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EXISTENCE THEOREMS FOR NONLINEAR BOUNDARY VALUE PROBLEMS

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1. Introduction. Let $C(I)$ denote the linear space of continuous functions from the compact interval $I = [a, b]$ into n-dimensional real arithmetic space $\mathbf{R}_n$, and let $C'(I)$ be the subspace of continuously differentiable functions on I. A general boundary value problem for a first-order system of n ordinary differential equations on I is given by

$$(1.1) \qquad x' + F(x, t) = 0, \qquad f(x) = 0.$$

We assume that f is a mapping from a subset of $C(I)$ into $\mathbf{R}_m$, where m and n are not necessarily equal. Problem (1.1), referred to as a *nonlinear boundary value problem*, includes as special cases such familiar boundary conditions as two-point and multi-point conditions. It also encompasses less common problems such as those involving integral conditions and conditions at an infinite set of points in I. This paper deals with the establishment of a constructive technique for proving the existence of solutions for such problems.

The nonlinear boundary value problem will first be reformulated as an operator equation between normed linear spaces. Banach's contraction mapping principle will then be used to obtain the desired existence criteria for solutions. The resulting theory is very extensive in that it applies to nonlinear differential equations subject to broad classes of both linear and nonlinear boundary conditions. The conditions for existence are correspondingly general in nature. For specific classes of problems and particular examples, conditions which are far more precise than those given here can readily be formulated. As an example we will consider a special case of (1.1) in which the boundary operator involves the values of a solution at a countably infinite subset of I.

2. An equivalent problem. Boundary value problem (1.1) will now be formulated as a functional analytic operator equation. We begin by making $C(I)$ into a Banach space with the uniform norm defined by

$$||x|| = \max_{t \in I} ||x(t)||, \quad x \in C(I).$$

The space $C'(I)$ will be treated as a normed linear subspace of $C(I)$ with the same norm. We shall also need to consider the product space $Y = C(I) \times \mathbf{R}_m$, which is a Banach space under the norm

$$||[\psi, v]|| = \max \{||\psi||, ||v||\}, \qquad [\psi, v] \in Y.$$

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In any normed linear space the open ball with centre at x and radius r will be denoted by $S(x, r)$ and its closure by $\bar{S}(x, r)$. The identity mapping on a linear space will be denoted by E.

We assume that the function F in (1.1) is continuously differentiable on $U \times I$ where U is an open subset of $\mathbf{R}_n$. The domain of the operator f is assumed to be an open subset D of $C(I)$, and we shall require that $x(t) \in U$, $t \in I$, for every choice of $x \in D$. Hence we can define an operator $T : D \to C(I)$ by

$$T(x)(t) = F(x(t), t), \qquad a \leqq t \leqq b.$$

Let A be an $n \times n$ matrix with continuous entries on $I = [a, b]$, and let L be a linear operator from $C(I)$ into $\mathbf{R}_m$. Define the linear operator M from $C'(I)$ into Y by

$$(2.1) \qquad Mx = [x' + Ax, Lx].$$

Let M^+ be any operator from Y into $C'(I)$ such that $MM^+ = E$, the identity mapping on Y. We are led to the following basic result, in which A and L are assumed to have the properties just described.

THEOREM 1. *Let R and S be the operators on D given by*

$$R(x) = Ax - T(x), \qquad S(x) = Lx - f(x).$$

Assume the operator M defined in (2.1) *has a right inverse M^+ and define the operator $Q : D \to C'(I)$ by*

$$(2.2) \qquad Q(x) = M^+[R(x), S(x)].$$

If x^* is a fixed point of Q, then x^* is a solution of the boundary value problem (1.1).

Proof. Let x^* be a fixed point of Q. Thus $Q(x^*) = x^*$ and we have that $x^* \in D \cap C'(I)$. Hence

$$x^* = M^+[R(x^*), S(x^*)]$$

and applying M to both sides, we obtain

$$Mx^* = [R(x^*), S(x^*)].$$

Hence, using (2.1), we have

$$(2.3) \qquad [x^{*\prime} + T(x^*), f(x^*)] = 0.$$

From the definition of the operations on Y and the form of (2.3) it follows immediately that x^* is a solution of (1.1). This completes the proof.

Theorem 1 makes it possible to find solutions of nonlinear boundary value problems by seeking fixed points for operators of the form (2.2). In particular we are interested in using contraction mapping methods to solve $Q(x) = x$. We shall apply Banach's well-known contraction mapping principle in the following form (for a proof, see [**3**, pp. 65-66]).

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THEOREM 2. *Let D be an open subset of a Banach space X. Suppose P is an operator from D into X. For $x_0 \in D$ assume there exist constants $\eta > 0$ and $\alpha \in [0, 1)$ such that*

(i) $||x_0 - P(x_0)|| \leqq \eta$;

(ii) $||P(x) - P(y)|| \leqq \alpha||x - y||$ *for all* $x, y \in \bar{S}(x_0, r_0)$

where

$$r_0 = \frac{\eta}{1 - \alpha}, \quad \bar{S}(x_0, r_0) \subset D.$$

Then the contraction mapping sequence $\{x_n\}$ for P starting at x_0, namely

$$x_{n+1} = P(x_n), \qquad n = 0, 1, 2, \ldots,$$

is defined, remains in $\bar{S}(x_0, r_0)$, and converges to a fixed point x^* of P in $\bar{S}(x_0, r_0)$. The rate of convergence is given by

$$||x^* - x_n|| \leqq \alpha^n r_0, \qquad n = 0, 1, 2, \ldots .$$

Remark. An operator that satisfies (ii) for $\alpha \in [0, 1)$ is called a *contraction mapping* on the set $\bar{S}(x_0, r_0)$.

3. The existence theorem. The following lemma deals with the existence of right inverses for the operator M.

LEMMA. *Suppose $M : C'(I) \to Y$ is defined as in* (2.1). *Let Φ be a fundamental matrix on I for*

$$x' + Ax = 0$$

and define the linear operator $N : \mathbf{R}_n \to \mathbf{R}_m$ by $N\xi = L(\Phi\xi)$, $\xi \in \mathbf{R}_n$. Let B be an $m \times n$ matrix representation of N and assume there exists an $n \times m$ matrix B^+ such that $BB^+ = E_m$ (the $m \times m$ identity matrix). *Then for any $x_0 \in C'(I)$ there exists an operator $M^+ : Y \to C'(I)$ such that both*

$$MM^+ = E \tag{3.1}$$

and

$$(M^+M)(x_0) = x_0. \tag{3.2}$$

Proof. Right inverses for M can be calculated by dealing with linear boundary value problems of the form

$$x' + Ax = \psi \tag{3.3}$$

$$Lx = v \tag{3.4}$$

where $[\psi, v] \in Y$. If we let $Hx = x' + Ax$, it follows that the inverse image of any $\psi \in C(I)$ under H is the set of solutions of (3.3) and is represented by the following linear variety of $C'(I)$:

$$\left\{ \int_a^t \Phi(t)\Phi^{-1}(s)\psi(s)ds \right\} + \mathcal{N}(H),$$

where $\mathcal{N}(H)$ denotes the null space of H. Thus H maps $C'(I)$ onto $C(I)$ and equation (3.3) has solutions for every choice of $\psi \in C(I)$. Furthermore the null space of H is isomorphic to $\mathbf{R}_n$ under the isomorphism defined by

$$(3.5) \qquad \xi \leftrightarrow \Phi\xi, \qquad \xi \in \mathbf{R}_n.$$

Because the operator H is onto, it has right inverses, and one such right inverse is given by

$$(H^+\psi)(t) = \int_a^t \Phi(t)\Phi^{-1}(s)\psi(s)ds, \quad a \leqq t \leqq b.$$

Hence, using (3.5), we see that all solutions of (3.3) can be represented by

$$(3.6) \qquad x = \Phi\xi + H^+\psi, \qquad \xi \in \mathbf{R}_n.$$

Therefore x will be a solution of (3.3), (3.4) if $\xi \in \mathbf{R}_n$ is a solution of

$$(3.7) \qquad B\xi = v - LH^+\psi.$$

By assumption B has a right inverse B^+, and thus

$$\xi = \xi_0 + B^+(v - LH^+\psi)$$

is a solution of (3.7) for each $\xi_0 \in \mathcal{N}(N)$. By (3.6) it follows that for every such ξ_0 an operator M^+ satisfying (3.1) is given by

$$M^+[\psi, v] = \Phi(B^+v + \xi_0 - B^+LH^+\psi) + H^+\psi.$$

We now seek a right inverse which also satisfies (3.2). Using (2.1) and an integration by parts, we obtain

$$(M^+M)(x_0) = \Phi\xi_0 + x_0 - \Phi\Phi^{-1}(a)x_0(a) + \Phi B^+B\Phi^{-1}(a)x_0(a).$$

Therefore if we choose

$$\xi_0 = \Phi^{-1}(a)x_0(a) - B^+B\Phi^{-1}(a)x_0(a),$$

it follows that $B\xi_0 = 0$ and (3.2) holds. This establishes the lemma.

It is necessary that certain relationships hold among the norms introduced on the various spaces of matrices in the problem. To be specific, let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be respectively the linear spaces of $j \times k$, $k \times l$, and $j \times l$ real matrices with corresponding norms $||\cdot||_1$, $||\cdot||_2$, and $||\cdot||_3$. Then the norms are said to be *compatible* if for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$ we have $||AB||_3 \leqq ||A||_1||B||_2$. If we make the natural identification of elements of $\mathbf{R}_n$ with $n \times 1$ matrices, then this notion of compatibility is a generalization of the concept defined in [**2**, p. 427]. We require that the arithmetic spaces $\mathbf{R}_n$ and $\mathbf{R}_m$ be given norms which are compatible with the norms introduced on the other spaces of matrices in the forthcoming development.

Since the function F in (1.1) is assumed to be continuously differentiable on $U \times I$, it follows that the operator T is continuously Fréchet differentiable on D. For any $x_0 \in D$, the value of $T'(x_0)$ at $x \in C(I)$ can be represented as

$$(T'(x_0)x)(t) = G(t)x(t), \qquad a \leqq t \leqq b,$$

where G is an $n \times n$ matrix of continuous functions and the indicated multiplication is ordinary multiplication of a matrix by a vector (see [**3**, p. 95]).

We are now ready to establish the main result concerning the existence of solutions for (1.1).

THEOREM 3. *Suppose D is an open subset of $C(I)$ and let $D' = D \cap C'(I)$. Assume the operator f is continuously Fréchet differentiable on D. For $x_0 \in D'$ define the linear operator $M : C'(I) \to Y$ by*

$$Mx = [x' + T'(x_0)x, f'(x_0)x],$$

and let Φ be a fundamental matrix on I for

$$x' + T'(x_0)x = 0.$$

Define the linear operator N on $\mathbf{R}_n$ by $N\xi = f'(x_0)(\Phi\xi)$, and let B be an $m \times n$ matrix representation of N. Assume that

(i) *B has a right inverse B^+.*

Letting

$$\xi_0 = \Phi^{-1}(a)x_0(a) - B^+B\Phi^{-1}(a)x_0(a), \tag{3.8}$$

define the operator $M^+ : Y \to C'(I)$ by

$$M^+[\psi, v] = H_1^+[\psi, v] + \Phi\xi_0$$

for all $[\psi, v] \in Y$ where

$$H_1^+[\psi, v] = \Phi B^+v - \Phi B^+f'(x_0)H^+\psi + H^+\psi,$$

$$(H^+\psi)(t) = \int_a^t \Phi(t)\Phi^{-1}(s)\psi(s)ds, \quad a \leqq t \leqq b.$$

Further suppose there exist non-negative constants α, β, K such that

(ii) $||[x_0' + T(x_0), f(x_0)]|| \leqq \alpha$;

(iii) $K_1||B^+|| + K_1K_2(b - a)(1 + K_1||B^+|| \, ||f'(x_0)||) \leqq \beta$

where

$$K_1 = \max_{t \in I} ||\Phi(t)||, \quad K_2 = \max_{t \in I} ||\Phi^{-1}(t)||;$$

(iv) $||[T'(x_0) - T'(x), f'(x_0) - f'(x)]|| \leqq K$ *for all* $x \in \bar{S}(x_0, r_0)$

where

$$r_0 = \frac{\alpha\beta}{1 - \beta K}, \quad \bar{S}(x_0, r_0) \subset D, \quad \beta K < 1.$$

Conclusion. The contraction mapping sequence $\{x_n\}$ *for*

$$Q(x) = M^+[T'(x_0)x - T(x), f'(x_0)x - f(x)]$$

starting at x_0, *namely*

$$x_{n+1} = Q(x_n), \qquad n = 0, 1, 2, \ldots,$$

is defined, remains in $\bar{S}(x_0, r_0)$, *and converges to a solution* $x^* \in \bar{S}(x_0, r_0)$ *of the boundary value problem* (1.1). *The rate of convergence is given by*

$$||x^* - x_n|| \leqq \frac{(\alpha\beta)(\beta K)^n}{1 - \beta K}, \quad n = 0, 1, 2, \ldots .$$

Proof. By an argument strictly analogous to that given in the lemma, it follows that hypothesis (i) and the definitions of Φ, ξ_0, B^+, and H_1^+ are sufficient to guarantee that M^+ is a right inverse for M. Thus by Theorem 1 it suffices to show that the operator Q defined on D satisfies the hypotheses of Theorem 2.

We first consider the operator H_1^+, which is clearly linear from Y into $C'(I)$. To show that H_1^+ is bounded, consider

$$||H_1^+|| = \sup_{[\psi,v]\in Y, [\psi,v]\neq[0,0]} \left\{\frac{||H_1^+[\psi, v]||}{||[\psi, v]||}\right\}.$$

Using the boundedness of $f'(x_0)$ and the compatibility of the norms, we obtain for each fixed $t \in I$ that

$$||H_1^+[\psi,v](t)|| \leqq K_1||B^+|| \, ||v|| + (b - a)K_1^2K_2||B^+|| \, ||f'(x_0)|| \, ||\psi|| + K_1K_2(b - a)||\psi||.$$

Therefore by hypothesis (iii) it follows that

(3.9) $\quad ||H_1^+|| \leqq \beta.$

We now show that Q is a contraction mapping on $\bar{S}(x_0, r_0)$. Denote $T'(x_0)$ by A and $f'(x_0)$ by L. Since T and f are continuously Fréchet differentiable on D, we have for any $x, y \in \bar{S}(x_0, r_0)$ that

$$||Q(x) - Q(y)|| = ||H_1^+[A(x - y) - \int_0^1 T'(\theta x + (1 - \theta)y)(x - y)d\theta, \; L(x - y) - \int_0^1 f'(\theta x + (1 - \theta)y)(x - y)d\theta]||.$$

Using inequality (3.9) and the boundedness of T' and f' at each point of $\bar{S}(x_0, r_0)$, we obtain

$$||Q(x) - Q(y)|| \leqq \beta \sup_{z\in\bar{S}(x_0,r_0)} ||[A - T'(z), L - f'(z)]|| \, ||x - y||.$$

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Therefore by hypothesis (iv) it follows that

$$||Q(x) - Q(y)|| \leqq \beta K||x - y||$$

for all $x, y \in \bar{S}(x_0, r_0)$. Since $\beta K < 1$ by assumption, Q is a contraction mapping on $\bar{S}(x_0, r_0)$.

From the definition of Q we obtain

$$||x_0 - Q(x_0)|| = ||x_0 - M^+[Ax_0 - T(x_0), Lx_0 - f(x_0)]||.$$

By (3.8) and the lemma we see that the operator M^+ satisfies $(M^+M)(x_0) = x_0$. Thus

$$\begin{aligned} ||x_0 - Q(x_0)|| &= ||(M^+M)(x_0) - M^+[Ax_0 - T(x_0), Lx_0 - f(x_0)]|| \\ &= ||H_1^+[x_0' + T(x_0), f(x_0)]|| \\ &\leqq ||H_1^+|| \, ||[x_0' + T(x_0), f(x_0)]||. \end{aligned} \tag{3.10}$$

Hence by (3.9) and hypothesis (ii), $||x_0 - Q(x_0)|| \leqq \alpha\beta$. Since $\alpha\beta/(1 - \beta K) = r_0$ by assumption, it follows that the hypotheses of Theorem 2 are satisfied and so an appeal to Theorem 1 finishes the proof.

4. Applications. The proof of Theorem 3 yields useful information concerning the application of the above technique in specific cases. It is essential to have a practical means for obtaining initial approximations to fixed points of the operator Q. Such initial guesses would be almost impossible to obtain directly since the form of Q itself depends upon the specific choice of initial point. However, from (3.10) we see that in general a "good" initial approximation to a solution of the nonlinear boundary value problem (1.1) will correspond to a suitable first guess for a fixed point of Q. The only restriction is that the norm of the operator H_1^+ must be reasonably small. Since it is quite natural and easy to work directly with problem (1.1), this procedure provides a practical means for obtaining a starting point for the iterations and for defining precisely the iteration function Q.

As with all local iterative techniques, the quality of the initial approximation x_0 is the key factor which governs whether or not this method can be applied to a particular boundary value problem. The choice of initial point determines the value of the number β in hypothesis (iii) since, by (3.9), β is a bound for the operator H_1^+. The value for β obtained from the calculation indicated in hypothesis (iii) is often only a crude bound for the operator. In specific examples an alternative approach is to calculate H_1^+ explicitly and then obtain the operator norm by a direct computation. Although such a procedure generally yields a smaller value for the operator norm, the expression for β in Theorem 3 has the advantage of being independent of any special properties of a particular ordinary differential equation or boundary operator.

The value for β then determines a range of permissible values for the constant K in hypothesis (iv). The final step in verifying the assumptions is the calculation of a suitable r_0 so that one of these values for K is a bound for the operator norm in (iv). As noted before, the quality of the initial approximation largely determines whether the technique can be applied in a specific case. In particular, since the operator

$$x \to [T'(x), f'(x)] \tag{4.1}$$

is continuous at x_0, a sufficiently small value for α in hypothesis (ii) will guarantee that a suitable r_0 can be found. The properties of the operators T' and f' are also important factors in the success or failure of the technique in applications. If, for instance, the operator (4.1) is approximately constant, then it can be expected that K will be small for a large range of values for r_0. In this case it is most likely that the technique can be applied for moderately large values of β with less restrictive conditions on the quality of the initial approximation.

To illustrate these points, we shall treat a problem which has been touched upon in linear form (see [**1**, pp. 148-149]) but for which no general nonlinear theory exists. Consider the nonlinear boundary value problem given by

$$\frac{dx}{dt} + F(x, t) = 0, \quad \sum_{i=1}^{\infty} B_i x(\tau_i) = c. \tag{4.2}$$

We assume that $F(x, t)$ has the properties outlined in Section 2 and that $c \in \mathbf{R}^m$, $\tau_i \in I$ for $i = 1, 2, \ldots$, and $\{B_i\}$ is a sequence of real $m \times n$ matrices such that

$$\sum_{i=1}^{\infty} ||B_i|| < \infty.$$

In this case we have

$$B = \sum_{i=1}^{\infty} B_i \Phi(\tau_i)$$

and $f'(x) = f'(x_0)$ for every $x \in D$. Hence, for the given initial approximation x_0, the calculation of r_0 and K in hypothesis (iv) depends only upon the operator T:

$$||[T'(x_0) - T'(x)]|| \leqq K \quad \text{for all } x \in \bar{S}(x_0, r_0).$$

If the nonlinearities in the operator T are such that T' is almost constant, then in general the method will yield the existence of a solution for boundary value problem (4.2) with only moderate restrictions needed on the quality of the initial guess.

We now deal with a specific example from the class of problems described by (4.2). The following boundary value problem on $I = [0, \frac{1}{2}]$ is chosen to illustrate the wide range of nonlinearities that can be treated by using the technique of Theorem 3:

$$(4.3) \qquad \ddot{x} + \log\,(20 + \dot{x}^2) + \frac{1}{(32)^2 + (x - \frac{1}{2})^2} = 3 - \sin^3(t^3),$$

$$(4.4) \qquad \sum_{n=1}^{\infty} \frac{1}{2^n}\, x(\tau_n) = \frac{1}{2}, \quad \tau_n = \frac{1}{3^n}, \quad n = 1, 2, \ldots .$$

The linear space $\mathbf{R}_3$ is given the norm

$$||v|| = \max\{|v_1|, |v_2|, |v_3|\}$$

and the linear space $\mathscr{A}$ of 1×3 real matrices is normed by

$$||A|| = |a_1| + |a_2| + |a_3|.$$

We choose

$$x_0 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

as the initial approximation, which yields $\alpha = 0.0033$ as a suitable value in hypothesis (ii). If we choose Φ to be the principal matrix solution of $x' + T'(x_0)x = 0$ on I, then the matrix B of hypothesis (i) is given by

$$B = (1\ 1/5\ 1/34).$$

Clearly B has right inverses, and we choose

$$B^+ = \begin{pmatrix} 170/209 \\ 170/209 \\ 170/209 \end{pmatrix}.$$

By calculating H_1^+ explicitly in this case, we find that $\beta = 3.00$ satisfies the requirement of hypothesis (iii). Another direct computation shows that

$$||[T'(x_0) - T'(x), f'(x_0) - f'(x)]|| < 0.224$$

for any choice of r_0. Since $K = 0.224$ yields $\beta K < 1$, the conditions of Theorem 3 are satisfied. Hence we can conclude that the nonlinear boundary value problem (4.3), (4.4) has a solution x^* which lies in $\bar{S}(x_0, 0.03)$ and is the limit of the contraction mapping sequence defined in Theorem 3.

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