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INTRODUCTION

Studies on celestial mechanics are greatly stimulated by the launchings of artificial celestial bodies. The present trend of the investigations is mainly directed towards the following five categories.

1. Adaptation of the classical perturbation theories to the numerical works for use of modern electronic computers, either in the method of rectangular co-ordinates or in Hansen's theory.

2. Improvement of the numerical theory of the motion of the major planets in the degree of accuracy attained both in observations and in numerical computations, principally by Clemence and his colleagues of the Naval Observatory in Washington. Radar echo observations of the planets are urging the revision of the astronomical constants.

3. Determination of the coefficients in the zonal and tesseral harmonic expansion of the Earth's gravitational potential based on the observations of artificial Earth satellites.

4. Extension of the classical analytical theories to the orbits of finite eccentricity and inclination. Von Zeipel's analytical theory has been proved to be effective for lunar theory, asteroidal theory and artificial satellite theory. Brouwer's work on the explanation of the gaps in the asteroidal distribution is worth mentioning. Recent development in stellar dynamics might throw light on the solution of the classical three-body problem.

5. Cultivations of mathematical theories on non-linear mechanics by Krylov, Bogoliubov and Mitropolsky, and of the existence-proof of periodic solutions and the stability problem by Diliberto and Siegel and their colleagues.

Progress in celestial mechanics has been promoted by several symposia, the IUTAM Symposium on the Dynamics of Satellites held in Paris in May 1962, the First International Symposium on Analytical Astrodynamics held in Los Angeles in June 1961, the International Symposium on Space Age Astronomy held in Pasadena in August 1961, the Symposium for the Use of Artificial Satellites for Geodesy held in Washington in April 1962, and the IAU Symposium no. 21, the System of the Astronomical Constants, held in Paris in May 1963. Above all the Summer Institute for Dynamical Astronomy held in summer each year in the U.S.A. directed by Brouwer is contributing to the progress of celestial mechanics in an unlimited degree. An IAU Symposium no. 25 on the Theory of Orbits in the Solar System and in Stellar Systems will be held in Thessaloniki in August 1964.

In May 1961 a National Commission on Theoretical Astronomy, affiliated to the Astronomical Council of the U.S.S.R. Academy of Sciences, was set up: Chairman, Corresponding Member of the U.S.S.R. Academy of Sciences M. F. Subbotin, Vice-Chairmen, Prof. G. N. Duboshin and Prof. G. A. Chebotarev; Secretary Dr E. A. Grebenikov.

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The Commission on Theoretical Astronomy has convened a scientific conference in general and applied problems of celestial mechanics, which was held on 20–25 November, 1961 in Moscow. Some foreign astronomers (Y. Kozai, J. Kovalevsky, K. Schmidt, E. Tengstrem and others) attended the Conference and took part in its proceedings.

Publications in book-form appeared during this tri-annual period are:

A. D. Dubiago, *The Determination of Orbits*, translated by R. D. Burke, G. Fordon, L. N. Rowell, F. T. Smith, The Rand Corporation, Macmillan, New York, 1961.

H. F. Chilmi, *Qualitative Methoden beim n-Körperproblem der Himmelsmechanik*, Heft 10 der Schriftenreihe der Institut für Mathematik, Akademische Verlag, Berlin, 1961.

J. M. A. Danby, *Fundamentals of Celestial Mechanics*, Macmillan, New York, 1962.

J. Meffroy, *La Figure d'Equilibre des Corps en Rotation*, Astronomie, Encyclopédie de la Pleiade, Gallimard, Paris, 1962.

Space Age Astronomy, ed. by A. J. Deutsch and W. B. Klemperer, Academic Press, New York, 1962.

Planets and Satellites, Vol. III of *The Solar System*, ed. by G. P. Kuiper and B. M. Middlehurst, Univ. of Chicago Press, 1961.

The Moon, Meteorites and Comets, Vol. IV of *The Solar System*, 1963.

S. W. McCusky, *Introduction to Celestial Mechanics*, Addison-Wesley, Reading, Mass., 1963.

The following books are forthcoming.

J. Kovalevsky, *Introduction à la Mécanique Céleste*, A. Colin, Paris, 1963.

K. Stumpff, *Himmelsmechanik*, Bd. II, Deutscher Verlag d. Wiss., Berlin.

Y. Hagihara, *Celestial Mechanics*, Vol. I, Blaisdell, New York.

,, , *Theories of Equilibrium Figures of a Rotating Homogeneous Fluid Mass*, Blaisdell, New York.

G. N. Duboshin has set out to write a number of textbooks on celestial mechanics. The first of these series 'Theory of Attraction', was published in 1961. The second book, 'Celestial Mechanics. Principal Problems and Methods', appeared in 1963. Late in 1960 'A Book of Tables and Nomograms for Processing Data obtained in AES Observations' appeared, compiled by I. D. Zhongolovich and V. M. Amelin. Mention should also be made of the second edition of the interesting popular science book, 'The Movement of Celestial Bodies' by Y. A. Riabov, 1962.

TWO-BODY PROBLEM

A method convenient for electronic computers for calculating undisturbed ephemerides in the position and velocity vectors of a planet or a comet with given initial values is presented by Stumpff (1). The usual Keplerian orbital elements need not be known. It can be applied without formal change to all types of orbits, elliptic or parabolic or hyperbolic. In place of Kepler's equation a transcendental so-called main equation appears which is valid for all types of orbits. For the convenience of computation Sundman's regularizing variable is used. The formulae for the determination of the six orbital elements, convenient for artificial satellites, are given by Kustaanheimo (2) according to the method of Stumpff. The classical theorems of Lambert, Euler and Encke, or the corresponding theorems of Stumpff, are formulated on the identity relating the major axis, the time interval, the sum of the corresponding two radius vectors, and the corresponding chord, or the vector sum of the corresponding two velocity vectors (3).

A method for computing the osculating elements of an artificial satellite has been proposed by Kovalevsky and Barlier (4). The method, originally due to Danjon (5) and Kovalevsky (6), is along the line of modified Laplace's. It consists in improving the development of the coordinates and then the Keplerian elements by variations.

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Instead of using Gauss's transcendental equation in the orbit determination Sconzo (**7**) had recourse to an elegant and analytical method due to Charlier. Lagrange's power series f and g are expressed by means of $p = \frac{1}{r^2}\frac{dr^2}{dt}$ and $q = \frac{1}{r^2}\frac{d^2r^2}{dt^2}$ in convergent manner. When p and q are known, the orbital elements can be obtained in the way of Laplace. The use of Lambert's theorem (**8**), once adopted by Subbotin, is discussed and applied to the orbit determination of an artificial Earth satellite.

Explicit expressions for the 36 partial derivatives of position and velocity, in an elliptical orbit with respect to the initial position and velocity, of which only 18 are independent, are derived by Sconzo (**9**). They are applied in formulating the equations of condition for the problem of differential orbit correction from Doppler frequency-shift measurements.

Martin (**10**) is continuing his research on the motion of two bodies with variable mass. The properties of conics with a single focus fixed are studied and are used for deducing the osculating conics in the motion of two bodies with variable mass. Formulae are obtained for the variation of the orbit as functions of the rate of the mass variation. Martin proposes a law for the mass decrease for applying to the evolution theory of a double star system.

Kozai (**11**) noticed the difference of the mean values with respect to various anomalies in Keplerian motion.

Kiang (**12**) made a detailed study on the effect of resisting medium on an elliptical orbit.

Gontkovskaya (**13**) has considered the problem of orbit determination from two heliocentric positions by solving a system of non-linear integral equations after Bucerius, with the estimation of the errors. The problems of the existence and uniqueness of the solutions in determining an orbit from two positions have been considered by Eliasberg (**14**). The Gauss method of determining an orbit from three observations has been adapted for the case of an artificial Earth satellite by Bazhenov (**15**). A method of improving the orbital elements of artificial Earth satellites from topocentric distances and radial velocities has been worked out by Batrakov (**16**). Brumberg (**17**) has considered the problem of constructing an orbit for the condition of random distribution of the initial positions, the initial velocities and the masses.

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DISTURBING FUNCTION

Meffroy (**1**) computed the periodic part of the disturbing function to find new secular terms of the third order in the expansion for the semi-major axis, in connection with Poisson's theorem on the invariability of the major axes of the planetary orbits.

Heinrich (**2**) writes that he has succeeded in eliminating the reciprocal distance of planets from the right-hand sides of the equations of motion by the operation, which he has previously discovered. The operation leads to an integral equation for a simple linear coupling of the major axes, which can be solved without the intervention of a small divisor. He has in mind to apply the theory to the motion of the Moon, the Trojans, Gauss's elliptic ring, and some of the characteristic asteroids.

Izsak (**3**) has published tables for the Laplace coefficients and their Newcomb derivatives. Izsak, Barnett, Efimba and Gerard (**4**) worked on the construction of Newcomb operators on a digital computer.

Mulholland (**5**) invented a method of computing the Laplace coefficients on electronic computers. He transformed the infinite series for the Laplace coefficients into forms better suited for computation by means of Gamma functions.

Kaula (**6**) and Musen, Bailie, Upton (**7**) published the analytical expansion of the lunar and solar disturbing functions for use in the theory of a close Earth satellite along the line of Kozai (**8**), Groves (**9**) and Kaula (**10**). The expansion will be naturally of use for the motion of a natural satellite with arbitrary inclination and eccentricity, when the ratio of the mean distances of the disturbed and the disturbing bodies is small enough. Kaula's (**6**) development is advantageous when it is desired to conserve computer storage space or to include the luni-solar perturbations in the same computation with perturbations due to tesseral harmonic terms of the Earth's potential. A quasi-potential for the solar radiation pressure effect for use in the equations of motion is also written in terms of the Keplerian elements.

Elenevskaya (**11**) has obtained a development of the disturbing function for an eccentricity approaching unity, by expanding in powers of $(1 - e)$.

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PLANETARY THEORY

For facilitating the ephemerides computation by means of an electronic computer the method of Strömgren has been improved by using a set of elements other than the Keplerian, since indeterminacy comes in for an orbit with vanishingly small eccentricity or inclination. Garafalo (1), Pines (2), Cohen and Hubbard (3), and Herget (4) discussed this point. A new method published by Herget (4) is greatly enlightening the works of the Central Bureau of the International Co-operation at Cincinnati in the ephemerides computation of asteroids. Instead of the usual concept of the elliptic elements M_0, n, a, e, ω, Ω and I the following are dealt with: $\mathbf{c} = p^{\frac{1}{2}}\mathbf{R}$, $X = e \cos \omega$, $Y = e \sin \omega$ and U_0, where U_0 is the sum $M + \omega$ for $t = 0$ and the vector $\mathbf{R}$ is directed to the normal of the orbital plane. Strömgren, Kaula (5), Musen (6) and Popovič (7) went farther in referring to matrices and dyadics.

Danby (8) used a kind of matrix notation for computing special perturbations and the matrizant of Keplerian motion in matrix method in the calculation and analysis of orbits, especially for Hill-Brouwer's method of computing the perturbation in rectangular co-ordinates. The solution is obtained directly in the form of the complementary function and the particular integral.

Musen and Carpenter (9) developed a new theory of general planetary perturbations in rectangular co-ordinates. Their theory has certain characteristics common with Hill's theory. They decomposed the perturbations along $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{R}$, in contrast to Hill's decomposition along $\mathbf{r}$, $\mathbf{R} \times \mathbf{r}$ and $\mathbf{R}$. This decomposition leads to a direct method of integration and the final formulae are in a convenient form for programming for electronic computers. Further the six integration constants are determined in a direct manner, in contrast to the seven constants appearing in Hill's double and triple integrals. The final expressions are obtained in the form of trigonometric series with the number of arguments equal to three times the number of planets in the problem. The potential is expanded in terms of multipoles, which facilitates the computation of the perturbations of any order. For the computation of the perturbations the components of the disturbing force are expanded into trigonometric series by means of numerical double harmonic analysis.

Musen (10) modified Strömgren's method for including the effects of higher orders. Although Strömgren's method has a mathematical elegance by using the vectorial elements, it has a disadvantage of taking into account only the first order perturbations, because Strömgren obtained the Gibbs rotation vector indicating the integrated value of the angular velocity of rotation of the osculating ellipse only in the first approximation. Musen obtained an accurate form for the rotation matrix in terms of the Gibbs vector and gave the differential equation for the perturbation of this vector.

Tables for the method of the perturbation of elements have been published by Merton (11). The tables contain the quantities required to facilitate the calculation of special perturbations in the orbits of comets by the rigorous method proposed by Merton (11) himself, in which the mean anomaly is taken as the independent variable.

P. Stumpff, Schubart and others are working, according to a letter from Schubart, on a program for obtaining the motion of up to ten bodies in a planetary system with high precision to be provided for the computer at Heidelberg. Simultaneous integration of the equations of motion with a constant step-length is used, although the step-length can be changed without stopping the computer (compare Commission 20).

Sehnal (13) applied Gauss's method for computing the secular variation to the computation of the secular perturbation of the quadrantid meteoric swarm, and then modified it (14) by expanding the disturbing function around its value for two circular orbits.

The long-range effects caused by the Moon and the Sun are of primary importance for proving the stability of highly eccentric orbits of an Earth satellite and for obtaining its lifetime (15). Musen (16) applied Halphen's method of computing the secular variation based on Gauss's idea. The method is also used by Musen (17) in computing the long-range or the secular effects in the motion of asteroids when there is no sharp commensurability between

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the mean motions of the asteroid and Jupiter. The disturbing function is averaged over the short-periods by Gauss's concept. Musen describes the method by referring to vectors, matrices and dyadics. The properties of the elliptic functions of Weierstrass and of the hypergeometric functions of Gauss are fully utilized. The formulae are more suitable for electronic computers than the usual analytical expressions for computing the long-period perturbations.

Clemence (**18**) applied Hansen's theory modified by Hill with further improvements and corrections for facilitating the use of a computing machine with very high accuracy to the motion of Mars. The long-period inequalities appear with arguments, among others, l (Venus) $-8\,l$ (Mars), $15\,l$ (Mars) $-8\,l$ (Earth), l (Mars) $-6\,l$ (Jupiter).

Clemence (**19**) remedied the discrepancy between the theory and observations in the longitudes by giving 1/3499·7 to the mass of Saturn.

The comparison of Clemence's new theory of the motion of Mars with observations extending from 1750 to 1960 to determine the definitive values of the constants has been continued by Duncombe. A preliminary solution of the equations of condition indicates a provisional value for the reciprocal of the mass of Venus of 408 945 $\pm$ 470.

Rectangular co-ordinates of Mars referred to the mean equinox and equator of 1950·0 with an interval of 4 days for the period 1800 – 1950 have been completed on the new theory. They are based on provisional elements and are consistent with the co-ordinates from 1950 to 2000, published in U.S. Naval Observatory Circular no. 90, 1960.

Clemence has completed the first-order portion of a new general theory of the heliocentric motion of the Earth and is working on higher order contributions.

Morando is studying a general theory on the motion of Vesta without secular or mixed secular terms.

Roemer (**20**) and Brandt (**21**) discussed the residuals of the acceleration in the motion of periodic comets in connection with the interaction of comet tails with the interplanetary medium.

In order to draw geophysical and geodetic conclusions from the motion of artificial satellites we need an accurate theory which permits easy inclusion of any gravitational term and which is suitable for machine computation.

Musen (**22**) has given Hansen's lunar theory in a form which permits a purely numerical treatment of the perturbations on artificial satellites. However, Musen for the purpose of numerical computation refers rather to Hansen's planetary theory and uses the fictitious mean anomaly instead of Hansen's W function, and sets up the process of iteration in a convenient form. After the process of iteration is completed, the function is formed and the perturbations in the mean anomaly and in the radius vector are determined. The orbit of the disturbing body is supposed to be a moving ellipse. The orbital plane of the disturbing body is taken as the basic reference plane according to Hansen. However, Musen (**23**) uses instead of the latitude four parameters in order to make all basic arguments linear in time from the outset, for applying to satellite orbits with high inclination. Bailie and Bryan (**24**) computed the osculating elements from this modified Hansen's theory, based on matrix transformation, for the motion of an artificial satellite.

The tendency of celestial mechanics of including the eccentricity in the disturbing function without expanding it in powers of the eccentricity as in the classical theory is one of the recent progresses never thought of in the past years due to the small eccentricities of the natural heavenly bodies. The discussion of the motion of asteroids with high eccentricity and inclination by Kozai (**25**) is one of the outcomes of this nature. Secular perturbations of an asteroid with high values of inclination and eccentricity have been studied by Kozai by referring to von Zeipel's theory. Since the conventional technique for developing the disturbing function cannot be adopted, Kozai expanded it in powers of the ratio of the semi-major axes of the asteroid

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and Jupiter. Short-period terms depending on the two mean anomalies are eliminated from the disturbing function by von Zeipel's transformation.

Hori is now engaged in the computation of the secular variations of periodic comets by avoiding the usual expansion in powers of eccentricities, inclinations and the ratio of the major-axes.

It should be mentioned that Jeffreys (**26**) pointed out the similarity of von Zeipel's theory and Brown's theory for eliminating all short-period terms simultaneously from the disturbing function.

Miachin (**27**) has presented a general technique for estimating an error in numerical methods of integrating the differential equations of celestial mechanics, with specific reference to the methods of Cowell and of Runge. Kulikov (**28**) has worked out a procedure of integrating the equations of motion in celestial mechanics by using electronic computers and Cowell's quadrature method with automatic pitch selection.

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CHARACTERISTIC ASTEROIDS

The long-period inequalities in the Keplerian elements of the characteristic asteroid Hilda

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due to the perturbation of Jupiter and Saturn for 273 years from 15 January 1875, are computed and the long-range behaviour is discussed by Akiyama (1), and those of the asteroid Thule due to the perturbation of Jupiter moving on a fixed ellipse for 400 years by Takenouchi. (2).

According to a letter from Schubart (3) he is working on the motion of asteroids with mean motions nearly commensurable with that of Jupiter. Following Poincaré's idea on the motion of the Hecuba group of asteroids the influence of the secular and the critical terms of the disturbing function is studied. These terms are obtained by numerically averaging the disturbing function of the restricted three-body problem in two or three dimensions with the aid of an IBM 7094 computer.

Message wrote me that he was engaged in the study of nearly commensurable mean motions in the restricted problem of three bodies, including some numerical investigation for the case $n/n' = 2/1$, with consideration of the effect of the long-period librations on the distribution of asteroids over mean motion near that case.

Wilkens wrote me that he was still working on the same problem.

Hagihara (4), by noticing that the mean motions of the existing asteroids belonging to the Hilda and the Thule groups are almost in commensurable ratios with that of Jupiter, at the same time as the several pairs of the satellites in the Saturnian system, while there are gaps corresponding to the ratios 2/1, 3/1, 5/2, 7/3 with large numbers of asteroids nearby, as well as gaps in the Saturnian ring, tentatively proposed a supposition that the large amplitudes of librations in the motion of the Hecuba group asteroids among others may be due to the disturbing effect of the neighbouring asteroids in the same group occasionally passing close by, as Brouwer mentioned in his paper on asteroidal families.

Rabe (5) announced his result of computation on the motion of the asteroid Griqua which has passed across the Hecuba gap quickly. Wilkens's computation has shown that the motion of an ideal asteroid in the exact commensurability-point is in libration around the exact commensurability of the mean motions. Similar circumstance may be thought of as to the Hilda and Thule group asteroids. Yet there exist asteroids in almost exact commensurability-points corresponding to the ratios 3/2 and 4/3.

Moser (6) proved the instability of the motion of characteristic asteroids for $p - q = 1, 2, 3, 4$, where $n/n' = p/q$, on purely gravitational ground.

Klose (7), by referring to the Jacobi integral, saw that the perturbation of the major axis is larger when the eccentricity increases than when it decreases. The distribution of major axes shows maxima at the commensurabilities of the types 2/1, 5/2 and 3/1. He plotted the distribution of the Jacobi constant and saw that the maxima in the distribution of major axes occurred by the drifting to smaller values of the major axes from the commensurability-points, accompanied by larger values of the eccentricity.

Brouwer's recent work (8) on the explanation of the gaps in the mean motions of asteroids is worth admiring. By von Zeipel's transformations he obtained a set of canonical equations in new variables that differed from the original by the sums of periodic terms with at least the first power of μ, the disturbing mass, as a factor. The new Hamiltonian is expanded in powers of μ. In the transformation process, divisors of the form $(j_1 + j_2 + j_3)n - j_1$ appear, where $n = 1/L^3$ and j_1, j_2, j_3, j_4 are integers. He distinguished two cases. (i) The ordinary case in which divisors of this form and small enough to endanger the convergence of the process do not appear. Two first integrals are obtained and the problem is reduced to that of two degrees of freedom. (ii) The commensurability case in which n is close to a commensurability $(p + q)/p$, where p and q are relatively prime positive integers. Then terms with arguments satisfying $(j_1 + j_2 + j_3)(p + q) = j_1 p$ will appear, which prevent the elimination of each term by a further transformation. The two integrals coalesce into a single integral in this case, but the problem is of three degrees of freedom. One of the integrals in the ordinary case is a substitute for the Jacobi integral.

Now suppose that the distribution of asteroids considered as a function of the Jacobi constant has no singularity, in the same way as Klose's. Brouwer found that the distribution function for L^*, which is the transform of L, is flat in the ordinary case, but becomes v-shaped in the commensurability case.

In the neighbourhood of a commensurability of a lower order, that is, in the ratio of two small integers, there exist an infinite number of commensurabilities in the ratios of very large integers, each with the v-shaped distribution functions. The higher order commensurabilities have adequate room for the v-shaped distribution in the vicinity of the 2/1 commensurability and also for commensurabilities such as 3/1, 5/2, 7/3, all with larger mean motions than 2/1. Brouwer says that for the 3/2 and even more for the 4/3 commensurability little room is available free from interference by other commensurabilities. This would, according to Brouwer, indicate the reason why the asteroids in this part of the belt seem to seek the commensurability region rather than avoid them. Brouwer thinks this conjecture is confirmed by the fact that the asteroids in the region between 440″ and 575″ mean motion show a distribution that is closely correlated with the distribution of the commensurabilities of comparatively lower orders. It can be understood that in the latter types of commensurabilities the v-shaped distribution may become almost flat due to the superposition or overlapping of the v-shaped distribution corresponding to the various commensurabilities of higher orders nearby. But the question is left unanswered why the asteroids in this part of the belt seem to seek the commensurability region. There is also the question of the relative density in the distribution of the commensurabilities of higher orders in the neighbourhood of the two types of the lower commensurabilities.

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TROJAN ASTEROIDS

Rabe (1) carried out the actual determination on electric computers of periodic libration orbits about the equilateral triangular equilibrium points of the restricted problem and their stabilities. He at first describes the general method developed for finding such periodic orbits and then deals with orbits of various dimensions including one of the horse-shoe shaped in the natural Trojan case (1) and, together with Schanzle, in the case of the Earth-Moon restricted problem (3). Rabe (2) obtained the Fourier series representation of all orbits for asteroids of the Trojan group by numerical harmonic analysis with an electronic computer. The convergence of the Fourier expansions is very satisfactory, according to Rabe, up to amplitudes of the order of those of the actual Trojan asteroids with the largest libration amplitudes. All orbits computed appear to be stable, even the horse-shoe shaped periodic orbit enclosing both the equilateral triangular equilibrium points L_4 and L_5 and the collinear equilibrium point L_3 opposite to Jupiter from the Sun. This latter class of orbit, anticipated by Brown, is of particular interest, because with their increasing amplitude but decreasing periods these orbits link the equilibrium

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points L_4 and L_5 with the satellite region of Jupiter. This settles the question as to the existence of the horse-shoe shaped orbits in Brown's conjecture. Similar computation has been carried out by Colombo, Lautman and Munford (4) in the elliptical problem by taking Jupiter's eccentricity into account, with application to the Earth-Moon system in mind.

Rabe later began to use the periodic solutions in the form of the Fourier series obtained by the harmonic analysis of the periodic orbits previously computed as reference or intermediary orbits for the analytical representation of non-periodic librational motions in the restricted as well as in the elliptical restricted problem with Jupiter's orbital eccentricity. This is facilitated in the Trojan cases by the availability of a sufficiently dense net of periodic solutions, so that we can interpret for any desired periodic reference orbit. In the elliptical problem the reference or the intermediary orbit is a suitable combination of a periodic solution of the restricted problem with a short-period scale factor corresponding to Jupiter's variable distance from the Sun. Here, of course, the resulting intermediary orbit is already non-periodic in nature, but is a useful basis for the description of librational motions on which Jupiter tends to impress its own elliptical terms. The results are of interest not only as far as the non-periodic motions of a librational nature are concerned, but also because they afford a discussion of the stability of the periodic reference orbit to a higher order than the first-order stability results from Hill's equation, which neglects the second and higher order terms.

Stumpff (5) extended Thüring's theory in 1929–31 on the mathematical treatment of the librational motion for approximating the periodic orbits for all amplitudes. Thüring's solution provides a starting point for an exact theory of the plane long-period Trojan orbits according to the method of the variation of constants. Special attention is devoted to the border line case in Brown's conjecture on the horse-shoe shaped orbit.

Deprit of the Louvain University is working at Cincinnati on the continuation of the horse-shoe shaped periodic Trojan orbits toward even larger amplitudes for establishing the end of this family as the immediate vicinity of Jupiter and of the collinear equilibrium points L_1 and L_2 is approached. For this purpose Deprit has to use regularizing variables on his computer program. Brouche (6) of the Louvain University computed 392 periodic orbits on an electronic computer in the Earth-Moon case in a systematic survey.

Danby (8) discusses the stability of the triangular points in the elliptic restricted three-body problem. Message's work on the librations of the Trojan asteroids will be published shortly in the lecture note of the Summer Institute on Dynamical Astronomy at Cornell in 1963. Sehnal (9) discussed the stability of Liapounov of an Earth satellite at the equilateral triangular point with the Moon and the Earth.

Jeffreys (7) argued against the supposition of Kuiper, that the Trojans had been satellites of Jupiter before Jupiter lost its mass to the present value, on the basis of the nature of L_4 and L_5. He says that, if Jupiter had lost mass, then the amplitude of the libration of the satellite should have increased. Further, Jeffreys attributes the secular acceleration of the natural satellites Jupiter V, Mars I, Saturn I to the interaction between satellites either to viscous or turbulent drag, the formation of shock waves or accretion, after showing that the mutual influence of satellites through tidal friction after Darwin is insufficient.

On the other hand Rabe (10) showed that, as far as the crowding of the Trojans in the neighbourhood of the centres is concerned, if no mass decrease of Jupiter is considered, there is no theoretical explanation for the preference of orbits with small amplitude and with small Jacobi constant. The result of Rabe's investigation as well as empirical fact, according to Rabe, gives considerable support to the suggested origin of the Trojans and the mass loss of Jupiter. Huang (11) advocates that a Trojan can escape from the Jovian system even without the mass decrease of Jupiter.

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LUNAR THEORY

Hori (**1**) developed Brouwer's project for the lunar theory on the basis of von Zeipel's method. As a first attempt he neglected the orbital inclination and obtained the solution in powers of m, but in a closed form with respect to the eccentricity. He computed as far as the order m^4 for the periodic terms and as far as m^5 for the secular terms. He has just completed the computation in closed forms with respect to both the eccentricity, the inclination and the solar eccentricity as far as the order m^4.

Stumpff's lunar theory (**2**) is based on the use of his so-called invariants and the main equation in his theory of the two-body problem. He considers the differential equations for Hill's variational orbit in his lunar theory in two dimensions, and obtains a relation of the form $f(\dddot{r}, \ddot{r}, \dot{r}, r; C) = 0$. The equation can be integrated numerically by iteration if the Moon's orbit is considered to be a disturbed Keplerian ellipse with any value of the eccentricity, for a certain fairly extended time interval in the vicinity of the initial epoch. The iteration process is limited to the solution of the transcendental main equation. In the second paper Stumpff expanded the co-ordinates in powers of Hill's m in a manner different from Hill's.

Schubart (**3**) published his work on the extension to three dimensions of the family of periodic orbits of Hill's lunar problem of two dimensions. For representing the third co-ordinate a new variable is introduced. The series obtained are proved to be convergent by the procedure of Siegel.

In 1961, at Berkeley, Eckert reported that Brown's harmonic series for the co-ordinates of the Moon had been substituted into the differential equations and the residuals obtained with precision, and outlined a method for the solution of the variation equations which would overcome the difficulties arising from the small divisors. Eckert wrote me that the method had been developed since and the programmes necessary to generate and solve the variation equations on the 7094 computer had been completed and applied to about 3500 residuals. The results appear to be completely satisfactory according to Eckert. The corrections obtained from the solution are now being applied to the initial series and the corrected series substituted in the differential equations. Eckert plans to repeat the entire process at least once with appropriate variations to estimate the stability of the results. He says that his machine programmes permit a complete solution with little effort compared to that already expanded on the problem.

Van der Waerden (**4**) considered the secular terms and fluctuations in the motions of the Sun and the Moon from two causes. The one is due to the tidal friction and it causes the secular retardation of the motion of the Moon. The other is the irregular rotation of the surface of the Earth due to random currents in the interior of the Earth.

Kovalevsky (**5**) is continuing his work on the theory of motion of the eighth satellite of Jupiter, according to the principle of successive approximations, but is obliged to wait until a more powerful computer can be utilized. He is also working on the long-period terms in the

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motion of the eighth to the twelfth satellites of Jupiter. Mello is engaged in the theoretical study on the interaction of three satellites of Jupiter.

Kondurar (6) has considered the motion of a satellite in Hill's problem of three bodies around a planet having a spheroidal shape.

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ARTIFICIAL SATELLITES

Apart from the theories of the motion of an artificial satellite by King-Hele, Brouwer, Garfinkel, Kozai and others, Vinti (1) has succeeded in integrating the equations of motion of

$$U = -\frac{\mu}{r}\left[1 - \sum_{n=1}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\sin\beta)\right]$$

by means of Hamilton-Jacobi's method in the separation of variables. According to this idea Izsak (2) has considered as his intermediary orbit for the motion of an artificial satellite the motion in the gravitational potential of the Earth by taking into account the coefficients of the zonal harmonics as far as the second order with respect to J_2. The ingenious assumption which Vinti made leads to the separation of variables if $J_{2n} = (-1)^{n+1} J_2^n$, $J_{2n+1} = (-1)^n J_1 J_2^n$. Observations show that the error $J_4 + J_2^2 \sim -10^{-6}$. Since J_1 is practically zero, the potential

$$U = -\frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty} (-J_2)^n \left(\frac{R}{r}\right)^{2n} P_{2n}(\sin\beta)\right]$$

is a good approximation. If oblate-spheroidal co-ordinates are used, then the Hamilton-Jacobi equation of the problem is of Stäckel's type. The solution can be expressed in elliptic integrals. By referring to the theory of elliptic functions with a linear fractional transformation in the complex domain Izsak expressed the solution in Fourier series and the integration constants as funtions of the Keplerian elements.

Vinti (3) tried to simplify Izsak's process of solution and expressed it in terms of the complete elliptic integrals by avoiding the use of complicated elliptic functions, that is, by obtaining the solution as the sum of integrals containing quartics which are evaluated by convergent Fourier series, and then transformed by uniformizing with the eccentric anomaly. Vinti (4) deduced the mean motion in such a conditionally periodic separable system. Then he (5) removed the restriction on the orbital inclination $I_c < I < 180° - I_c$, where I_c might be as large as 1° 54′ for an orbit sufficiently close to the Earth. Many of the formulae for the periodic terms are simplified when the orbit is equatorial or almost equatorial.

Bonavito (6) announces that Vinti's solution goes very fast on the IBM 7090. Shapiro (7) discussed the prediction of satellite orbits.

Callander (8) shows the motion of an artificial close satellite to be quasi-periodic by obtaining the bounds of the variation of the radial distance and the variation of the inclination. Poritsky (9) has shown that the satellite orbit is forever confined to the interior of a certain toroidal region by judging from an integral analogous to the energy integral obtained after a reduction of the differential equations.

Barrar (**10**) estimated to be of the order J_2^3 the difference between the two procedures: to average the potentials and solve the problem or to solve the problem for the given potential and average the answers.

Hori (**11**) studied the hyperbolic case of the motion of artificial satellites. He extended Delaunay's variables to this hyperbolic case and transformed the canonical equations by von Zeipel's theory.

Popovič (**12**) wrote me that he had given the disturbing force of an ellipsoid of revolution with coefficients J_2 and J_4. Zagar (**13**) describes the general method of perturbation of the orbit of an artificial satellite by the spheroidal figure of the Earth. He has organized a working group for the mathematical study of the trajectories of artificial celestial bodies and for the associated physical and geophysical problems. The orbit perturbations of an artificial satellite caused by the Earth's gravitational force and the influence of high atmosphere have been studied, together with the method of calculation on an IBM electronic computer for the orbit problem. Kovalevsky (**14**) discussed the general method for computing the effects of more general origin, including non-sphericity of the Earth potential, atmospheric drag and radiation pressure, and recommended von Zeipel's method but without attention to the convergence of the solution.

Sconzo (**15**) published the differential orbit correction and ephemerides tables for the motion of an artificial Earth satellite, based on Brouwer's theory and the drag effect on empirical procedure. Using the best available observational data, such as by Baker-Nunn cameras, several sets of orbital mean elements at different epochs are obtained. They are expressed as empirical functions of time by means of a general least square fitting procedure. Then the values of the orbital elements at any assigned epoch and then the ephemerides are computed.

Since very accurate observations on the motion of artificial satellites are now available, Petty and Breakwell (**16**) and Struble (**17**) derived the second-order periodic perturbations as functions of the true longitude. Kozai (**18**) went farther and discussed the second-order periodic perturbations with third-order secular perturbations in satellite motions derived by von Zeipel's method. The potential of the Earth is developed into a series of zonal spherical harmonics by assuming that J_2 is a small quantity of the first order, J_3 and J_4 are of the second order, and J_5, J_6, J_7, and J_8 are of the third order. The final expressions of the short-period perturbations are given in the radius, the argument of latitude, the inclination, and the longitude of the ascending node. Izsak (**19**) transformed the variables from Delaunay's to the canonical system ($\dot{r}$, G, H; r, u, h), where u is the argument of latitude, for computing the short-period perturbation of the co-ordinates.

Kovalevsky (**20**) described various analytical methods for computing the coefficients of the expansion of the Earth potential. Most of the results are inconsistent, because the differences are due to the imperfections in the definition of the integration constants, especially the so-called mean elements. He proposes a mathematical definition of the constants, and some principle which should be followed in any complete study of the orbit in order to compare the theory with observations. He prefers the osculating elements to the mean elements for their determination by observations.

Garfinkel (**21**) describes a historical survey of the theory of the motion of an artificial satellite. He wrote me that he had built up an improved theory of the motion (**22**).

The coefficients of zonal and tesseral spherical harmonics of the Earth's gravitational potential have been derived from the minitrack and field-reduced Baker-Nunn observations of artificial satellites by Kozai (**23, 31**) Izsak (**24, 33**), Kaula (**25, 26**), King-Hele (**27, 32**), Smith (**28**),

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Newton *et al* (**29**). The mathematics has been worked out by Kozai (**30**) both for the zonal and the tesseral coefficients. The tesseral and sectorial harmonics cannot cause secular perturbations of the first order in any orbital element of the satellite, but they can cause periodic perturbations of two kinds: short-period perturbations of which the arguments depend on the mean anomaly and the long-period perturbations of which the periods are nearly integral fractions of a day. The amplitudes of the long-period perturbations are usually almost ten times larger than those of the short-period.

The up-to-date determinations are listed in Table 1 for the zonal and in Table 2 for the

Table 1. Coefficients of zonal spherical harmonics

Kozai, 1962 (**31**)	King-Hele, Cook, Rees, 1963 (**32**)
$J_2 = +1082{\cdot}48 \times 10^{-6}$	$J_2 = +1082{\cdot}78 \times 10^{-6}$
$J_4 = -1{\cdot}84 \times 10^{-6}$	$J_4 = -0{\cdot}78 \times 10^{-6}$
$J_6 = +0{\cdot}39 \times 10^{-6}$	$J_6 = +0{\cdot}70 \times 10^{-6}$
$J_8 = -0{\cdot}02 \times 10^{-6}$	$J_8 = +0{\cdot}24 \times 10^{-6}$
	$J_{10} = -0{\cdot}50 \times 10^{-6}$
	$J_{12} = +0{\cdot}28 \times 10^{-6}$
$J_3 = -2{\cdot}562 \times 10^{-6}$	$J_3 = -2{\cdot}44 \times 10^{-6}$
$J_5 = -0{\cdot}064 \times 10^{-6}$	$J_5 = 0$
$J_7 = -0{\cdot}470 \times 10^{-6}$	$J_7 = -0{\cdot}45 \times 10^{-6}$
$J_9 = +0{\cdot}117 \times 10^{-6}$	

Table 2. Coefficients of tesseral spherical harmonics

Izsak, 1963 (**33**)		Kaula, 1963 (**26b**)	
		$\Delta\bar{C}_{00} = -2{\cdot}46$	$\Delta\bar{C}_{20} = -0{\cdot}03$
$C_{22} = +9{\cdot}68 \times 10^{-7}$	$S_{22} = -4{\cdot}00 \times 10^{-7}$	$\bar{C}_{22} = +1{\cdot}88$	$\bar{S}_{22} = -1{\cdot}38$
		$\bar{C}_{30} = +0{\cdot}97$	
$C_{31} = +1{\cdot}12 \times 10^{-6}$	$S_{31} = +6{\cdot}16 \times 10^{-8}$	$\bar{C}_{31} = +1{\cdot}52$	$\bar{S}_{31} = +0{\cdot}14$
$C_{32} = +9{\cdot}12 \times 10^{-8}$	$S_{32} = -1{\cdot}83 \times 10^{-7}$	$\bar{C}_{32} = -0{\cdot}02$	$\bar{S}_{32} = +0{\cdot}42$
$C_{33} = +7{\cdot}17 \times 10^{-8}$	$S_{33} = +1{\cdot}24 \times 10^{-7}$	$\bar{C}_{33} = +0{\cdot}70$	$\bar{S}_{33} = +0{\cdot}76$
		$\bar{C}_{40} = +0{\cdot}67$	
$C_{41} = -2{\cdot}88 \times 10^{-7}$	$S_{41} = -3{\cdot}21 \times 10^{-7}$	$\bar{C}_{41} = -0{\cdot}33$	$\bar{S}_{41} = +0{\cdot}37$
$C_{42} = +3{\cdot}51 \times 10^{-8}$	$S_{42} = +1{\cdot}23 \times 10^{-7}$	$\bar{C}_{42} = +0{\cdot}01$	$\bar{S}_{42} = +0{\cdot}35$
$C_{43} = +2{\cdot}15 \times 10^{-8}$	$S_{43} = +1{\cdot}48 \times 10^{-8}$	$\bar{C}_{43} = +0{\cdot}17$	$\bar{S}_{43} = +0{\cdot}41$
$C_{44} = +9{\cdot}72 \times 10^{-9}$	$S_{44} = +1{\cdot}63 \times 10^{-8}$	$\bar{C}_{44} = -0{\cdot}01$	$\bar{S}_{44} = +0{\cdot}18$

tesseral harmonic coefficients. The formula for the zonal harmonic expansion has been given. For the tesseral harmonic expansion, Izsak (**33**) adopts

$$U = \frac{\mu}{r}\left\{1 + \sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n} [C_{nm}\cos m\lambda + S_{nm}\sin m\lambda]\, P_{nm}(\sin\beta)\right\},$$

$$P_{nm}(x) = (1-x^2)^{m/2}\,\frac{d^mP_n(x)}{dx^m}\;,$$

in accordance with the resolution of the IAU Commission 7 at Berkeley in 1961, while Kaula refers to his formula (**26**)

$$R_{nm} = \frac{\mu R}{a^{n+1}} \sqrt{\frac{(n-m)!\,(2n+1)\kappa_m}{(n+m)!}} \sum_{p=0}^{n} F_{nmp}(i) \sum^{\infty} G_{npg}(\rho)$$
$$\times \left[\begin{Bmatrix} \overline{C}_{nm} \\ -\overline{S}_{nm} \end{Bmatrix}_{(n-m)\ odd}^{(n-m)\ even} \cos\{(n-2p)\omega + (n-2p+\epsilon)M + m(\Omega-\theta)\} \right.$$
$$\left. + \begin{Bmatrix} \overline{S}_{nm} \\ \overline{C}_{nm} \end{Bmatrix}_{(n-m)\ odd}^{(n-m)\ even} \sin\{(n-2p)\,\omega + (n-2p+\epsilon)\,M + m(\Omega-\theta)\} \right],$$

where $\kappa_0 = 1$, $\kappa_m = 2$, $m \neq 0$. $\Delta\overline{C}_{00}$, $\Delta\overline{C}_{20}$ mean the corrections to $0{\cdot}3986032 \times 10^{21}$ $(1{\cdot}0 - 0{\cdot}001\,082\,36\,P_2)$ c.g.s. The mean equatorial radius by Kaula is 6 378 196 ± 11 metres, while Kozai's value is $R = 6{\cdot}378\,165 \times 10^6$ metres.

There is scarcely any work published in the theory of rotation of an artificial satellite during its flight along its orbit. Colombo (**34**) analyzed the rotation of Explorer XI around its axis from the point of view of the gravitational and the magnetic torques and computed the period of its tumbling. Hagihara (**35**) computed analytically the effects on the rotation due to the gravitational, gasdynamical and magnetic torques with complicated algebra.

Belorizky determined, in a homogeneous ellipsoid of rotation of small oblateness, the precise parallel where the force of attraction is the same as if the ellipsoid were replaced by a sphere of the same density. The theory is applied to the Hayford ellipsoid currently accepted. At the parallel of 35° 25′ 10″, one finds that the attraction is that of a sphere with the mass of the Earth less 3×10^{-6} of its value.

Kalitzin (**36**) studied Poincaré's limit of the maximum angular velocity of a rotating celestial body when the angular velocity depends on the distance from the rotation axis and examined Roche's model with central condensation. He proved the existence of a unique solution for a generalized problem of a non-uniformly rotating fluid in the case of small angular velocities, by means of an integro-differential equation, which is a generalization of the equation of Clairaut and Liapounov.

The following has been taken from the report of Soviet astronomers.

(*a*) *Motion of AES* (*Artificial Earth Satellite*) *in the Gravitational Field of the Earth.* A precise solution for the problem of the motion of an artificial satellite in the normal gravitational field of the Earth has been obtained by Kislik (**37**). For solving the same question Aksenov, Grebenikov and Demin (**38**) have used a generalized problem of two fixed centres. Zhongolovich (**39**) has presented closed formulae with respect to the eccentricity and inclination of the orbits for AES perturbations. Zhongolovich and Pellinen (**40**) have given a new description of the term 'mean AES elements'. Chebotarev (**41**) has considered the AES motion for the case of small eccentric orbits.

(*b*) *Periodic Orbits of AES.* Volkov (**42**) has proved the existence of the first, second and third sort periodic solutions of Poincaré for the problem of a particle moving in the gravitational field of a flattened planet and its satellite. Orlov (**43**) has proved the existence of almost-circular periodic orbits in the gravitational field of a spheroid for the case of a critical inclination of the orbit relative to the equatorial plane of the spheroid.

(*c*) *Perturbations in AES movement caused by the Moon and the Sun.* Noteworthy results concerning the stability and evolution of orbits of artificial satellites of a planet have been obtained by Lidov (**44**). As a particular numerical example he has considered the orbit evolution of a polar Earth satellite whose semi-major axis and eccentricity are equal, respectively, to the semi-major axis and eccentricity of the Moon. It has been shown that such a satellite would drop to the surface of the Earth in four years' time after only 52 revolutions. Egorova (**45**) has obtained the first order perturbations in AES orbital elements precise to the first order of the Earth's compression, and secular perturbations up to the second order of the compression.

Perturbations by the Sun and the Moon have been made precise to the second order in the ratios of the mean motions of the Moon and Sun to the mean motion of the satellite. Chebotarev and Gontkovskaya (46) have considered in detail the perturbations caused by the Moon and the Sun in the motion of Lunik 3, the third Soviet space rocket.

(*d*) *Translational-Rotational Motion of AES.* Duboshin (47) has considered a general problem of integrating the differential equations for the rotational motion of an artificial celestial body whose inertia-centre is moving along an assigned trajectory. The same question has been considered by Beletski (48). Kondurar (49) has attacked a general problem of the progressive-rotational motion of two absolutely rigid bodies whose ellipsoids of inertia differ from a sphere. For small characteristics, the dynamical compression of the bodies is assumed. He has also discussed (50) the influence of the satellite shape on the motion of the centre of masses around a spherical planet under the assumption that a non-disturbed orbit is either a circle or an ellipse. The satellite is supposed to be a rotating body with dynamical symmetry about the axis of rotation. Magnaradze (51) has considered the translational-rotational motion of a space vehicle relative to the Earth by taking the air resistance into account. The space vehicle has a rod-like shape and a variable mass. Volkov (52) has developed an approximate general solution in the vicinity of the periodic solution for the translational-rotational motion of a satellite in the gravitational field of a sphere. The stability of a rotational motion relative to the various disturbing factors has been studied by Beletski (53). He has also determined the characteristics of the rotation and the orientation of the third Soviet Satellite over an extensive lapse of time.

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Drag effect

Brouwer and Hori (1) have discussed the motion of an artificial satellite through a medium causing drag on the motion by generalizing von Zeipel's method. The atmosphere is assumed to be spherically symmetric at least upward from the perigee height and stationary with respect to the Earth. The law of density is assumed to be isothermal. Hori is now revising this theory.

The theory of Cook *et al.* (2) is limited to nearly circular orbits with a slightly different law for the density.

Izsak (3) computed the periodic drag effect by the method of the variation of constants. Vinti (4) considered the effect of atmospheric drag on the secular variation of orbital inclination following the method of Garfinkel (5). The motion is separated into an initial elliptic stage, a quasi-steady spiral stage, and a final ballistic stage. The secular change is deduced separably for the spiral and the elliptic stages.

There are two representations for the effect of atmospheric drag which differ depending on the initial values of the orbital parameters. Other theories neglect the atmospheric rotation and hence commit errors of several percents. Westerman (6) presented a technique for the method which yields a unique expression for the secular change in each standard element, and computed (7) the life-time of an artificial satellite.

Jacchia (8) analyzed the observed drag effect for deducing the variable atmospheric density and especially the drag during the November 1960 events from the point of view of the solar-terrestrial relationship. Macé pointed out the effect of the atmospheric turbulence.

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Radiation pressure

Mello (1) has established an analytical theory of the motion of an artificial satellite under the action of the solar radiation pressure by taking into account the circumstance that the action is non-effective during the passage of the satellite in the shadow of the Earth. The effect is considered by multiplying the term in the perturbation function due to the radiation pressure by a factor called the shadow function. The expansions obtained are analogous to those in the satellite theory. Mello used Tchebychev's polynomials for the expansion. He concluded the non-existence of the secular terms in the major-axis, eccentricity and inclination, and computed the secular terms in the longitudes of the node and the perigee and the mean anomaly, and also the principal effect of long periods. He is planning to compute the observations of Echo and other satellites by modifying the theory for the case of small eccentricity.

Musen (2) and Kozai (3) independently worked out the effect of the solar radiation pressure on the motion of an artificial Earth satellite and computed the secular effect. Bryant (4) computed the effect by the method of Krylov-Bogoliubov. Sehnal (5) discussed the Poynting-Robertson

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effect on the motion of an artificial satellite. Brouwer (6) discussed analytically the resonance caused by radiation pressure on the motion.

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Critical inclination

Brouwer's theory (1) on the motion of an artificial Earth satellite is based on von Zeipel's method of eliminating all short-period terms by a canonical transformation. Brouwer's elements are transformed by Cain (2) to the mean elements in the ordinary sense. Lyddane and Cohn (3) computed numerically the motion of an artificial Earth satellite by Cowell's integration method for verifying Brouwer's theory. The first-order terms in the semi-major axis are not sufficient. By taking the second-order terms in the semi-major axis they could verify Brouwer's theory satisfactorily. Lyddane (4) referred to Poincaré's canonical variables instead of Delaunay's in order to remedy Brouwer's theory on von Zeipel's method from the difficulty for $e = 0$, $I = 0$. The same difficulty has been dealt with by Smith (5) by ordinary co-ordinate transformation.

The expressions in Brouwer's theory show that the method fails when $1 - 5\cos^2 I = 0$. It corresponds to the inclination $I = 63°26'$, which is called the critical inclination. There have been several hot discussions as regards to the critical inclination whether it is a real existence or it is just an illusion caused by the wrong treatment of the problem. As far as the present method of perturbation theory is concerned, that is, in separating the perturbation into the short-period, the long-period and the secular, the appearance of the critical inclination is essential, although it is tacitly assumed that the perturbation is small enough to be divisible into its parts, then integrated separately and finally summed over the separately integrated results, irrespective of the convergence of the solution.

Hagihara (6) referred to his general theory (7) of libration based on Poincaré-Andoyer's theory on the motion of the Hecuba group asteroids. After carrying out von Zeipel's transformation an integral $F =$ constant is obtained, where F is the new Hamiltonian. A pair of double points of the curve representing this integral corresponds to the critical inclination. It is shown that one of the double points corresponding to the critical inclination is a centre in Poincaré's terminology and is stable, while the other is a saddle point and is unstable. The orbits near the centre are libratory and those near the saddle point are of revolution. The width of the libration in inclination is very narrow. The solution is obtained in elliptic functions in both cases. It is noticed that there appears no critical inclination in Vinti's treatment with the assumption $J_2^2 + J_4 = 0$. Garfinkel (8) obtained similar results by a different method based on von Zeipel's transformation by making Vinti's parameter $J_2^2 + J_4$ explicitly.

Hagihara considered only the terms J_2 and J_4. Kozai (9) extended the discussion to include J_3 and J_5. Aoki (10) included all these terms from the start. By referring to the elliptic functions of Weierstrass he solved the problem completely in each of the different cases arising from the values of the constants in the problem.

Hori (11) expanded the solution in powers of the square root of J_2 in order to solve the problem near the critical point. He obtained the solution in the form of the elliptic integrals of the first and the second kinds.

Izsak (12) noticed that the expansion in powers of $\sqrt{J_2}$ fails when we include higher order terms. He answers the question of how far we have to carry out the approximation so that higher order terms shall no longer change the qualitative feature of the motion. Izsak derived the perturbation equations for the modified action-angle variables of Vinti's dynamical problem and computed the secular and the long-period parts of the Hamiltonian, in order to obtain equations simpler than the usual.

Petty and Breakwell (13), and Struble (14) also discussed the problem of the critical inclination.

Message, Hori and Garfinkel (15) have shown that the solution for the critical case agrees with that for the non-critical case as far as the dominant terms in the immediate vicinity of the critical point. This should be interpreted as the fitting of the two asymptotic solutions.

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24-hour Earth satellite

The asymmetric character of the Earth's equator discovered by Kozai, Izsak, Kaula and others leads to observable effects on the orbits of close Earth satellites. This effect is important for stationary communication satellites, as well as for space telescopes or any observable platform designed to stay fixed at a given geographic longitude on the equator. The motion can be discussed in the same way as for a satellite with critical inclination. The influence of the principal longitude-dependent term of the Earth's potential on the orbit of a 24-hour satellite has been studied by Sehnal (1). He noticed the influence of tesseral harmonics on the long-period perturbation and obtained the periodic shift from the stationary point by considering the action of the Moon, then he discussed the long-period terms produced by the Earth's equatorial ellipticity and the secular terms of inclination due to the action of the Moon by the method of the variation of constants.

Blitzer, Boughton, Kang and Page (2) and Blitzer, Kang and McGuire (3) studied the influence of the principal longitude-dependent term by assuming the orbit to be nearly circular. There is a stable equilibrium point on the minor axis of the equatorial ellipse and an unstable equilibrium point on the major axis, the motion in latitude being simple periodic. Libration occurs of the order 850 days around the stable point. The nearby motion to the unstable point is a revolution.

Musen and Bailie (4, 5) studied the condition of stability even for an orbit with high inclination. They referred to von Zeipel's method after Brouwer, and computed the period and the amplitude of the libration and the mean motion of the revolution.

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Morando (6) considered higher tesseral harmonics R_{mn} determined by Kozai as far as $m, n = 4$. He followed Hori's treatment of the motion of an artificial satellite with critical inclination by applying von Zeipel's method. The equilibrium positions obtained for $I = 0$ are from R_{22}, R_{31}, R_{33}, R_{42}, R_{44}. The equilibrium positions for $I \neq 0$ and e small are obtained for each harmonic R_{22}, R_{33}, R_{44}. Morando (7) also discussed the case of resonance in the form 24 hours $\times$ p/q where p and q are relatively prime integers.

Roy (8) writes me that he has completed a study of the usefulness of interplanetary orbits for probes, the periods of which are commensurable with one year.

Weimer (9) discussed the stability of synchronous orbits of a sphere and an ellipsoid under mutual gravitation. A synchronous orbit is one for which the rotational period of the ellipsoid is equal to its orbital period. A stationary orbit is one in which the sphere appears stationary as seen from the ellipsoid. It is found that the only stationary orbit is that for which either the major axis or the minor axis of the equator of the ellipsoid is pointing always towards the sphere. He also examined the condition for stability of these stationary orbits.

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Lunar probe

Kozai (1) applied his theory on the motion of an asteroid with high inclination and eccentricity to the motion of a lunar orbiter. The Moon is regarded as a tri-axial ellipsoid, the major axis of the equator being directed towards the Earth according to Jeffreys (2). The disturbing function consists of secular and short-period terms due to the oblateness of the Moon, bi-monthly and short-period terms due to the tri-axiality, and secular and periodic terms due to the Earth. The Earth's luni-centric orbit is assumed to be circular in the plane of the Moon's equator. If the orbiter is far from the Moon's surface, then a stable equilibrium solution exists. If it is a bit near, then there is an unstable equilibrium point. When the distance from the Moon's surface is far enough, a lunar orbiter has a good chance to impact the Moon's surface. Kozai seems to be in favour of the meteor impact hypothesis for the generation of the lunar craters from this point.

Huang (2) at first discussed the escape of an artificial Earth satellite from the Earth only on the basis of the Jacobi integral. Then he studied (3) numerically on an electronic computer the ideal orbits of a space vehicle for various Moon probes under the approximation of the restricted three-body problem: orbits which enclose the Earth and the Moon, which have periods commensurable with the period of the Moon, and which pass at relatively short distance from the Earth as well as from the Moon for a number of times. He computed (4) two families of periodic orbits that enclose both the Earth and the Moon in the plane of the Earth-Moon orbit. The stability of such orbits is also discussed.

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In order to obtain an analytical series expansion for a Moon probe enclosing both the Moon and the Earth inside its orbit by describing a trajectory in the form of the figure o or of the figure 8, Lagerstrom and Kevorkian (5) have tried to match two different asymptotic solutions, one for a Moon satellite and the other for an Earth satellite, in a similar manner to the method of fit of two asymptotic solutions, one for the inside of an atom and the other for a distant point, in quantum-mechanical computation of the wave functions for an electron configuration.

Kevorkian (6) discussed, as a preliminary to this fresh idea, a uniformly valid asymptotic representation to the motion of a satellite in the vicinity of a planet within the framework of the restricted three-body problem. It is shown that, depending on the proximity of the satellite to the planet, there exist two distinct sets of approximations to the restricted three-body equations. For satellite orbits where the gravitational attraction of the planet is of the same order as the centrifugal and Coriolis forces due to the planet's motion around the Sun, one is led to Hill's equations for the motion of the Moon. For orbits which are close enough to the planet for the gravitational attraction of the planet to be the dominant force, a similar set of equations is obtained for which the intermediary orbit is Keplerian in a non-rotating frame centred at the planet. Kevorkian shows that, by choosing the co-ordinates with respect to which the intermediary orbit remains stationary in the mean, one is able to derive a uniformly valid asymptotic solution to the approximation equations.

According to a letter from Roy, he is engaged in studying the orbit of a close artificial satellite of the Moon under the action of the Moon's gravitational potential and the gravitational field of the Sun and the Earth. Forga in Paris is also working on the motion of a Moon satellite.

The motion of an imaginary artificial Moon satellite has been studied by Chebotarev, Brumberg and Kirpichnikov (7) by using a numerical method. Brumberg (8) has formulated a general analytical theory of the artificial Moon satellite motion by taking the non-sphericity of the Moon into account, up to the first order in the periodic perturbations and up to the second order in the secular perturbation. Lemekhova (9) has used Delaunay's method for studying the motion of an artificial Moon satellite. Periodic orbits close to a circular orbit for an artificial Moon satellite have been developed by Aksenov and Demin (10) for the perturbations due to the Earth and the Moon, the Moon's figure not being taken into account.

Brumberg (11) has presented a new technique for finding an orbit with the optimum energy for the two-impulse transfer of a rocket from one given point of the phase space of co-ordinates and velocities to another pre-assigned point. He has also presented (12) the formulae for solving the boundary value problem by the method of steepest descent for the action integral, and given a numerical example of determining a non-disturbed interplanetary trajectory.

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RELATIVITY EFFECTS

Radar echoes from Venus obtained at the Lincoln Laboratory, MIT, (1) and at the Jet Propulsion Laboratory, Cal. Tech., (2) have been interpreted by Clemence (3) to mean that the distance from the Earth to the Sun is about 75 000 km greater than is indicated by the dynamical method. He shows, by improving de Sitter's treatment on the relativity corrections of the orbital elements and by basing on Hansen's theory, that the relativistic effects previously neglected yields correction to the radius vector of a planet amounting at most to about a kilometre. While the correction to the longitude of Mercury is too small to be detected by any known optical technique, the correction to the radius vector in the principal periodic term may provide a new test of general relativity, by radar echo observations.

On the other hand Kustaanheimo (4, 5) deals with the possibility of demonstrating the relativistic curvature of space by the observations of satellites of large eccentricity. He proves that the period of revolution of a satellite is increased by an observable relativistic effect. If a Keplerian orbit and an orbit of the Schwarzschild metrics are defined by means of two orbital constants which have the same numerical values in both theories, then the sidereal periods of the two orbits are different. If the two orbital constants are detectable by observations, then the relativistic period is longer than the Newtonian period.

Briggs (6) discussed the steady-state distribution of meteoric particles under the operation of Poynting-Robertson effect.

Clemence (7) discussed controlled experiments in celestial mechanics for clearing the possible dependence of the Earth's gravitational field on its orbital velocity, the secular change of the gravitational constant, the orbital constants of the Earth, the mass of the Moon and the mechanical ellipticity of the Moon. He pointed out the work of Eckert on the motion of the Moon together with the motions of the near planets Mars and Venus, the measurement of the annual change of the radial velocity of hydrogen cloud, the measurement of the radial velocity of Venus on optical wave length, and finally the lunar probes, as the powerful means for the clarification. This is not a relativistic correction.

Schmidt-Kaler (8) discussed the free falls in Einstein's theory of gravitation.

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THREE-BODY PROBLEM

Klemperer (1) proved the existence of equilibrium configurations of n bodies for limited ranges of rhombic configurations and of arrays in the shape of hexagonal and octagonal regular rosettes, by extending the works of Dziobek and Bilimovič. Such arrangements comprise similar heavier bodies and an equal number of similar lighter bodies in regularly alternating fashion.

Miyahara (2) showed certain conditions, under which a contact transformation from a

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system of Hamiltonian equations with two degrees of freedom to that of the normal form is convergent in the neighbourhood of an equilibrium point.

Sibahara proved several theorems concerning the three-body problem after Chazy and Merman. He showed a possibility of a motion which is hyperbolic-elliptic for $t = -\infty$ and hyperbolic for $t = \infty$ (3), or is hyperbolic-elliptic for $t = \pm\infty$ (4), and derived a sufficient condition for a hyperbolic-elliptic motion of a collision-free system with negative energy (5). He found a lower limit of the shortest mutual distance among the three bodies for the same system to be stable in Lagrange's sense (6), and tried to extend Birkhoff-Merman's theorems to a system with positive energy. Also Sibahara and Yoshida (7) derived a condition for a triangular solution and a three-body collision.

Stumpff (8) attempted to bring the classical Lagrange's theory of the three-body problem in new trappings of modern mathematical representation. Lagrange's theory needs nine integrals for solving the problem by simple quadratures, which are invariant with respect to co-ordinate transformation. Stumpff used four vectors and five angles. He selected nine quantities expressed symmetrically by the velocities and the co-ordinates, and named them the fundamental invariants of the reduced three-body problem. The coefficients of Lagrange's quartic-equation are expressed symmetrically in terms of these fundamental invariants. He proposed three third-order differential equations as preferable for numerical integration.

Szebehely (9) studied the relations between the zero-velocity curves and the orbits in the restricted three-body problem. A condition which the force function must satisfy is derived giving a criterion for identifying certain periodic orbits with the zero-velocity curves. This kind of study is important for proving the existence of the surface of section in Birkhoff's sense of Poincaré's invariant-point theorem on the existence of periodic solutions in the three-body problem. Szebehely (10) then extended the application of zero velocity or Hill's curves in the dynamical systems of two degrees of freedom, the one to more general dynamical systems and the other to much wider applications than to establish various regions of possible motions in Hadamard's theory. A general set of dynamical problems, which possess zero-velocity curves was presented and the general problem of using the Hill curves for orbit generation was solved. Szebehely (11) further studied the general relation between the zero-velocity curves and the orbits for certain classes of dynamical systems including the restricted three-body problem and established the conditions for giving precise analytical meaning to the similarity, which the periodic orbits numerically obtained show to the Hill curves. Szebehely's extension of the Hill curves is called isotach since those curves are described with constant speed. It is shown that the totality of separable potential function, both of the product and of the additive types, possessing isotach orbits is generated by the solution of a second-order ordinary differential equation, and that a necessary condition for such extended Hill curves to be orbits for a dynamical system performing small vibrations is the equality of the characteristic values of the system. The result is applied to the restricted three-body problem in a particular case.

Eckstein (12) has examined the bounded Hill's curve in the $3 + 1$ body problem, after Lindow and Schaub.

Arenstorf (13) has derived a new regularizing transformation of the co-ordinates and the time in the planar restricted problem of three bodies. This transformation, a generalization of Birkhoff's transformation of 1915 and of Thiele's transformation of 1895, transfers the Hamiltonian function to a rational function of the new canonical variables and the original co-ordinates are rational functions of the new co-ordinates of order four.

Contopoulos (14) found the third integral, besides the energy and the angular momentum integrals, in the motion of stars under the gravitational potential of a galaxy as a whole. He calculated explicitly this third integral by means of von Zeipel's method. The question is the convergence of the integral in general cases, for example, in the three-body problem when expanded in series form as is usually done. Search for such a kind of integrals might throw some light on the solution of the classical problem in celestial mechanics.

Arnold (**15**) has obtained a basic result for presenting the solution of the problem of disturbed movement of planets in a trigonometric form for all initial data, with the exception of a certain manifold whose relative measure is small together with the disturbing mass. For the planar restricted problem of three bodies it gives a stability of periodic solutions in the sense of Liapounov and that of any solution in the sense of Lagrange.

Sitnikov (**16**) has proved the existence of oscillatory movements in the problem of three bodies. Alekseev (**17**) has shown a possibility of exchange in the problem of three bodies, that is, the change of hyperbolic-elliptic motion in Chazy's sense. Merman (**18**) has established the non-stability of periodic solutions in the case of a main resonance, according to Liapounov and Levi-Civita.

Ovenden and Roy (**19**) have formulated the Jacobi integral and the angular momentum integrals in the elliptical restricted problem of three bodies in terms of certain auxiliary functions depending on time. It is shown that long-range prediction on the Jacobi integral in the circular case would give wrong result for the elliptical case.

It is known that, when there is a cyclic co-ordinate, a Hamiltonian system posesses an integral common to the whole class of potential functions. Schmeidler (**20**) obtained a general criterion for the existence of such an integral on the basis of Lie's idea of transformation groups and function groups. He applied this criterion to problems of stellar dynamics and showed the impossibility of the existence of such an integral.

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PERIODIC SOLUTION AND STABILITY PROBLEM

Diliberto's mathematical theory (1) on the periodic surfaces originates from a conjecture that the existence of a family of periodic surfaces, which, if true, would establish the stability of the orbits and the validity of the algorithm for finding approximate solutions. The union of the trajectories through an invariant curve is called a periodic surface.

The equations of motion for a dynamical system of two degrees of freedom, including the motion of an artificial satellite around the oblate Earth's gravitational field, are transformed to

$$\frac{d\theta_i}{d\varphi} = 1 + \lambda\, \Theta_i\,(\theta_1,\, \theta_2,\, r_1,\, r_2), \qquad (i = 1, 2)$$
$$\frac{dr_i}{d\varphi} = \lambda R_i\,(\theta_1,\, \theta_2,\, r_1,\, r_2),$$

where $\lambda = 0$ corresponds to the undisturbed motion. A periodic surface of this system of equations is the graph of a pair of analytic functions

$$r_i = S_i\,(\theta_1,\, \theta_2,\, \lambda)$$

defined for all (θ_1, θ_2), and for some neighbourhood of $\lambda = 0$, with period 2π in θ_i and such that, if $p_i = S_i\,(r_1, r_2, \lambda)$, then the solution $r_i\,(\varphi)$, $\theta_i(\varphi)$ taking on the initial values $r_i\,(\varphi_0) = p_i$, $\theta_i(\varphi_0) = r_i$ satisfy

$$r_i(\varphi) = S_i[\theta_1(\varphi),\, \theta_2(\varphi),\, \lambda]$$

for $-\infty < \varphi < +\infty$. The torus which is an example of a periodic two-surface depends on two parameters p_1 and p_2, the radii of its normal cross sections. The central mathematical problem is whether there exists a nested complete family of periodic two-surfaces when the perturbation parameter λ is not zero.

Diliberto *et al.* (2) described their expansion technique for finding the periodic surfaces, and a simple requisite condition which a co-ordinate system must satisfy so as to allow the possibility of such expansions and the determination of at least one co-ordinate system which satisfies the prerequisite condition. It was shown that the energy integral has a simple form in terms of the co-ordinates used. From this it follows that, if one component of the surface is known to a given order, then the other component can be found to the same order directly from the energy integral. Diliberto *et al.* initiated numerical studies for the application of the theory to the study of the orbits of artificial satellites and gave the formulae for the approximate numerical solution.

Thus they announced, for the case where the only term in the perturbation potential is the second zonal harmonic and is sufficiently small, the existence of the family of periodic orbits of arbitrary inclination, the existence of a quadruple family of periodic orbits of arbitrary eccentricity, and the property that these two families plus that at zero inclination include all periodic orbits with common axial angular momentum with periods continuously approaching that of the corresponding Keplerian orbits as the perturbation tends to zero.

For the motion of a satellite around an axially symmetric planet Haseltine (3) proved that there are three one-parameter families, in the sense that, if a value of the parameter is chosen, then the corresponding periodic orbit exists when the perturbation potential is small enough.

Kyner (4) discussed the mathematical problem of the orbits about an oblate planet. He referred to the method of averaging for constructing a new set of approximating formulae, which cannot be a solution but have the novel feature of being accompanied by error estimates. This method of averaging is developed by Struble (5) independently of the more general theory developed by Bogoliubov and Mitropolski. The first-order formulae are free from the difficulties common to most general perturbation method as in the solutions at the critical inclination and for small eccentricities. Thus Kyner proved the existence of periodic solutions after the fashion of the recent works of Cesari and Hale but by a method different from Diliberto's. It was shown that the only possible generating orbits are circular orbits of arbitrary inclination, orbits in the equatorial plane, and orbits at the critical inclination. Macmillan's classical paper is criticized.

According to Diliberto (**6**), Krein studied linear differential equations with periodic coefficients and proved that there are certain Hamiltonian systems such that arbitrary periodic Hamiltonian perturbations of them have only stable motions. By Floquet's theorem this implies that in terms of a perturbation parameter the given Hamiltonian system is diagonalizable to a constant matrix whose roots are all purely imaginary. Diliberto proves that for a linear perturbation these characteristic roots are analytic in the parameter. This establishes the stability of the motion.

Some time ago Siegel and Moser (**7**) discussed the stability of a motion of two degrees of freedom and obtained several elegant theorems. Kolmogorov and Arnold proved the existence of almost-periodic solutions of Hamiltonian systems. Moser (**8**) invented a new technique, which he called a smoothing operator, for constructing the solution of a non-linear differential equation by extending the usual iteration process of Picard. He then proceeded (**9**) to prove a criterion of stability of periodic solutions, and studied the perturbation theory for almost-periodic solutions for undamped non-linear differential equations. Recently Moser (**10**) referred to the invariant-point theorem by Poincaré and Birkhoff for the existence of periodic solutions, that is, by studying the area-preserving mappings of a ring-domain into itself. Moser (**11**) proved a theorem which guarantees the existence of closed invariant curves of such a mapping. Closed invariant curves correspond to almost-periodic solutions of the differential equation which generates the mapping, and are important for the study of stability of periodic solutions.

Cronin (**12**), on the other hand, introduced the idea of topological degree for a criterion of stability of periodic solutions in the perturbation problems, which is identical with the criterion of Andronov and Witt on the van der Pol equations.

Conley (**13**) proved the existence of some new long-period periodic solution in the plane restricted three-body problem. Auslander (**14**) and Seibert (**15**) considered the prolongation of orbits by his method of continuing the orbits beyond their omega limit sets in the sense of Birkhoff and generalized the stability in the sense of Liapounov.

Choudhry (**16**) has discussed the existence of direct and retrograde symmetric periodic orbits, as referred to the rotating axes, in the restricted three-body problem in three dimensions. The periodic orbits obtained by analytic continuation from the generating periodic orbits touch the generating orbits, so that they correspond to the periodic solutions of Schwarzschilds's type.

Huang (**17**), on the other hand, obtained a stability criterion of the periodic orbits in the restricted three-body problem based on the eigenvalues of a fourth order matrix.

New classes of periodic orbits in the restricted three-body problem have been found by Aksenov (**18**) enclosing both of the finite mass bodies, and by Demin (**19**) in the vicinity of trajectories for the problem of two fixed centres with Thiele's variables, both according to Poincaré's method. Krasinski (**20**) has developed a theory of double collision trajectories in the restricted three-body problem, symmetric and asymmetric, with Levi-Civita's regularization, similar to Poincaré's theory of periodic orbits. Petrovskaya (**21**) has found the values of disturbing masses and other characteristics which give the convergence of the series representing the periodic solutions of the planar restricted three-body problem. Volkov (**22**) has obtained symmetric periodic solutions for the three-body problem with finite dimensions as a continuation of the works of Kondurar. Merman (**23**) proved the existence of almost-periodic solutions in the planar restricted three-body problem.

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