=1}^{n}[\varepsilon_i, e_i]$ and L is usp, then M is usp.*

PROOF: (a) Let (F, Δ) be the insulator for M. Then define

$$G = \{[d_i\gamma f, \mu] : 1 \leqslant i \leqslant m, \gamma, \mu \in \Delta, f \in F\}.$$

Now suppose that $l, l' \in L$, $l \neq 0$ and $lGl' = 0$. Then $ld_j \neq 0$ for at least one j, for $ld_i = 0$, $1 \leqslant i \leqslant m$ implies $\sum_i [ld_i, \delta_i] = 0$, whence $\ell \sum_i [d_i, \delta_i] = 0$, that is $\ell = 0$. Hence, for all $x \in M$, $f \in F$, $\gamma, \mu \in \Delta$, $(ld_j)\gamma f\mu(l'x) = 0$. Since $ld_j \neq 0$, $l'x = 0$ for all $x \in M$, that is $l' = 0$. Hence G is an insulator for L, so L is usp.

(b) Let F be the insulator for L. Suppose that $F = \{l_1, \ldots, l_r\}$ and that $l_j = \sum_{i=1}^{s(j)}[x_{ij}, \gamma_{ij}]$. Define

$$G = \{x_{ij} : 1 \leqslant i \leqslant s(j), 1 \leqslant j \leqslant r\},$$
$$\phi = \{\varepsilon_1, \ldots, \varepsilon_n\}$$
$$\Lambda = \{\gamma_{ij} : 1 \leqslant i \leqslant s(j), 1 \leqslant j \leqslant r\}$$

∎

Let $x, y \in M$ be such that $x \neq 0$ and $x\phi G\Lambda y = 0$. Then $[x, \varepsilon_j] \neq 0$ for at least one ε_j, otherwise $\sum_i x\varepsilon_i e_i = 0$, that is $x = 0$. Now for all $1 \leqslant i \leqslant n$, $\gamma \in \Gamma$, we have that $[x, \varepsilon_j]F[y, \gamma] = 0$. If follows that $[y, \gamma] = 0$ for all $\gamma \in \Gamma$. In particular, $[y, \varepsilon_i] = 0$, $1 \leqslant i \leqslant n$, whence $\sum_i y\varepsilon_i e_i = 0$, that is $y = 0$. Hence, by Lemma 2.1, M is usp, as required.

In [**11**] the usp radical of a ring R, $\tau(R)$, is defined to be the intersection of its usp ideals. Similarly, we define the *usp radical* of a Γ-ring M, $\tau(M)$, to be the intersection of its usp ideals.

LEMMA 2.3. ([**9**], Theorem 2). *Let M be a Γ-ring with left and right unities, and let L be the left operator ring of M. Then, if $A \lhd L$, $A = (A^+)^{+'}$, and if $B \lhd M$, $B = \left(B^{+'}\right)^+$. Hence the mapping $A \to A^+$ defines a one-to-one correspondence between the sets of ideals of L and M.*

LEMMA 2.4. ([**2**], Corollary 2.2). *Let M be a Γ-ring with left operator ring L. If $A \lhd M$, then the left operator ring of the factor Γ-ring M/A is isomorphic to $L/A^{+'}$.*

LEMMA 2.5. *Let M be a Γ-ring with left and right unities, and let L be the left operator ring of M. Then the mapping $A \to A^+$ defines a one-to-one correspondence between the sets of usp ideals of L and M.*

PROOF: Suppose A is a usp ideal of L. Then L/A is a usp ring, and by Lemma 2.4 the left operator ring of M/A^+ is isomorphic to $L/(A^+)^{+'} = L/A$ by Lemma 2.3. Hence, by Theorem 2.2, M/A^+ is an usp Γ-ring, hence A^+ is a usp ideal of M.

Suppose now that B is a usp ideal of M. Then M/B is a usp Γ-ring, and the left operator ring of M/B is isomorphic to $L/B^{+'}$ by Lemma 2.4. Hence, by Theorem 2.2, $L/B^{+'}$ is a usp ring, whence $B^{+'}$ is a usp ideal of L. ∎

The result now follows from 2.3.

THEOREM 2.6. *Let M be a Γ-ring with left and right unities, and with left and right operator rings L and R respectively. Then*

$$\tau(L)^+ = \tau(M) = \tau(R)^*.$$

PROOF: $\tau(L)^+ = \tau(M)$ follows directly from Lemma 2.5. $\tau(M) = \tau(R)^*$ follows from the right duals of the results in this section. ∎

REMARK: Let M be an arbitrary Γ-ring with left and right operator rings L and R, respectively.

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(1) The equality $\tau(L)^+ = \tau(R)^*$ does not hold in general. For example, let U and V be, respectively, finite and infinite dimensional vector spaces over the same field F. Let $M = \mathcal{L}(U, V)$ and $\Gamma = \mathcal{L}(V, U)$. Then M is a Γ-ring with the operations of pointwise addition and composition of mappings. Let L and R denote the left and right operator rings of M. It may be shown that $L \cong \mathcal{L}(U, U)$, while R is isomorphic to the ring of finite rank operators on V. Since F is usp, L is usp (see [**11**], Lemma 9). Hence $\tau(L) = 0$. But R is not strongly prime (see [**13**], p.81), and hence not usp. Since R is a simple ring, $\tau(R) = R$. Hence $\tau(L) = 0^+ = 0$ and $\tau(R)^* = M$.

(2) In ([**3**], Proposition 2.4) it is shown that, if $\mathcal{R}$ is an N-radical class of rings in the sense of Sands [**14**], then $\mathcal{R}(L)^+ = \mathcal{R}(R)^*$. The above example shows that τ is not an N-radical in the variety of rings.

LEMMA 2.7. ([**5**], Lemma 1.4). *Let R be a ring. Then a subset P of R is a prime ideal of R if and only if P is a prime ideal of R considered as Γ-ring with $\Gamma = R$.*

LEMMA 2.8. *Let R be a ring. Then a subset P of R is a usp ideal of the ring R if and only if P is a usp ideal of R considered as a Γ-ring with $\Gamma = R$.*

PROOF: Let P be a usp ideal of R. If $P = R$, clearly P is a usp ideal of the Γ-ring R. So suppose $P \neq R$. Let F be a finite subset of R such that $xFy \subseteq P$ implies $x \in P$ or $y \in P$, for all $x, y \in R$.

Suppose $xF^3y \subseteq P$, and $x \notin P$. Then $F^2y \subseteq P$, which implies $y \in P$ or $F \subseteq P$. If $F \subseteq P$, $uFv \subseteq P$ for all $u, v \in R$, which is impossible, since $P \neq R$. Hence $y \in P$, and so P is a usp ideal of the Γ-ring R.

Let Q be a usp ideal of the Γ-ring R. Then, by Lemma 2.7 Q is an ideal of the ring R. Let F, G be finite subsets of R such that $xFGFy \subseteq Q$ implies $x \in Q$ or $y \in Q$, for all $x, y \in Q$. Let $H = FGF = \{fgf' \colon f, f' \in F, g \in G\}$. Then H is finite, and $xHy \subseteq P$ implies that $x \in P$ or $y \in P$, for all $x, y \in P$. Hence P is a usp ideal of R, and the proof is complete. ∎

THEOREM 2.9. *Let R be a ring and let $\tau(R)$, $\tau'(R)$ denote, respectively, the usp radical of the ring R and the usp radical of R considered as a Γ-ring with $\Gamma = R$. Then $\tau(R) = \tau'(R)$.*

PROOF: This follows directly from the definitions of $\tau(R)$, $\tau'(R)$ and Lemma 2.8. ∎

3. MATRIX GAMMA RINGS

Let M be a Γ-ring, and let m, n be positive integers. Denote by M_{mn} and Γ_{nm} the sets of $m \times n$ matrices with entries from M and $n \times m$ matrices with entries from Γ, respectively.

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Let $(x_{ij}), (y_{ij}) \in M_{mn}$ and $(\gamma_{ij}) \in \Gamma_{nm}$. We define $(z_{ij}) = (x_{ij})(\gamma_{ij})(y_{ij})$, where $z_{ij} = \sum_p \sum_q x_{ip}\gamma_{pq}y_{qj}$.

Then M_{mn} is a Γ_{nm}-ring with respect to matrix addition and the operation defined above. If $x \in M$, the notation xE_{pq} will be used to denote a matrix in M_{mn} with x in the p-th row and q-th column, and zeros in all other positions. The notation γE_{pq}, where $\gamma \in \Gamma$, will have a similar meaning. If $A \subseteq M$, A_{mn} will denote the set of $m \times n$ matrices with entries from A. If $\phi \subseteq \Gamma$, ϕ_{nm} is similarly defined.

Theorem 3.1. *M is a usp Γ-ring if and only if M_{mn} is a usp Γ_{nm}-ring.*

Proof: Suppose M is a usp Γ-ring. Let (F, Δ) be an insulator for M. Put

$$G = (F \cup \{0\})_{mn},$$
$$\phi = (\Delta \cup \{0\})_{nm}.$$

Suppose now that $(x_{ij}), (y_{ij})$ are nonzero elements of M_{mn}. We will show that $(x_{ij})\phi G \phi(y_{ij}) \neq 0$. Let x_{pq}, y_{st} be nonzero entries from $(x_{ij}), (y_{ij})$, respectively. Then there exist $f \in F$ and $\gamma, \mu \in \Delta$ such that $x_{pq}\gamma f \mu y_{st} \neq 0$. Consider the product $(x_{ij})(\gamma E_{q1})(fE_{11})(\mu E_{1s})(y_{ij})$. The element in the p-th row and q-th column in this product is $x_{pq}\gamma f \mu y_{st}$. It follows that (G, ϕ) is the required insulator for M_{mn}.

Conversely, suppose that M_{mn} is usp. Let (F, Δ) be the insulator for M_{mn}. Let G be the set of those elements of M which are entries from some matrix in F, and let ϕ be the set of those elements of Γ which are entries from some matrix in Δ. Suppose $0 \neq x, y \in M$. Then there exist $(\gamma_{ij}), (\mu_{ij}) \in \Delta$, $f_{ij} \in F$ such that $(xE_{11})(\gamma_{ij})(f_{ij})(\mu_{ij})(yE_{11}) \neq 0$. Clearly this implies that the entry in the first row and first column of the above product is nonzero. But this entry is $x\gamma_{11}f_{11}\mu_{11}y$. It follows that (G, ϕ) is the required insulator for M. ∎

Lemma 3.2. ([10], Theorem 2). *Let M be a Γ-ring, and let m, n be positive integers. Then a subset Q of M_{mn} is a prime ideal of M_{mn} if and only if $Q = P_{mn}$, for some prime ideal P of M.*

Lemma 3.3. ([8], Lemma 4). *Let M be a Γ-ring and let $I \lhd M$. Then $(M/I)_{mn}$ is isomorphic to M_{mn}/I_{mn}, for all positive integers m and n.*

Theorem 3.4. *Let M be a Γ-ring, and let m, n be positive integers. Then $\tau(M_{mn}) = (\tau(M))_{mn}$.*

Proof: Let P be a usp ideal of M. Then M/P is a usp Γ-ring, whence $M_{mn}/P_{mn} \cong (M/P)_{mn}$ is a usp $\Gamma_{nm} - ring$, by Theorem 3.1 and Lemma 3.3. Consequently, P_{mn} is a usp ideal of M_{mn}. Suppose Q is a usp ideal of M_{mn}. Then Q is a prime ideal of M_{mn}, whence $Q = P_{mn}$ for some prime ideal P of M, by Lemma 3.2.

Hence $M_{mn}/Q = M_{mn}/P_{mn} \cong (M/P)_{mn}$. But M_{mn}/Q is a usp Γ_{nm}-ring, whence M/P is a usp-Γ-ring by Theorem 3.1. Hence P is a usp ideal of M.

We have shown that a subset Q of M_{mn} is a usp ideal of M_{mn} if and only if $Q = P_{mn}$ for some usp ideal P of M. The result now follows directly from the definition of τ. ∎

4. Special radicals

Following Heyman and Roos [**7**], a class $\mathcal{M}$ of Γ-rings is called a special class:

(i) $\mathcal{M}$ consists of prime Γ-rings.
(ii) $\mathcal{M}$ is hereditary, that is $M \in \mathcal{M}$, and $A \lhd M$ implies $A \in \mathcal{M}$.
(iii) $\mathcal{M}$ is essentially closed, that is $\mathcal{M}$ is a Γ-ring, $A \lhd \cdot M$, and $A \in \mathcal{M}$, implies $M \in \mathcal{M}$.

If $\mathcal{R}$ is a radical class of Γ-rings, and $\mathcal{M}$ is a special class such that for any Γ-ring M, $\mathcal{R}(M) = \cap\{A \lhd M : M/A \in \mathcal{M}\}$, then $\mathcal{R}$ is the upper radical determined by the class $\mathcal{M}$, and is called a special radical. The general radical theory of Γ-rings closely parallels that the associative rings. For details, we refer to [**3**].

LEMMA 4.1. *Let M be a Γ-ring and $I \lhd M$. If P is a usp ideal of M, then $P \cap I$ is a usp ideal of I.*

PROOF: Let (F, Δ) be the insulator of P in M. It is easy to show that if $a \in I \backslash P$ is a fixed element, then (F_1, Δ) with $F_1 = F\Delta a \Delta F$ is a insulator for $I \cap P$ in I. ∎

THEOREM 4.2. *The class $\mathcal{M}$ of all usp Γ-rings is a special class and hence τ is a special radical.*

PROOF:

(i) Clearly, every element of $\mathcal{M}$ is prime.
(ii) $\mathcal{M}$ is hereditary follows from Lemma 4.1.
(iii) Let $A \lhd \cdot M$ with $A \in \mathcal{M}$. Since prime Γ-rings are essentially closed, we have from ([**5**], Lemma 2.2), that $l(A) = r(A) = 0$. Let (F, Δ) be the insulator of A. For every $0 \neq a, b \in M$, there exists $0 \neq x_1, x_2 \in A$ and $0 \neq \alpha_1, \alpha_2 \in \Gamma$ such that $x_1\alpha_1 a \neq 0$ and $b\alpha_2 x_2 \neq 0$. Since $x\alpha_1 a$ and $b\alpha_2 x_2$ are nonzero elements of A we have $x\alpha_1 a \Delta F \Delta b \alpha_2 x_2 \neq 0$. Whence $a\Delta F \Delta b \neq 0$. Therefore, $M \in \mathcal{M}$ with insulator (F, Δ). ∎

If M is a Γ-ring, then M is called $us(1)$ prime it has an insulator of the form $(\{x\}, \{\gamma\})$ where $x \in M$ and $\gamma \in \Gamma$.

As in Theorem 4.2 we can show that the class $\mathcal{M}_1$ of all $us(1)$ prime Γ-rings is a special class. The upper radical determined by this class will be denoted τ_1. Clearly, for any Γ-ring M, $\tau(M) \subseteq \tau_1(M)$. In [**12**] a ring R is defined to be $us(1)$ prime if

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R has an insulator consisting of a single element. The $us(1)$ radical of R, $\tau_1(R)$, is the upper radical determined by the class of $us(1)$ prime rings, which is shown in [**12**] to be special. Using reasoning similar to that employed in the proof of Lemma 2.8 and Theorem 2.9, we can prove:

THEOREM 4.3. *Let R be a ring and let $\tau_1(R)$, $\tau_1'(R)$ denote, respectively, the $us(1)$-prime radicals of the ring R and of R considered as a Γ-ring with $\Gamma = R$. Then $\tau_1(R) = \tau_1'(R)$.*

REMARK: For rings, usp does not, in general, imply $us(1)$-prime. For example, let F be a field. It is trivial that F is $us(1)$-prime (choose $f = \{1\}$). By the ring analogy of Theorem 3.1, the ring F_n of $n \times n$ matrices with entries from F is usp. However, if $n \geqslant 2$, F_n is not $us(1)$-prime. Let f be any matrix in F_n. Suppose that $0 \neq a \in F_n$ is a singular matrix. Then af is singular, whence there exists $0 \neq b \in F_n$ such that $afb = 0$. Since F_n is a simple ring this implies that $\tau(F_n) = 0$ while $\tau_1(F_n) = F_n$. In view of Theorems 2.9 and 4.3, this implies that for a Γ-ring M, the equality $\tau(M) = \tau_1(M)$ does not hold in general.

The following radicals, inter alia, have been introduced for a Γ-ring M: Jacobson $\mathcal{J}(M)$ [**6**], Brown-McCoy $\mathcal{B}(M)$ [**1**], superprime $\sigma(M)$ [**5**], Levitzki $\mathrm{L}(M)$ [**6**], nil $\mathcal{N}(M)$ [**6**], strongly prime $\mathcal{S}(M)$ [**4**]. We refer to these papers for the definitions and properties of the radicals.

It is known ([**11**], Theorem 19) in the ring case that τ is independent of both the Jacobson and Brown-McCoy radicals. In view of Theorem 2.9 and its analogies for the Jacobson and Brown-McCoy radicals ([**6**], Theorem 10.1 and [**1**], Theorem 5.1 respectively), the same is true in the Γ-ring case. It follows directly from the definitions that $\mathcal{S}(M) \subseteq \tau(M) \subseteq \tau_1(M)$. In ([**4**], Corollary 3.4), it is shown that $\mathrm{L}(M) \subseteq \mathcal{S}(M)$.

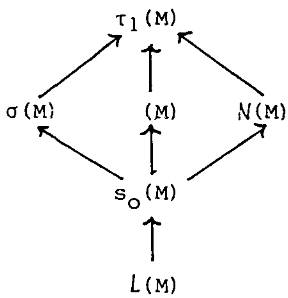
Recall [**5**] that a Γ-ring M is called right-superprime if for every nonzero ideal I of M there exists $x \in I$, $\alpha \in \Gamma$ such that if $y \in M$, $x\alpha y = 0$ implies $y = 0$. The superprime radical σ is now the upper radical determined by the class of all superprime Γ-rings.

M is called a nil Γ-ring if for all $x \in M$, $\gamma \in \Gamma$ there exists a positive integer n such that $(x\gamma)^n x = x\gamma x \ldots \gamma x = 0$. The nil radical $\mathcal{N}(M)$ of an arbitrary Γ-ring M is the sum of all the nil ideals of M.

THEOREM 4.4. *If M is any Γ-ring, then $\sigma(M) \subseteq \tau_1(N)$ and $\mathcal{N}(M) \subseteq \tau_1(N)$.*

PROOF: Let M be a $us(1)$ prime Γ-ring with insulator $(\{f\}, \{\gamma\})$ where $f \in M$ and $\gamma \in F$. Let A be any nonzero ideal of M. If $0 \neq a \in A$, then $a\gamma f \in A$ and if $b \in M$, $a\gamma f\gamma b = 0$ implies $b = 0$. Hence M is superprime and, therefore, $\sigma(M) \subseteq \tau_1(M)$. Let $M \in \mathcal{N}$, that is $\mathcal{N}(M) = M$. If $M \notin \tau_1$, then there exists a homomorphic image, M', of M which is $us(1)$-prime. Since M is nil, M' is also a nil Γ-ring. Let $(\{f\}, \{\gamma\})$ be the insulator of M'. Since $f \in M'$ we can find a positive integer n such that $(f\gamma)^n f = 0$ and $(f\gamma)^{n-1} f \neq 0$. Clearly $[(f\gamma^{n-1})f]\gamma f\gamma[(f\gamma)^{n-1}]f = 0$ which contradicts the choice of F as insulator. Whence $M \in \tau_1$ and, therefore, $\mathcal{N}(M) \subseteq \tau_1(M)$. ∎

The diagram below summarises the relationships between the radicals discussed in the paper. All inclusions are sharp, and radicals not linked are not comparable.



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Dr. G.L. Booth
Department of Mathematics
University of Zululand
Private Bag X1001
3886 Kwadlangezwa
Republic of South Africa

Dr. N.J. Groenwald
Department of Mathematics
University of Port Elizabeth
P.O. Box 1600
6000 Port Elizabeth
Republic of South Africa