THE K-THEORY OF THE $C^\star$-ALGEBRAS OF 2-RANK GRAPHS ASSOCIATED TO COMPLETE BIPARTITE GRAPHS

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Abstract

Using a result of Vdovina, we may associate to each complete connected bipartite graph κ a two-dimensional square complex, which we call a tile complex, whose link at each vertex is κ. We regard the tile complex in two different ways, each having a different structure as a 2-rank graph. To each 2-rank graph is associated a universal $C^\star$-algebra, for which we compute the K-theory, thus providing a new infinite collection of 2-rank graph algebras with explicit K-groups. We determine the homology of the tile complexes and give generalisations of the procedures to complexes and systems consisting of polygons with a higher number of sides.

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1. Introduction

In [15], it was shown how to construct a two-dimensional CW-complex whose link at each vertex is a complete bipartite graph. In [6], generalising the work of [10], certain combinatorial objects called *higher-rank graphs* were defined and then associated with a generalisation of a graph algebra [9, Ch. 1]. We combine these two methods to build an infinite family of $C^\star$-algebras corresponding to complete bipartite graphs.

We begin in Section 2 by detailing Vdovina's construction of the CW-complexes, called *tile complexes*; the data we use to build these is called a *tile system*. In Sections 3 and 5, we associate adjacency matrices to the tile systems in two different ways: by considering the tiles as pointed and as unpointed geometrical objects, as in [5]. Since these adjacency matrices commute, they characterise the structure of a higher-rank

graph, and induce a universal $C^\star$-algebra: the *higher-rank graph algebra*. We use a result of [2] to calculate the K-groups of these algebras (Theorems 3.10 and 5.3).

In Section 6, we show that the tile complexes have torsion-free homology groups given by $H_1 \cong \mathbb{Z}^{\alpha+\beta+2}$, $H_2 \cong \mathbb{Z}^{(\alpha-1)(\beta-1)}$ and $H_n = 0$ otherwise.

Finally, we explore extensions of these methods to *2t-gon systems*, constructed analogously from two-dimensional complexes consisting entirely of $2t$-gons. In all, we associate 2-rank graph $C^\star$-algebras to five systems, and compute their K-theory in the following theorems:

(1) pointed and unpointed tile systems (Theorems 3.10 and 5.3);
(2) pointed and unpointed $2t$-gon systems for even t (Theorem 7.4 and Corollary 7.6);
(3) pointed $2t$-gon systems for arbitrary t (Theorem 7.11).

The respective systems in (2) directly generalise those in (1); however, there is another intuitive way of building $2t$-gon systems from polyhedra, (3). We discuss the naturality of these generalisations in Section 7.

Our approach differs from that of Robertson and Steger [10], who focused on complexes with one vertex. Furthermore, we use the terminology of higher-rank graphs in order to demonstrate the large intersection between the fields of k-graphs and geometry.

Throughout this paper, α, β are positive integers, and $\kappa(\alpha, \beta)$ denotes the complete connected bipartite graph on α white and β black vertices.

2. The tile system associated to a bipartite graph

DEFINITION 2.1. Let $t \in \mathbb{Z}$ satisfy $t \geq 2$, and let $A_1, \ldots, A_n$ be a sequence of solid t-gons, with directed edges labelled from some set $\mathcal{U}$. By gluing together like-labelled edges (respecting their direction), we obtain a two-dimensional complex P. We call such a complex a *t-polyhedron*.

The *link* at a vertex z of P is the graph obtained as the intersection of P with a small 2-sphere centred at z.

THEOREM 2.2 (Vdovina [15]). *Let G be a connected bipartite (undirected) graph on α white and β black vertices, with edge set E(G). Then, for each $t \geq 1$, we can construct a 2t-polyhedron P(G) which has G as the link at each vertex.*

We refer to [15], in which it was shown how to build such a $2t$-polyhedron. The general method is as follows.

Write $U' = \{u_1, \ldots, u_\alpha\}$ for the set of white vertices of G, and $V' = \{v_1, \ldots, v_\beta\}$ for the set of black vertices.

Let U be a set with $2t\alpha$ elements, indexed $u_i^1, u_i^2, \ldots, u_i^t, \bar{u}_i^1, \bar{u}_i^2, \ldots \bar{u}_i^t$ for each $u_i \in U'$, and let V be the corresponding set with $2t\beta$ elements. Define fixed-point-free involutions $u_i^r \mapsto \bar{u}_i^r$ and $v_i^r \mapsto \bar{v}_i^r$ in U and V, respectively.

Each edge of the graph G joins an element of U' to an element of V'; for each edge $e = u_p v_q$, we construct a $2t$-gon A_e with a distinguished base vertex.
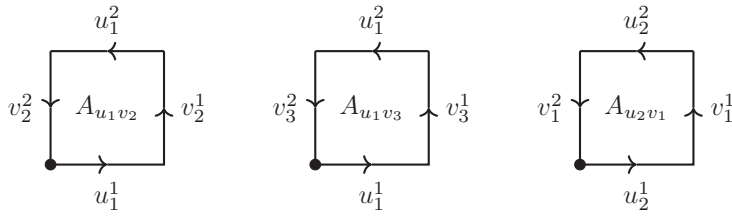
FIGURE 1. Construction of a 2t-polyhedron. Give each side of a sequence of solid 2t-gons a direction and a label from one of two sets U, V, and then glue together corresponding sides with respect to their direction.

Label the boundary of A_e anticlockwise, starting from the base, by the sequence $u_p^1, v_q^1, u_p^2, v_q^2, \ldots, u_p^t, v_q^t$, giving each side of the boundary a forward-directed arrow. We denote this pointed oriented 2t-gon by $A_e = [u_p^1, v_q^1, \ldots, u_p^t, v_q^t]$. Then glue the A_e together in the manner of Definition 2.1 in order to obtain a 2t-polyhedron $P(G)$ (Figure 1).

DEFINITION 2.3. In this paper, we mainly concern ourselves with 4-polyhedra, that is, those constructed by gluing together squares. We refer to 4-polyhedra as *tile complexes*. For a connected bipartite graph G, write $TC(G)$ for the tile complex $P(G)$, and define the set

$$\mathcal{S}(G) := \{A_e = [u_p^1, v_q^1, u_p^2, v_q^2], [\bar{u}_p^1, \bar{v}_q^2, \bar{u}_p^2, \bar{v}_q^1],$$
$$[u_p^2, v_q^2, u_p^1, v_q^1], [\bar{u}_p^2, \bar{v}_q^1, \bar{u}_p^1, \bar{v}_q^2] \mid e = u_p v_q \in E(G)\}. \qquad (2\text{-}1)$$

We call the elements of $\mathcal{S}(G)$ *pointed tiles*. We define an equivalence relation which, for each A_e, identifies the four corresponding pointed tiles in (2-1). We denote by $\mathcal{S}'(G)$ the quotient of $\mathcal{S}(G)$ with respect to this relation, and we write $A_e' = (u_p^1, v_q^1, u_p^2, v_q^2)$ for the equivalence class of A_e in $\mathcal{S}'(G)$. Then $\mathcal{S}'(G)$ is the set of geometric squares (that is, disregarding basepoint and orientation) in $TC(G)$. We call elements of $\mathcal{S}'(G)$ *unpointed tiles*.

   Notice that by placing the basepoint at the bottom-left vertex, we can arrange that the horizontal sides of each pointed tile are labelled by elements of U, and the vertical sides by elements of V, such that $\mathcal{S}(G) \subseteq U \times V \times U \times V$. Indeed, the four tuples in (2-1) correspond to the four symmetries of a pointed tile that preserve this property (Figure 2).

   Note also that, by design, any two pointed tiles in $\mathcal{S}(G)$ are distinct, and any two adjacent sides of a tile uniquely determine the remaining two sides.

DEFINITION 2.4. Let G be a connected bipartite graph on α white and β black vertices. Let U, V be sets with $|U| = 4\alpha$, $|V| = 4\beta$, as constructed above, and let $\mathcal{S} = \mathcal{S}(G) \subseteq U \times V \times U \times V$ be the corresponding set of pointed tiles. We call the datum $(G, U, V, \mathcal{S})$ a *tile system*.
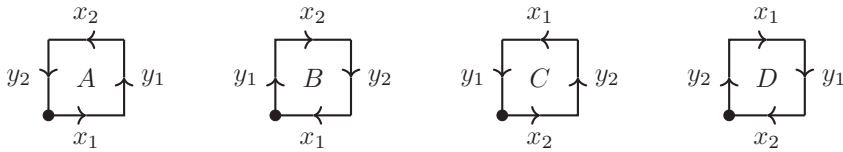
FIGURE 2. Visualisation of tiles: $A = [x_1, y_1, x_2, y_2]$, $B = [\bar{x}_1, \bar{y}_2, \bar{x}_2, \bar{y}_1]$, and so on. These four pointed squares represent different pointed tiles, but the same unpointed tile.
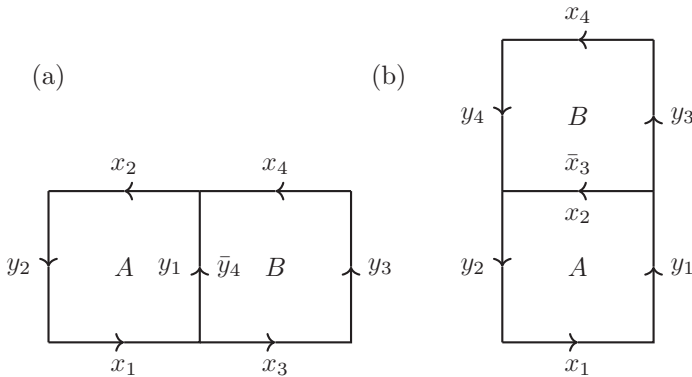

FIGURE 3. Horizontal and vertical adjacency: (a) $M_1(A, B) = 1$, (b) $M_2(A, B) = 1$.

This construction is closely related to, and indeed modelled on, that of a *VH-datum*, introduced in [16] and developed further in [1].

3. The $C^\star$-algebra corresponding to a tile system

DEFINITION 3.1. Let $(G, U, V, \mathcal{S})$ be a tile system, and $A = [x_1, y_1, x_2, y_2]$ and $B = [x_3, y_3, x_4, y_4]$ be pointed tiles in $\mathcal{S}$. We define two $4\alpha\beta \times 4\alpha\beta$ matrices M_1, M_2 with AB th entry $M_i(A, B)$ as follows:

$$M_1(A, B) = \begin{cases} 1 & \text{if } y_1 = \bar{y}_4 \text{ and } x_1 \neq \bar{x}_3, \\ 0 & \text{otherwise}, \end{cases}$$

$$M_2(A, B) = \begin{cases} 1 & \text{if } x_2 = \bar{x}_3 \text{ and } y_1 \neq \bar{y}_3, \\ 0 & \text{otherwise}, \end{cases}$$

as demonstrated in Figure 3. We call M_1 the *horizontal adjacency matrix* and M_2 the *vertical adjacency matrix*. If $M_i(A, B) = 1$, we say that B is *horizontally* or *vertically adjacent* to A.

DEFINITION 3.2. Let $(G, U, V, \mathcal{S})$ be a tile system, and let A, B, C be pointed tiles in $\mathcal{S}(G)$ such that $M_1(A, B) = 1$ and $M_2(A, C) = 1$. We say that the tile system $(G, U, V, \mathcal{S})$
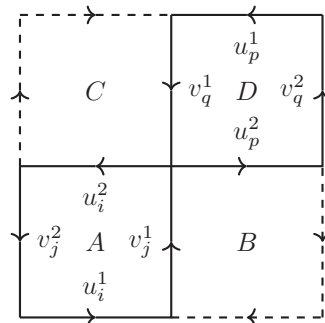
FIGURE 4. Proposition 3.3. Given tiles A and D as shown, tiles B and C are uniquely determined; hence, $M_1 M_2 = M_2 M_1$. Likewise, given an initial tile A, a horizontally adjacent tile B and a vertically adjacent tile C, there is a unique tile D adjacent to both B and C: this is the UCE property.

satisfies the *unique common extension property* (*UCE property*) if there exists a *unique* $D \in \mathcal{S}$ such that $M_2(B, D) = M_1(C, D) = 1$.

PROPOSITION 3.3. *Consider the complete bipartite graph $\kappa = \kappa(\alpha, \beta)$ on $\alpha \geq 2$ white and $\beta \geq 2$ black vertices, and let $(\kappa, U, V, \mathcal{S}(\kappa))$ be a tile system with corresponding adjacency matrices M_1, M_2. Then:*

(1)   *M_1 and M_2 are symmetric and commute with each other;*
(2)   *each row and column of M_1 and M_2 contains at least one nonzero element;*
(3)   *$(\kappa, U, V, \mathcal{S}(\kappa))$ satisfies the UCE property.*

PROOF. It is straightforward to verify that the matrices M_1 and M_2 are symmetric. Now, consider the pointed tile $A = [u_i^1, v_j^1, u_i^2, v_j^2] \in \mathcal{S}(\kappa)$ (any other tiles may be dealt with in a similar manner), and let $D = [u_p^2, v_q^2, u_p^1, v_q^1] \in \mathcal{S}(\kappa)$ for some $p \neq i$, $q \neq j$. By the completeness of κ and the fact that $\alpha, \beta \geq 2$, there are tiles $B = [\bar{u}_i^1, \bar{v}_j^2, \bar{u}_p^2, \bar{v}_j^1]$ and $C = [\bar{u}_i^2, \bar{v}_q^1, \bar{u}_i^1, \bar{v}_q^2]$ in $\mathcal{S}(\kappa)$ such that $M_1(A, B) = M_2(B, D) = 1$ and $M_2(A, C) = M_1(C, D) = 1$ (Figure 4), proving (2). Since any two adjacent sides of a tile determine the remaining sides, B and C are unique. So $M_1 M_2(A, D) = M_2 M_1(A, D) \in \{0, 1\}$ for all $A, D \in \mathcal{S}(\kappa)$.

Similarly, given $A, B, C \in \mathcal{S}(\kappa)$ as above, D is the unique tile adjacent to both B and C, and so $(\kappa, U, V, \mathcal{S}(\kappa))$ has the UCE property. □

We see shortly that a tile system is actually an example of a so-called *k-rank graph* (specifically a 2-rank graph), as introduced in [6] to build on work by [10].

3.1. Higher-rank graphs.

DEFINITION 3.4. Let Λ be a category such that $\mathrm{Ob}(\Lambda)$ and $\mathrm{Hom}(\Lambda)$ are countable sets (that is, a *countable small category*), and identify $\mathrm{Ob}(\Lambda)$ with the identity morphisms in $\mathrm{Hom}(\Lambda)$. For a morphism $\lambda \in \mathrm{Hom}_\Lambda(u, v)$, we define range and source maps $r(\lambda) = v$ and $s(\lambda) = u$, respectively.

Let $d : \Lambda \to \mathbb{N}^k$ be a functor, called the *degree map*, and let $\lambda \in \mathrm{Hom}(\Lambda)$. We call the pair (Λ, d) a *k-rank graph* (or simply a *k-graph*) if, whenever $d(\lambda) = \mathbf{m} + \mathbf{n}$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{N}^k$, we can find *unique* elements $\mu, \nu \in \mathrm{Hom}(\Lambda)$ such that $\lambda = \nu\mu$, and $d(\mu) = \mathbf{m}$, $d(\nu) = \mathbf{n}$. Note that, for μ, ν to be composable, we must have $r(\mu) = s(\nu)$.

For $\mathbf{n} \in \mathbb{N}^k$, we write $\Lambda^{\mathbf{n}} := d^{-1}(\mathbf{n})$; by the above property, $\Lambda^{\mathbf{0}} = \mathrm{Ob}(\Lambda)$, and we call the elements of $\Lambda^{\mathbf{0}}$ the *vertices* of (Λ, d); see [6].

We direct the reader to [14], for example, for further details and standard examples of higher-rank graphs.

If E is a directed graph on n vertices, we can construct an $n \times n$ vertex matrix $M_E(i, j)$ with ij th entry 1 if there is an edge from i to j, and 0 otherwise.

If E, F are directed graphs with the same vertex set, whose associated vertex matrices M_E, M_F commute, then [6] showed that we can construct a 2-rank graph out of E and F. We use their method to prove the following proposition.

PROPOSITION 3.5. *Let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on $\alpha \geq 2$ white and $\beta \geq 2$ black vertices, and let $(\kappa, U, V, \mathcal{S}(\kappa))$ be a tile system with adjacency matrices M_1, M_2. Then $(\kappa, U, V, \mathcal{S}(\kappa))$ has a 2-rank graph structure.*

PROOF. Following the method of Theorem 2.2, label the elements of the sets U, V

$$U = \{u_1^1, u_1^2, \dots, u_\alpha^1, u_\alpha^2, \bar{u}_1^1, \bar{u}_1^2, \dots, \bar{u}_\alpha^1, \bar{u}_\alpha^2\},$$
$$V = \{v_1^1, v_1^2, \dots, v_\beta^1, v_\beta^2, \bar{v}_1^1, \bar{v}_1^2, \dots, \bar{v}_\beta^1, \bar{v}_\beta^2\},$$

where $u_1, \dots, u_\alpha$ and $v_1, \dots, v_\beta$ are the white and black vertices of κ, respectively. Construct the tile complex $TC(\kappa)$, and consider the set $\mathcal{S}(\kappa) \subseteq U \times V \times U \times V$ of pointed tiles of $TC(\kappa)$. Since κ is complete, there is for an edge joining each u_i and v_j, thus:

$$\mathcal{S}(\kappa) = \{[u_i^1, v_j^1, u_i^2, v_j^2], [\bar{u}_i^1, \bar{v}_j^2, \bar{u}_i^2, \bar{v}_j^1],$$
$$[u_i^2, v_j^2, u_i^1, v_j^1], [\bar{u}_i^2, \bar{v}_j^1, \bar{u}_i^1, \bar{v}_j^2] \mid 1 \leq i \leq \alpha, 1 \leq j \leq \beta\}.$$

Consider the corresponding adjacency matrices M_1 and M_2 as described in Definition 3.1, and note that they commute by Proposition 3.3. We can draw directed graphs E, F with the same vertex set $E^0 = F^0 = \mathcal{S}(\kappa)$. We draw a directed edge in E from A to B whenever $M_1(A, B) = 1$, and in F from A to B whenever $M_2(A, B) = 1$ (Figure 5). Write r_E, s_E (respectively, r_F, s_F) for the maps describing the respective range and source of edges in E^1 (respectively, F^1).

Define the following edge sets: $E^1 * F^1 := \{(\lambda, \mu) \in E^1 \times F^1 \mid r_E(\lambda) = s_F(\mu)\}$ and $F^1 * E^1 := \{(\mu, \lambda) \in F^1 \times E^1 \mid r_F(\mu) = s_E(\lambda)\}$. Since M_1, M_2 commute, there is a unique bijection $\theta : E^1 * F^1 \to F^1 * E^1$, mapping $(\lambda, \mu) \mapsto (\mu', \lambda')$ such that $s_E(\lambda) = s_F(\mu')$ and $r_F(\mu) = r_E(\lambda')$.
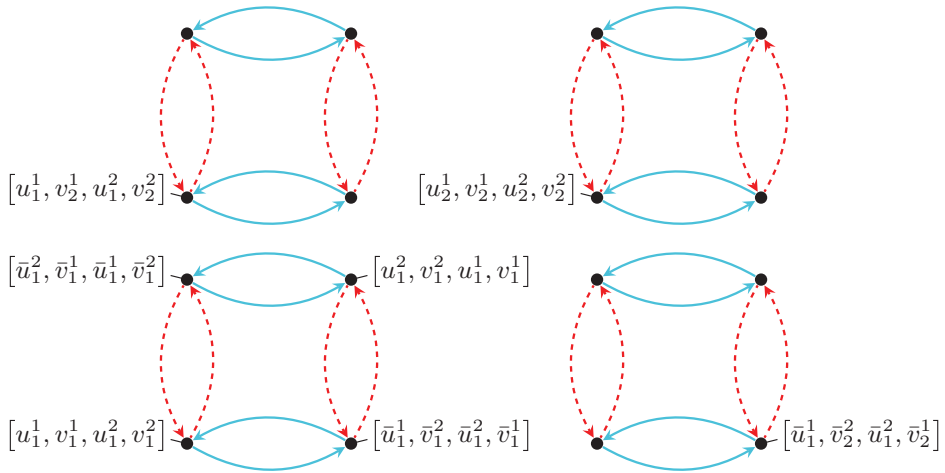
FIGURE 5. Visualisation of the tile system corresponding to the complete bipartite graph $\kappa(2,2)$. Each vertex is labelled with an element of $\mathcal{S}(\kappa)$; a few labels have been shown here. A solid (dashed) arrow joins vertex A to B if and only if $M_1(A,B) = 1$ ($M_2(A,B) = 1$, respectively). Notice the commuting squares, giving the tile system a 2-rank graph structure: from any vertex A, follow a solid arrow, and then a dashed arrow to another vertex D, say. Then θ defines a unique dashed-solid path from A to D. The 1-skeleton of the 2-rank graph $\Lambda(\kappa(\alpha,\beta))$ is strongly connected only when $\alpha,\beta \geq 3$.

We construct a 2-rank graph (Λ, d) in the following way. Let $\Lambda^0 = \mathcal{S}(\kappa)$, and for each $(m,n) \in \mathbb{N}^2$, define the set $W(m,n) := \{(p,q) \in \mathbb{N}^2 \mid p \leq m, q \leq n\}$. Then an element of $\Lambda^{(m,n)}$ is a triple $(A, \lambda, \mu) = ((A(p,q))_{p,q}, (\lambda(p,q))_{p,q}, (\mu(p,q))_{p,q})$ such that:

(a) $A(p,q) \in \mathcal{S}(\kappa)$ for some $(p,q) \in W(m,n)$;
(b) $\lambda(p,q) \in E^1$ for some $(p,q) \in W(m-1,n)$;
(c) $\mu(p,q) \in F^1$ for some $(p,q) \in W(m,n-1)$;
(d) $s_E(\lambda(p,q)) = s_F(\mu(p,q)) = A(p,q)$;
(e) $r_E(\lambda(p,q)) = A(p+1,q)$ and $r_F(\mu(p,q)) = A(p,q+1)$;
(f) $\theta(\lambda(p,q), \mu(p+1,q)) = (\mu(p,q), \lambda(p,q+1))$,

whenever these conditions make sense. We write $\Lambda := \bigcup_{m,n \geq 0} \Lambda^{(m,n)}$, and define range and source maps $r(A,\lambda,\mu) := A(0,0)$, $s(A,\lambda,\mu) := A(m,n)$, respectively. Note that two finite paths μ, ν in such a directed graph E can be concatenated to give a path $\nu \cdot \mu$ if and only if $s_E(\mu) = r_E(\nu)$; consequently, 'change the direction' of the sources and ranges of the arrows here.

If φ, ψ are paths of nonzero length m, n in E, F respectively, with $r_E(\varphi) = s_F(\psi)$, then there is a unique element $\varphi\psi = (A,\lambda,\mu) \in \Lambda^{(m,n)}$ such that $\varphi = \lambda(0,0) \cdots \lambda(m-1,0)$ and $\psi = \mu(m,0) \cdots \mu(m,n-1)$. If, instead (or as well), $r_F(\psi) = s_E(\varphi)$, then there is a unique element $\psi\varphi$ such that $\varphi = \lambda(0,n) \cdots \lambda(m-1,n)$ and $\psi = \mu(0,0) \cdots \mu(0,n-1)$ (Figure 6).
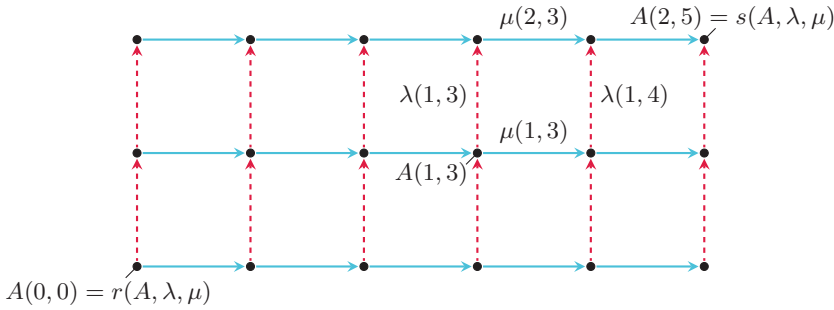
FIGURE 6. An element (A, λ, μ) of $\Lambda^{(m,n)}$ can be represented as an $m \times n$ grid. The isomorphism θ defines commuting squares. Here is an element of $\Lambda^{(2,5)}$.

Then, given two elements $(A_1, \lambda_1, \mu_1) \in \Lambda^{(m_1,n_1)}$ and $(A_2, \lambda_2, \mu_2) \in \Lambda^{(m_2,n_2)}$ such that $A_1(m_1, n_1) = A_2(0, 0)$, we can find a unique element $(A_1, \lambda_1, \mu_1)(A_2, \lambda_2, \mu_2) = (A_3, \lambda_3, \mu_3)$ in $\Lambda^{(m_1+m_2,n_1+n_2)}$ that satisfies

(a) $A_3(p, q) = A_1(p, q)$, and $A_3(m + p, n + q) = A_2(p, q)$;
(b) $\lambda_3(p, q) = \lambda_1(p, q)$, and $\lambda_3(m + p, n + q) = \lambda_2(p, q)$;
(c) $\mu_3(p, q) = \mu_1(p, q)$, and $\mu_3(m + p, n + q) = \mu_2(p, q)$,

whenever these conditions make sense. In this way, composition is defined in Λ, and, by construction, we have associativity and the factorisation property of Definition 3.4. Thus Λ, together with obvious degree functor $d : (A, \lambda, \mu) \mapsto (m, n)$ for $(A, \lambda, \mu) \in \Lambda^{(m,n)}$, has the structure of a 2-rank graph, and we write $(\Lambda, d) = \Lambda(\kappa)$.   □

DEFINITION 3.6. Let (Λ, d) be a k-rank graph, let $\mathbf{n} \in \mathbb{N}^k$, and let $v \in \Lambda^0$. Write $\Lambda^{\mathbf{n}}(v)$ for the set of morphisms in $\Lambda^{\mathbf{n}}$ which map onto v, that is, $\Lambda^{\mathbf{n}}(v) := \{\lambda \in \Lambda^{\mathbf{n}} \mid r(\lambda) = v\}$. We say that (Λ, d) is *row-finite* if each set $\Lambda^{\mathbf{n}}(v)$ is finite, and that (Λ, d) has *no sources* if each $\Lambda^{\mathbf{n}}(v)$ is nonempty.

As an extension of the concept of a *graph algebra* (see [9]), we can associate a $C^\star$-algebra to a k-rank graph.

DEFINITION 3.7. Let $\Lambda = (\Lambda, d)$ be a row-finite k-rank graph with no sources. We define $C^\star(\Lambda)$ to be the universal $C^\star$-algebra generated by a family $\{s_\lambda \mid \lambda \in \Lambda\}$ of *partial isometries* satisfying the following properties.

(a)   For all $u, v \in \Lambda^0$, we have $(s_v)^2 = s_v = s_v^*$ and $s_u s_v = 0$ whenever $u \neq v$.
(b)   If $r(\lambda) = s(\mu)$ for some $\lambda, \mu \in \Lambda$, then $s_{\mu\lambda} = s_\mu s_\lambda$.
(c)   For all $\lambda \in \Lambda$, we have $s_\lambda^* s_\lambda = s_{s(\lambda)}$.
(d)   For all vertices $v \in \Lambda^0$ and $\mathbf{n} \in \mathbb{N}^k$,

$$s_v = \sum_{\lambda \in \Lambda^{\mathbf{n}}(v)} s_\lambda s_\lambda^*.$$

Note that, without the row-finiteness condition, property (d) is not well defined.

THEOREM 3.8 (Evans [2, Proposition 4.4]). *Let Λ be a row-finite 2-graph with no sources, finite vertex set Λ^0, and vertex matrices M_E, M_F. Then*

$$K_0(C^\star(\Lambda)) \cong \mathbb{Z}^{r_0} \oplus \mathrm{tor}(\mathrm{coker}(\mathbf{1} - M_E^T, \mathbf{1} - M_F^T)),$$
$$K_1(C^\star(\Lambda)) \cong \mathbb{Z}^{r_1} \oplus \mathrm{tor}(\mathrm{coker}(\mathbf{1} - M_E, \mathbf{1} - M_F)),$$

where

$$r_0 := \mathrm{rk}(\mathrm{coker}(\mathbf{1} - M_E^T, \mathbf{1} - M_F^T)) + \mathrm{rk}(\mathrm{coker}(\mathbf{1} - M_E, \mathbf{1} - M_F)),$$
$$r_1 := \mathrm{rk}(\mathrm{coker}(\mathbf{1} - M_E^T, \mathbf{1} - M_F^T)) + \mathrm{rk}(\mathrm{coker}(\mathbf{1} - M_E, \mathbf{1} - M_F)),$$

$|\Lambda^0| = n$, $\mathbf{1}$ *is the $n \times n$ identity matrix, $(\,*\,,\,*\,)$ denotes the corresponding block $n \times 2n$ matrix, $\mathrm{rk}(\mathfrak{G})$ is the torsion-free rank of a finitely generated Abelian group $\mathfrak{G}$, and $\mathrm{tor}(\mathfrak{G})$ is the torsion part of $\mathfrak{G}$.*

COROLLARY 3.9. *Let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on $\alpha \geq 2$ white and $\beta \geq 2$ black vertices, and let $(\kappa, U, V, \mathcal{S}(\kappa))$ be a tile system with adjacency matrices M_1, M_2 as in Definition 3.1. For simplicity, we write $C^\star(\kappa) = C^\star(\Lambda(\kappa))$. Then*

$$K_0(C^\star(\kappa)) = K_1(C^\star(\kappa)) = \mathrm{coker}(\mathbf{1} - M_1^T, \mathbf{1} - M_2^T) \oplus \mathbb{Z}^r,$$

where $r := \mathrm{rk}(\mathrm{coker}(\mathbf{1} - M_1^T, \mathbf{1} - M_2^T))$.

PROOF. Firstly, $\alpha, \beta < \infty$ by assumption, and by the UCE property of the tile system (Proposition 3.3) we know that each row and column of M_1 and M_2 has at least one nonzero element. Hence, $\Lambda(\kappa)$ is row-finite, has no sources, and satisfies $|\Lambda(\kappa)^0| = 4\alpha\beta$, whence the result follows from Theorem 3.8.   □

THEOREM 3.10 (K-groups for pointed tile systems). *Let $a, b \geq 0$. Let $\kappa(a + 2, b + 2)$ be the complete bipartite graph on $a + 2$ white and $b + 2$ black vertices. Without loss of generality, we assume that $a \leq b$. Write $l := \mathrm{lcm}(a, b)$, and $g := \mathrm{gcd}(a, b)$. Then, for $\epsilon = 0, 1$, we have the following assertions.*

(1)   *If $a = b = 0$, then $K_\epsilon(C^\star(\kappa(a + 2, b + 2))) = K_\epsilon(C^\star(\kappa(2, 2))) \cong \mathbb{Z}^8$.*

(2)   *If $a = 0, 1$ and $b \geq 1$, then*

$$K_\epsilon(C^\star(\kappa(a + 2, b + 2))) \cong (\mathbb{Z}/b)^2 \oplus \mathbb{Z}^{4(b+1)}.$$

(3)   *If $a, b \geq 2$ and a, b are coprime, then*

$$K_\epsilon(C^\star(\kappa(a + 2, b + 2))) \cong (\mathbb{Z}/a)^{b-a} \oplus (\mathbb{Z}/ab)^{a+1} \oplus \mathbb{Z}^{2(a+1)(b+1)}.$$

(4)   *If $a, b \geq 2$ and a, b are not coprime, then*

$$K_\epsilon(C^\star(\kappa(a + 2, b + 2))) \cong (\mathbb{Z}/a)^{b-a} \oplus (\mathbb{Z}/l)^{a+1} \oplus (\mathbb{Z}/g)^{a+2} \oplus \mathbb{Z}^{2(a+1)(b+1)},$$

   where $(\mathbb{Z}/a)^0$ is defined to be the trivial group in the case that $a = b$.

PROOF. We begin by proving (3) and (4), since (1) and (2) are special cases thereof.

Assume that $a, b \geq 2$. Write $\alpha = a + 2$ and $\beta = b + 2$, and for $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$, let A_{ij} denote the pointed tile $[u_i^1, v_j^1, u_i^2, v_j^2] \in \mathcal{S}(\kappa)$. Similarly, write $B_{ij} := [\bar{u}_i^1, \bar{v}_j^2, \bar{u}_i^2, \bar{v}_j^1]$, $C_{ij} := [\bar{u}_i^2, \bar{v}_j^1, \bar{u}_i^1, \bar{v}_j^2]$, $D_{ij} := [u_i^2, v_j^2, u_i^1, v_j^1]$ for the tiles with the same edge labels as the horizontal reflection, vertical reflection, and rotation by π of A_{ij}, respectively. Then $\mathcal{S}(\kappa) = \{A_{ij}, B_{ij}, C_{ij}, D_{ij} \mid 1 \leq i \leq \alpha, 1 \leq j \leq \beta\}$, and

$$\text{coker} = \Big\langle S \in \mathcal{S}(\kappa) \Big| S = \sum_{T \in \mathcal{S}(\kappa)} M_1(S, T) \cdot T = \sum_{T \in \mathcal{S}(\kappa)} M_2(S, T) \cdot T \Big\rangle, \qquad (3\text{-}1)$$

where $\text{coker} := \text{coker}(\mathbf{1} - M_1^T, \mathbf{1} - M_2^T)$. Now fix some $p \in \{1, \ldots, \alpha\}$ and $q \in \{1, \ldots, \beta\}$, and notice that:

- $M_1(A_{pq}, T) = 1$ if and only if $T = B_{iq}$, and $M_1(B_{pq}, T) = 1$ if and only if $T = A_{iq}$, for some $i \neq p$;
- $M_1(C_{pq}, T) = 1$ if and only if $T = D_{iq}$, and $M_1(D_{pq}, T) = 1$ if and only if $T = C_{iq}$, for some $i \neq p$;
- $M_2(A_{pq}, T) = 1$ if and only if $T = C_{pj}$, and $M_2(B_{pq}, T) = 1$ if and only if $T = D_{pj}$, for some $j \neq q$;
- $M_2(C_{pq}, T) = 1$ if and only if $T = A_{pj}$, and $M_2(D_{pq}, T) = 1$ if and only if $T = B_{pj}$, for some $j \neq q$.

Hence, the relations of (3-1) are equations of the form $A_{pq} = \sum_{i \neq p} B_{iq} = \sum_{j \neq q} C_{pj}$, and so on for each B_{pq}, C_{pq}, D_{pq}. In particular, we can write $B_{pq} = \sum_{i \neq p} A_{iq}$ and $C_{pq} = \sum_{j \neq q} A_{pj}$ so that

$$A_{pq} = (\alpha - 1)A_{pq} + (\alpha - 2)\sum_{i \neq p} A_{iq} \quad \text{and} \quad A_{pq} = (\beta - 1)A_{pq} + (\beta - 2)\sum_{j \neq q} A_{pj}.$$

Define $J_q := \sum_{i=1}^{\alpha} A_{iq}$, and $I_p := \sum_{j=1}^{\beta} A_{pj}$. Then $(\alpha - 2)J_q = (\beta - 2)I_p = 0$, and, viewing the sum of all the tiles A_{ij} both as the sum of all the I_i and of the J_j, we conclude that $g\Sigma = 0$, where $\Sigma := \sum_{i,j} A_{ij}$.

Now, we can also write D_{pq} (and all of the relevant relations) in terms of the A_{ij}, namely $D_{pq} = \sum_{i \neq p} \sum_{j \neq q} A_{ij}$. Hence, we can remove all the B_{pq}, C_{pq}, and D_{pq} from the list of generators of coker, yielding

$$\text{coker} = \langle A_{pq} \mid (\alpha - 2)J_q = (\beta - 2)I_p = 0, J_q = \sum_i A_{iq},$$
$$I_p = \sum_j A_{pj}, \text{ for } 1 \leq p \leq \alpha, 1 \leq q \leq \beta \rangle. \qquad (3\text{-}2)$$

We have the following equalities:

$$A_{p1} = I_p - \sum_{j=2}^{\beta} A_{pj}, \quad A_{1q} = J_q - \sum_{i=2}^{\alpha} A_{iq}, \quad I_1 = \Sigma - \sum_{i=2}^{\alpha} I_i, \quad J_1 = \Sigma - \sum_{j=2}^{\beta} J_j.$$

FIGURE 7. The 2-graph $\Lambda(\kappa(2,2))$, depicted in Figure 5, consists of four copies of $C_2 \times C_2$, where C_2 is the cyclic 1-graph with two vertices.

Furthermore, A_{11} may be expressed in terms of Σ, I_p, J_q, and A_{pq} for $p, q \geq 2$, and so, after a sequence of Tietze transformations on (3-2), we find that

$$\text{coker} = \langle \Sigma, I_p, J_q, A_{pq} \mid (\alpha - 2)J_q = (\beta - 2)I_p = g\Sigma = 0, \text{ for } 2 \leq p \leq \alpha, 2 \leq q \leq \beta \rangle,$$

where $g := \gcd(\alpha - 2, \beta - 2)$. This, after substituting $a = \alpha - 2$ and $b = \beta - 2$, gives a presentation for $(\mathbb{Z}/b)^{a+1} \oplus (\mathbb{Z}/a)^{b+1} \oplus (\mathbb{Z}/g) \oplus \mathbb{Z}^{(a+1)(b+1)}$. In particular, we have $a + 1$ copies of $(\mathbb{Z}/b) \oplus (\mathbb{Z}/a)$. It is well known that if a and b are not coprime, $(\mathbb{Z}/b) \oplus (\mathbb{Z}/a) \cong (\mathbb{Z}/l) \oplus (\mathbb{Z}/g)$; in case (4), this, together with Corollary 3.9, immediately gives the desired result. In case (3), where a and b are coprime, we instead have that $(\mathbb{Z}/b) \oplus (\mathbb{Z}/a) \cong (\mathbb{Z}/ab)$, and we are done.

Now consider case (1), where $\alpha = \beta = 2$. Then, following the method above, coker is generated by $\{A_{pq} \mid p, q = 1, 2\}$ with trivial relations, and so $\text{coker} \cong \mathbb{Z}^4$. Hence, by Corollary 3.9, $K_\epsilon(C^\star(\kappa)) \cong \mathbb{Z}^8$.

Similarly, when $\alpha = 2$ and $\beta \geq 3$, it is straightforward to show that

$$\text{coker} = \langle I_p, A_{pq} \mid (\beta - 2)I_p = 0, \text{ for } p = 1, 2 \text{ and } 2 \leq q \leq \beta \rangle,$$

and, when $\alpha = 3$ and $\beta \geq 3$,

$$\text{coker} = \langle \Sigma, I_p, J_q, A_{pq} \mid J_q = (\beta - 2)I_p = \Sigma = 0, \text{ for } p = 2, 3 \text{ and } 2 \leq q \leq \beta \rangle,$$

both of which are presentations of $(\mathbb{Z}/(\beta - 2))^2 \oplus \mathbb{Z}^{2(\beta-1)}$; hence, by Corollary 3.9, (2) is proved. □

EXAMPLE 3.11. Recall the tile system corresponding to $\kappa(2,2)$, given in Figure 5. From the diagram, we can see that the (1-skeleton of the) 2-rank graph $\Lambda(\kappa(2,2))$ comprises four connected components, each being the Cartesian product $C_2 \times C_2$, depicted in Figure 7. It is well known that the k-graph $C^\star$-algebra of C_2 is isomorphic to $M_2(C(\mathbb{T}))$. Furthermore, there is a natural isomorphism $C^\star(C_m \times C_n) \cong M_{mn}(C(\mathbb{T}^2))$, and so $C^\star(\kappa(2,2)) \cong (M_4(C(\mathbb{T}^2)))^4$. The K-groups of this $C^\star$-algebra are both $\mathbb{Z}^8$, in agreement with Theorem 3.10.

THEOREM 3.12. *Let $\alpha, \beta \geq 3$, and let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on α white and β black vertices. Then the order of the class of the identity $[\mathbf{1}]$ in $K_0(C^\star(\Lambda(\kappa)))$ is equal to $g := \gcd(\alpha - 2, \beta - 2)$.*

PROOF. From [3], it follows that the order of $[\mathbf{1}]$ in $K_0(C^\star(\kappa))$ is equal to the order of the sum of pointed tiles in $\mathcal{S}(\kappa)$; by considerations in the proof of Theorem 3.10, we know this to be g. □

4. Aperiodicity and Kirchberg–Phillips classification

Kumjian and Pask in [6] developed conditions under which the $C^\star$-algebra of a k-rank graph is both simple and purely infinite. In this section we show that the conditions are satisfied by the algebras $C^\star(\kappa)$, and thus, by the Kirchberg and Phillips results [4, 8], that the $C^\star(\kappa)$ are completely classified by their K-theory. We detail the following definitions from [6].

Let $k \geq 1$, and let Ω_k be the countable small category with object set $\mathrm{Ob}(\Omega_k) := \mathbb{N}_0^k$ and morphism set $\mathrm{Hom}(\Omega_k)$ given by

$$\{(\mathbf{m}, \mathbf{n}) = (m_1, \ldots, m_k, n_1, \ldots, n_k) \in \mathbb{N}_0^k \times \mathbb{N}_0^k \mid m_i \leq n_i \text{ for all } 1 \leq i \leq k\}.$$

We identify $\mathrm{Ob}(\Omega_k)$ with the set of identity morphisms $\{(\mathbf{m}, \mathbf{m}) \mid \mathbf{m} \in \mathbb{N}_0^k\}$, and, hence, identify Ω_k with $\mathrm{Hom}(\Omega_k)$. Define range and source maps $r(\mathbf{m}, \mathbf{n}) := \mathbf{m}$ and $s(\mathbf{m}, \mathbf{n}) := \mathbf{n}$, respectively. Then Ω_k, together with the degree map $d(\mathbf{m}, \mathbf{n}) := \mathbf{n} - \mathbf{m}$, is a k-rank graph, which we can visualise as a nonnegative integer lattice in $\mathbb{R}^k$ (see Figure 6).

DEFINITION 4.1. Let Λ be a k-rank graph. We define the *infinite path space* Λ^∞ of Λ to be $\Lambda^\infty := \{\varphi : \Omega_k \to \Lambda \mid \varphi \text{ is a } k\text{-graph morphism}\}$.

Given a vertex $v \in \Lambda^0$, we write $\Lambda^\infty(v)$ for the set of infinite paths that begin at v, that is, $\Lambda^\infty(v) := \{\varphi \in \Lambda^\infty \mid \varphi(\mathbf{0}) = v\}$.

Let $\mathbf{p} \in \mathbb{Z}^k$, and let $\varphi \in \Lambda^\infty$. We say that $\mathbf{p}$ is a *period* for φ if, for every $(\mathbf{m}, \mathbf{n}) \in \Omega_k$ with $\mathbf{m} + \mathbf{p} \geq \mathbf{0}$, we have $\varphi(\mathbf{m} + \mathbf{p}, \mathbf{n} + \mathbf{p}) = \varphi(\mathbf{m}, \mathbf{n})$. We call φ *periodic* if we can find a nonzero period for φ.

Given $\mathbf{q} \in \mathbb{N}_0^k$ and a path $\varphi \in \Lambda^\infty$, we write $\varphi_{\mathbf{q}}(\mathbf{m}, \mathbf{n}) := (\mathbf{m} + \mathbf{q}, \mathbf{n} + \mathbf{q})$. We say that φ is *eventually periodic* if we can find some nonzero $\mathbf{q} \in \mathbb{N}_0^k$ such that $\varphi_{\mathbf{q}}$ is periodic. We say that an infinite path φ is *aperiodic* if it is neither periodic nor eventually periodic.

We say that Λ satisfies the *aperiodicity condition* (also referred to in the literature as *Condition (A)*) if, for every vertex $v \in \Lambda^0$, we can find an aperiodic path $\varphi \in \Lambda^\infty(v)$. We say that Λ is *cofinal* if, for every vertex $v \in \Lambda^0$ and every infinite path $\varphi \in \Lambda^\infty$, we can find $\lambda \in \Lambda$ and $\mathbf{n} \in \mathbb{N}_0^k$ such that $r(\lambda) = v$ and $s(\lambda) = \varphi(\mathbf{n})$.

The aperiodicity condition is a generalisation of the condition on 1-graphs that every cycle has an entrance. Similarly, cofinality is a generalisation of the property of 1-graphs that every vertex be reachable from somewhere on every infinite path.

LEMMA 4.2. *Consider the complete bipartite graph $\kappa = \kappa(\alpha, \beta)$, where $\alpha, \beta \geq 3$, and let $\Lambda(\kappa)$ be the corresponding 2-rank graph as constructed in the proof of Proposition 3.5. Then $\Lambda(\kappa)$ satisfies the aperiodicity condition.*

In order to get a feeling as to why this is true, consider Figure 8, which shows a representation of $\Lambda(\kappa(3, 3))$. Each vertex is labelled by a pointed tile from $\mathcal{S}(\kappa(3, 3))$, and since each tile is vertically adjacent to two others (and horizontally adjacent to
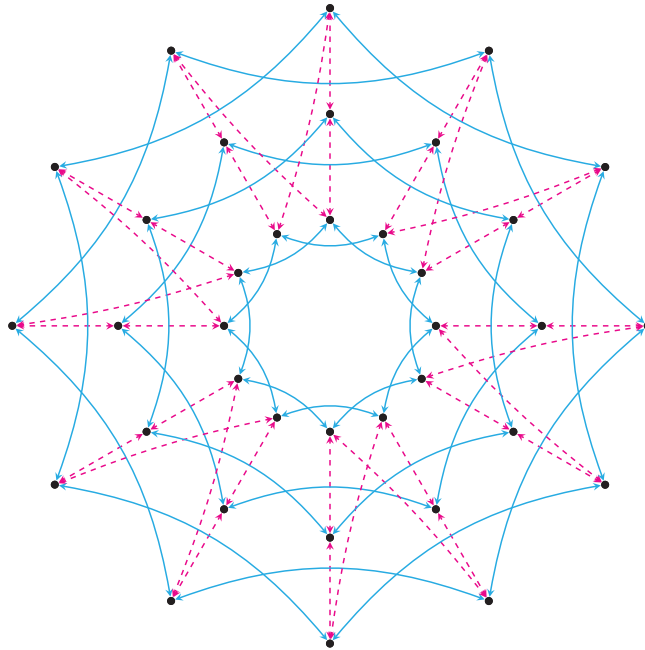
FIGURE 8. A representation of $\Lambda(\kappa(3, 3))$. It is always possible to exit a cycle.

two others), there are two solid arrows and two dashed arrows emanating from each vertex of $\Lambda(\kappa(3, 3))$. This suggests that, analogously to the 1-graph condition, we can always find an entrance to any cycle in Λ, namely by stopping mid-cycle at a vertex, and diverting the path down the second of the two available edges. Hence, as long as $\alpha, \beta \geq 3$, there is enough choice at each vertex to be able to exit a cycle.

PROOF. Firstly, write $\Lambda = \Lambda(\kappa)$, and let $A \in \Lambda^{\mathbf{0}}$ be an arbitrary vertex. We construct an aperiodic infinite path beginning from A in the following way.

Let $x : \Omega_1 \to \bigcup_{m \geq 0} \Lambda^{(m,0)}$ be a 1-graph morphism such that $x(0) = A$. The vertex A represents a pointed tile in $\mathcal{S}(\kappa)$, which is horizontally adjacent to $\beta - 1$ other pointed tiles. Hence, A is connected by bidirectional blue arrows to $\beta - 1$ other vertices in Λ. Choose two of these vertices, B_1 and B_2, say, and let x be such that

$$x(m, m) = \begin{cases} A & \text{if } m \text{ is even,} \\ B_1 & \text{if } m = r^2 + r + 1 \text{ for some } r \geq 1, \\ B_2 & \text{otherwise,} \end{cases}$$

for all $m \in \mathbb{N}_0$. Since x forms an aperiodic sequence, there is no $p \in \mathbb{Z}$ such that $x(m, m) = x(m + p, m + p)$ for all m, nor any $q \in \mathbb{N}$ such that x_q is periodic; hence,

x is an aperiodic path. Similarly, define $y : \Omega_1 \to \bigcup_{n \geq 0} \Lambda^{(0,n)}$ by

$$y(n, n) = \begin{cases} A & \text{if } n \text{ is even,} \\ C_1 & \text{if } n = s^2 + s + 1, \text{ for some } s \geq 1, \\ C_2 & \text{otherwise,} \end{cases}$$

for some vertices labelled by pointed tiles C_1, C_2 that are vertically adjacent to A. Then y is also an aperiodic path. By the UCE property, x and y uniquely determine an infinite path $\varphi : \Omega_2 \to \Lambda$ with $\varphi((m, 0), (m, 0)) = x(m, m)$ and $\varphi((0, n), (0, n)) = y(n, n)$.

Denote by D the unique pointed tile (other than A) adjacent to both B_1 and C_1. This cannot also be adjacent to B_2, nor to C_2, so $\varphi((m, n), (m, n)) = D$ precisely when $m = r^2 + r + 1$ and $n = s^2 + s + 1$, for some $r, s \geq 1$. As above, there is no $\mathbf{p} \in \mathbb{Z}^2$ such that $\varphi((m, n), (m, n)) = \varphi((m, n) + \mathbf{p}, (m, n) + \mathbf{p})$, nor any $\mathbf{q} \in \mathbb{N}_0^2$ such that $\varphi_{\mathbf{q}}$ is periodic. Since our initial vertex A was arbitrary, we are done. □

The following definitions are required for the rest of the section. For the reader who desires more detail, we recommend [11, Ch. 5].

DEFINITION 4.3. Let $\mathcal{A}$ be a unital $C^\star$-algebra, and let $\mathcal{B} \subset \mathcal{A}$ be a $C^\star$-subalgebra. We say that $\mathcal{B}$ is *hereditary* if, for all $a, b \in \mathcal{A}$, if $b \in \mathcal{B}$ and $a \leq b$, then $a \in \mathcal{B}$.

We say that $\mathcal{A}$ is *simple* if it has no nontrivial closed two-sided ideals.

If $\mathcal{A}$ is simple, we say that it is *purely infinite* if every nonzero hereditary $C^\star$-subalgebra of $\mathcal{A}$ contains a projection which is Murray–von Neumann equivalent to a proper subprojection of itself. Equivalently, $\mathcal{A}$ is purely infinite if every nonzero hereditary $C^\star$-subalgebra contains a projection equivalent to $\mathbf{1}$.

THEOREM 4.4 (Kumjian and Pask [6]). *Let Λ be a k-rank graph that satisfies the aperiodicity condition. Then the associated universal $C^\star$-algebra $C^\star(\Lambda)$ is simple if and only if Λ is cofinal.*

THEOREM 4.5 (Kumjian and Pask [6], Sims [13, Proposition 8.8]). *Let Λ be a k-rank graph that is cofinal and satisfies the aperiodicity condition. Suppose that, for every $v \in \Lambda^0$, we can find $\lambda \in \Lambda$ with $r(\lambda) = v$, and some cycle $\mu \in \Lambda$ with an entrance, such that $d(\mu) \neq \mathbf{0}$, and $s(\lambda) = r(\mu) = s(\mu)$. Then $C^\star(\Lambda)$ is purely infinite.*

PROPOSITION 4.6. *Consider $\kappa = \kappa(\alpha, \beta)$ for $\alpha, \beta \geq 3$, and let $\Lambda(\kappa)$ be the corresponding 2-rank graph. Then the $C^\star$-algebra $C^\star(\kappa)$ from Definition 3.7 is simple and purely infinite.*

PROOF. Firstly observe that $\Lambda(\kappa)$ is cofinal, since the 1-skeleton of $\Lambda(\kappa)$ is strongly connected. From Theorem 4.4 it follows that $C^\star(\kappa)$ is simple.

Now, let $A \in \Lambda(\kappa)^0$ be an arbitrary vertex. Since each edge of the 1-skeleton of $\Lambda(\kappa)$ is bidirectional, we can set μ to be a path that begins at A and traverses a single solid edge to some vertex B, before immediately returning to A. Then $d(\mu) = (2, 0)$, and, since $\alpha, \beta \geq 3$, B is the range of some other solid edge, and so μ is a cycle with an entrance. Then, by strong-connectedness, the conditions of Theorem 4.5 are satisfied, and $\Lambda(\kappa)$ is purely infinite. □

In [2] it is shown that, given a row-finite k-rank graph Λ with no sources, the $C^\star$-algebra $C^\star(\Lambda)$ is separable, nuclear, unital, and satisfies the universal coefficient theorem. Furthermore, we have shown in Proposition 4.6 that, given a complete bipartite graph $\kappa = \kappa(\alpha, \beta)$ with $\alpha, \beta \geq 3$, the $C^\star$-algebra $C^\star(\kappa)$ associated to its 2-rank graph is simple and purely infinite. Hence, we can deduce the following result.

COROLLARY 4.7. *Consider the complete bipartite graph $\kappa = \kappa(\alpha, \beta)$ for $\alpha, \beta \geq 3$, with corresponding 2-rank graph $\Lambda(\kappa)$. Then the isomorphism class of the associated $C^\star$-algebra $C^\star(\Lambda(\kappa))$ is completely determined by the K-groups $K_0(C^\star(\kappa)) = K_1(C^\star(\kappa))$ and the position of the class of the identity in $K_0(C^\star(\kappa))$.*

5. Unpointed tiles

There is an alternative way we could have defined the adjacency matrices above, giving rise to a different 2-rank graph structure.

Define an *unpointed tile system* $(G, U, V, \mathcal{S}')$ in the same way as in Definition 2.4, but replacing $\mathcal{S} = \mathcal{S}(G)$ with the set of unpointed tiles $\mathcal{S}' = \mathcal{S}'(G)$. We show that analogues of the results in Section 3 also hold for unpointed tile systems.

DEFINITION 5.1. Let $(G, U, V, \mathcal{S}')$ be an unpointed tile system, and let $A', B' \in \mathcal{S}'$ be unpointed tiles, that is, equivalence classes of some pointed tiles $A, B \in \mathcal{S}$ (see Definition 2.3). Recall the matrices M_1, M_2 from Definition 3.1. We define functions $M_1', M_2' : \mathcal{S}' \times \mathcal{S}' \to \{0, 1\}$ as follows:

$$M_1'(A', B') = \begin{cases} 1 & \text{if } M_1(A_\bullet, B_\bullet) = 1, \text{ for some } A_\bullet \sim A, \ B_\bullet \sim B, \\ 0 & \text{otherwise,} \end{cases}$$

$$M_2'(A', B') = \begin{cases} 1 & \text{if } M_2(A_\bullet, B_\bullet) = 1, \text{ for some } A_\bullet \sim A, B_\bullet \sim B, \\ 0 & \text{otherwise.} \end{cases}$$

We define adjacency matrices M_1', M_2' accordingly.

PROPOSITION 5.2. *Consider the complete bipartite graph $\kappa = \kappa(\alpha, \beta)$ on $\alpha \geq 2$ white and $\beta \geq 2$ black vertices, and let $(\kappa, U, V, \mathcal{S}'(\kappa))$ be an unpointed tile system. Then the corresponding adjacency matrices M_1' and M_2' commute, and $(\kappa, U, V, \mathcal{S}'(\kappa))$ has the UCE property. Hence, $(\kappa, U, V, \mathcal{S}'(\kappa))$ has a 2-rank graph structure.*

PROOF. Given two unpointed tiles $A', B' \in \mathcal{S}'(\kappa)$, consider their respective sets of pointed tiles $\mathcal{A}, \mathcal{B} \in \mathcal{S}(\kappa)$ as defined in Definition 5.1. Notice that $M_1'(A', B') = 1$ if and only if, for *every* $A_\bullet \in \mathcal{A}$, we can find some $B_\bullet \in \mathcal{B}$ such that $M_1(A_\bullet, B_\bullet) = 1$. The

same is true for M'_2. Write $A' = (u^1_i, v^1_j, u^2_i, v^2_j)$, and define sets

$$X_A := \{T \in \mathcal{S}'(\kappa) \mid M'_1(A, T) = 1\}, \quad Y_A := \{T \in \mathcal{S}'(\kappa) \mid M'_2(A, T) = 1\}.$$

Then X_A contains precisely those tiles of the form $(u^1_k, v^1_j, u^2_k, v^2_j)$, where $k \neq i$, and Y_A only those of the form $(u^1_i, v^1_l, u^2_i, v^2_l)$, where $l \neq j$. The proof then proceeds in a similar fashion to that of Proposition 3.3, and the 2-rank graph structure follows immediately from [6, Section 6] as in Theorem 3.5. □

We write $\Lambda'(\kappa)$ for the 2-rank graph induced by the adjacency matrices M'_1 and M'_2. It is not difficult to verify that $\Lambda'(\kappa)$ is row-finite, with a finite vertex set and no sources. Hence, we can apply Theorem 3.8, and derive the following result.

THEOREM 5.3 (K-groups for unpointed tile systems). *Let* $a, b \geq 0$, *and let* $\kappa(a + 2, b + 2)$ *be the complete bipartite graph on* $a + 2$ *white and* $b + 2$ *black vertices. Again, without loss of generality, we can assume that* $a \leq b$. *Write* $C^\star(\kappa) := C^\star(\Lambda'(\kappa))$. *Then, for* $\epsilon = 0, 1$, *we have the following assertions.*

(1) *If* $a = b = 0$, *then* $K_\epsilon(C^\star(\kappa(a + 2, b + 2))) = K_\epsilon(C^\star(\kappa(2, 2))) \cong \mathbb{Z}^2$.
(2) *If* $a = 0$ *and* $b \geq 1$, *then*

$$K_\epsilon(C^\star(\kappa(a + 2, b + 2))) \cong (\mathbb{Z}/2)^b \oplus (\mathbb{Z}/(2b)).$$

(3) *If* $a, b \geq 1$, *then*

$$K_\epsilon(C^\star(\kappa(a + 2, b + 2))) \cong (\mathbb{Z}/2)^{(a+1)(b+1)-1} \oplus (\mathbb{Z}/2g),$$

where $g := \gcd(a, b)$.

PROOF. Again, we start by proving (3) as the first two cases follow. Write $\alpha := a + 2$, $\beta := b + 2$, and let $\alpha, \beta \geq 3$. For $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$, write A'_{ij} for the unpointed tile $(u^1_i, v^1_j, u^2_i, v^2_j) \in \mathcal{S}'(\kappa)$. Then

$$\mathrm{coker} = \mathrm{coker}(\mathbf{1} - (M'_1)^T, \mathbf{1} - (M'_2)^T)$$
$$= \left\langle A'_{ij} \in \mathcal{S}'(\kappa) \middle| A'_{ij} = \sum_{T' \in \mathcal{S}'(\kappa)} M'_1(A'_{ij}, T') \cdot T' = \sum_{T' \in \mathcal{S}'(\kappa)} M'_2(A'_{ij}, T') \cdot T' \right\rangle. \quad (5\text{-}1)$$

Fix $p \in \{1, \ldots, \alpha\}$, $q \in \{1, \ldots, \beta\}$, and notice that:

- $M'_1(A'_{pq}, T') = 1$ if and only if $T' = A'_{iq}$ for some $i \neq p$;
- $M'_2(A'_{pq}, T') = 1$ if and only if $T' = A'_{pj}$ for some $j \neq q$.

Hence, the relations of (5-1) are equivalent to $A'_{pq} = \sum_{i \neq p} A'_{iq} = \sum_{j \neq q} A'_{pj}$. Define

$$J_{pq} := \left(\sum_{i=2}^{\alpha} A'_{iq} \right) - A'_{pq} \quad \text{and} \quad I_{pq} := \left(\sum_{j=2}^{\beta} A'_{pj} \right) - A'_{pq}$$

for $p, q \geq 2$. Then

$$
\begin{aligned}
2J_{pq} &= 2\Big(\sum_{i=2}^{\alpha} A'_{iq}\Big) - 2A'_{pq} \\
&= 2(A'_{2q} + \cdots + A'_{\alpha q} - A'_{pq}) + A'_{1q} - A'_{1q} \\
&= (A'_{1q} + A'_{2q} + \cdots + A'_{\alpha q} - A'_{pq}) + (-A'_{1q} + A'_{2q} + \cdots + A'_{\alpha q}) - A'_{pq} \\
&= A'_{pq} + 0 - A'_{pq} = 0,
\end{aligned}
$$

and, similarly, $2I_{pq} = 0$. Now $J_{pq} = 0$ or $I_{pq} = 0$ only if $A'_{pq} = A'_{1q}$ or $A'_{pq} = A'_{p1}$, respectively. But, since $\alpha, \beta \geq 3$, these equivalences are not relations listed at (5-1), and so $\mathrm{ord}(J_{pq}) = \mathrm{ord}(I_{pq}) = 2$. Notice that we can write each A'_{1q} and A'_{p1} in terms of the other A'_{ij} for $p, q \geq 2$; hence, we can remove these from the list of generators by a sequence of Tietze transformations.

Also notice that $A'_{2q} = J_{2q} - \sum_{i=3}^{\alpha} A'_{iq}$. Proceeding inductively, we can write each A'_{pq} in terms of the J_{iq} and the A'_{iq} for $i > p$. Similarly, we can express each A'_{pq} in terms of the I_{pj} and the A'_{pj} for $j > q$. Hence, we can rewrite the generators of coker as A'_{11}, I_{pq}, J_{pq} for $p, q \geq 2$. But $A'_{11} = -(A'_{p1} + J_{p1}) = -(A'_{1q} + I_{1q})$ for all $p, q \geq 2$, so

$$
(\alpha - 2)A'_{11} = -\sum_{i=3}^{\alpha}(A'_{i1} + J_{i1}) = -\Big(J_{21} + \sum_{i=3}^{\alpha} J_{i1}\Big),
$$

and $2(\alpha - 2)A'_{11} = 0$. Similarly, $2(\beta - 2)A'_{11} = 0$, and, hence, $2gA'_{11} = 0$, where $g := \gcd(\alpha - 2, \beta - 2)$.

Observe that, since I_{pq} is defined in terms of the A'_{pj}, and each A'_{pj} can be written in terms of the J_{ij}, we can remove the I_{pq} from the list of generators of coker. Finally, we can rewrite (5-1) as

$$
\begin{aligned}
\mathrm{coker} = \langle J_{2q}, J_{p2}, J_{pq}, A'_{11} \mid 2J_{2q} = 2J_{p2} = 2J_{pq} = 2gA'_{11} = 0, \\
\text{for } 3 \leq p \leq \alpha, 3 \leq q \leq \beta \rangle,
\end{aligned}
$$

and, after substituting $a = \alpha - 2$, $b = \beta - 2$, this gives a presentation for $(\mathbb{Z}/2)^{(a+1)(b+1)-1} \oplus (\mathbb{Z}/2g)$. There is no torsion-free part, so this proves (3). If $\alpha = 2$, then $A'_{1q} = A'_{2q}$ for all $1 \leq q \leq \beta$, so we can write

$$
\mathrm{coker} = \Big\langle A'_{1q} \Big| A'_{1q} = \sum_{j \neq q} A'_{1j}, \text{ for } 1 \leq q \leq \beta \Big\rangle.
$$

We adjust the proof above accordingly to obtain the result of (2). Finally, in case (1), where $\alpha = \beta = 2$, we have $A'_{11} = A'_{12} = A'_{21} = A'_{22}$ with no further relations, so that $\mathrm{coker} = \langle A'_{11} \rangle \cong \mathbb{Z}$, and the result follows from Theorem 3.8. $\qquad \square$

THEOREM 5.4. *Let $\alpha, \beta \geq 3$, let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on α white and β black vertices, and write $g := \gcd(\alpha - 2, \beta - 2)$. Then the order of the class of the identity $[\mathbf{1}]$ in $K_0(C^\star(\Lambda'(\kappa)))$ is equal to g if g is odd, and $g/2$ if g is even.*

PROOF. We use the same notation as in the proof of Theorem 5.3. As with Theorem 3.12, we know that the order of $[\mathbf{1}]$ in $K_0(C^\star(\kappa))$ is equal to the order of the sum of all tiles A'_{ij}. We write Σ for this sum.

We have $A'_{pq} = \sum_{i \neq p} A'_{iq} = \sum_{j \neq q} A'_{pj}$, and so $\Sigma = (\alpha - 1)\Sigma = (\beta - 1)\Sigma$. From this, it follows that $g\Sigma = 0$. We also have $A'_{pq} = \sum_{i \neq p} \sum_{j \neq q} A'_{ij}$, so

$$\Sigma = A'_{pq} + \sum_{i \neq p} A'_{iq} + \sum_{j \neq q} A'_{pj} + \sum_{i \neq p} \sum_{j \neq q} A'_{ij} = 4A'_{pq}$$

for any fixed p, q. But $2gA'_{pq} = 0$, and so, if $g = 2h$ for some integer h, then $h\Sigma = 4hA'_{pq} = 0$, and we are done. □

The proof of the next proposition is analogous to that of Proposition 4.6.

PROPOSITION 5.5. *Consider the complete bipartite graph $\kappa = \kappa(\alpha, \beta)$ for $\alpha, \beta \geq 3$, and the associated 2-rank graph $\Lambda'(\kappa)$. Then the isomorphism class of the universal $C^\star$-algebra $C^\star(\Lambda'(\kappa))$ is completely determined by its K-theory and the position of the class of the identity in $K_0(C^\star(\Lambda'(\kappa)))$.*

6. The homology of a tile complex

THEOREM 6.1. *Let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on $\alpha \geq 2$ white and $\beta \geq 2$ black vertices, let $(\kappa, U, V, \mathcal{S}'(\kappa))$ be an unpointed tile system, and let $TC(\kappa)$ be its associated tile complex. Then the homology groups of $TC(\kappa)$ are given by*

$$H_n(TC(\kappa)) \cong \begin{cases} 0 & \text{for } n = 0, \\ \mathbb{Z}^{\alpha+\beta-2} & \text{for } n = 1, \\ \mathbb{Z}^{(\alpha-1)(\beta-1)} & \text{for } n = 2, \\ 0 & \text{for } n \geq 3. \end{cases}$$

PROOF. As $TC(\kappa)$ is a path-connected, two-dimensional CW-complex by construction, $H_n(TC(\kappa)) \cong 0$ for $n = 0$ and $n \geq 3$.

The proof is based on that of [7, Proposition 3]. The boundary of each square in $TC(\kappa)$ is given by an element of $\mathcal{S}'(\kappa)$; write these elements as $(u_i^1, v_j^1, u_i^2, v_j^2)$. By construction, $TC(\kappa)$ has four vertices: each is the origin of all directed edges labelled u_i^1, v_j^1, u_i^2 and v_j^2. Each tile is homotopy equivalent to a point; pick tile $(u_1^1, v_1^1, u_1^2, v_1^2)$ and contract it, thereby identifying the four vertices. Call the resulting tile complex $TC_1(\kappa)$. This is a two-dimensional CW-complex whose edges are loops, and whose 2-cells comprise for $2 \leq i \leq \alpha$, $2 \leq j \leq \beta$

- $(\alpha - 1)(\beta - 1)$-many unpointed tiles $A'_{ij} = (u_i^1, v_j^1, u_i^2, v_j^2)$;
- $(\alpha - 1)$-many 2-gons X'_i with boundaries described analogously by (u_i^1, u_i^2);
- $(\beta - 1)$-many 2-gons Y'_j with boundaries described by (v_j^1, v_j^2).

Consider the chain complex associated to $TC_1(\kappa)$:

$$\cdots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

Since $TC_1(\kappa)$ is two-dimensional and has one vertex, this boils down to

$$0 \xrightarrow{0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{0} 0,$$

and so $H_1(TC_1(\kappa)) \cong C_1/\operatorname{im}(\partial_2)$ and $H_2(TC_1(\kappa)) \cong \ker(\partial_2)$. We have $\partial_2(A'_{ij}) = u_i^1 + v_j^1 + u_i^2 + v_j^2$, $\partial_2(X'_i) = u_i^1 + u_i^2$, and $\partial_2(Y'_j) = v_j^1 + v_j^2$. Clearly $\ker(\partial_2)$ is generated by $\{A'_{ij} - X'_i - Y'_j \mid 2 \le i \le \alpha, 2 \le j \le \beta\}$, which implies that $\ker(\partial_2) \cong \mathbb{Z}^{(\alpha-1)(\beta-1)}$.

Similarly, an Abelian group presentation for $H_1(TC_1(\kappa))$ is given by

$$H_1(TC_1(\kappa)) \cong \langle u_i^1, v_j^1, u_i^2, v_j^2 \mid u_i^1 + v_j^1 + u_i^2 + v_j^2 = u_i^1 + u_i^2$$
$$= v_j^1 + v_j^2 = 0 \text{ for } 2 \le i \le \alpha, 2 \le j \le \beta\rangle,$$

which, after substituting $u_i^2 = -u_i^1$ and $v_j^2 = -v_j^1$, gives

$$H_1(TC_1(\kappa)) \cong \langle u_i^1, v_j^1 \mid 2 \le i \le \alpha, 2 \le j \le \beta\rangle.$$

This is a presentation for $\mathbb{Z}^{\alpha+\beta-2}$, and, since $TC_1(\kappa)$ is homotopy equivalent to $TC(\kappa)$, we are done. □

7. Pointed and unpointed $2t$-gon systems

In this section we suggest generalisations of the methods above for constructing $C^\star$-algebras associated to $2t$-gon systems, both for even and arbitrary $t \ge 1$.

When $t = 2$, we have an innate idea of what it means for two $2t$-gons to be 'stackable' – functions we called horizontal and vertical adjacency in Definition 3.1. We extend this notion to all even $t \ge 2$ in as natural a way as possible. The following definition directly generalises those at the beginning of Section 2.

DEFINITION 7.1. Let G be a connected bipartite graph on α white and β black vertices. Let U, V be sets with $|U| = 2t\alpha$, $|V| = 2t\beta$, gifted with fixed-point-free involutions $u \mapsto \bar{u}$, $v \mapsto \bar{v}$, respectively. Using U and V, construct the $2t$-polyhedron $P(G)$ from Theorem 2.2, which comprises $2t$-gons $A_e = [u_p^1, v_q^1, \ldots, u_p^t, v_q^t]$ and has G as its link at each vertex. Write $\mathcal{S}'(G) := \{A_e \mid e \in E(G)\}$ for the set of $2t$-gons comprising $P(G)$. We call elements of $\mathcal{S}'_t(G)$ *unpointed $2t$-gons*, and denote them by $A_e = (x_1, y_1, \ldots, x_t, y_t)$.

Analogously to Section 2, we write $[x_1, y_1, \ldots, x_t, y_t]$ for a *pointed $2t$-gon*, that is, a $2t$-gon labelled anticlockwise and starting from a distinguished basepoint by the sequence $x_1, y_1, \ldots, x_t, y_t$, for some $x_i \in U$, $y_i \in V$. Write $\mathcal{S}_t = \mathcal{S}_t(G)$ for the set of $2t\alpha\beta$ pointed $2t$-gons. We call the tuple $(G, U, V, \mathcal{S}_t)$ a *$2t$-gon system*. Similarly, we call the tuple $(G, U, V, \mathcal{S}'_t)$ an *unpointed $2t$-gon system*.
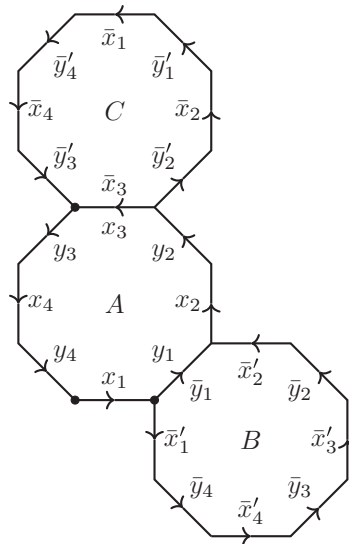
FIGURE 9. U-, V-adjacency: Consider the pointed octagons $A = [x_1, y_1, \ldots, x_4, y_4]$, $B = [\bar{x}'_1, \bar{y}_4, \ldots, \bar{x}'_2, \bar{y}_1]$, and $C = [\bar{x}_3, \bar{y}'_2, \ldots, \bar{x}_4, \bar{y}'_3]$ in $\mathcal{S}_4$. We say that A and B are V-adjacent, and A and C are U-adjacent. There is a unique octagon $D = [x'_3, y'_3, \ldots, x'_2, y'_2]$ that is both U-adjacent to B and V-adjacent to C.

Consider the adjacency matrices M_1 and M_2 from Definition 3.1. We can view two pointed tiles (4-gons) $A = [x_1, y_1, x_2, y_2]$ and B as being horizontally adjacent, that is, $M_1(A, B) = 1$, if and only if, after reflecting A through an axis connecting the midpoints of x_1 and x_2, and then replacing x_1, x_2 by some $x'_1 \neq x_1$, $x'_2 \neq x_2$ respectively, we can obtain B. Likewise, if and only if we can obtain B by reflecting A through an axis joining the midpoints of the y edges, and then changing the labels of those edges, do we say that A and B are vertically adjacent.

DEFINITION 7.2. Let t be an even integer, let $(G, U, V, \mathcal{S}_t)$ be a $2t$-gon system, and let $A = [x_1, y_1, \ldots, x_t, y_t] \in \mathcal{S}_t$ be a pointed $2t$-gon.

Reflect A through an axis joining the midpoints of the sides labelled x_1 and $x_{(t/2)+1}$ to obtain a new pointed $2t$-gon $[\bar{x}_1, \bar{y}_t, \bar{x}_t, \bar{y}_{t-1}, \ldots, \bar{x}_2, \bar{y}_1]$. A pointed $2t$-gon $B \in \mathcal{S}_t$ is V-adjacent to A whenever $B = [\bar{x}'_1, \bar{y}_t, \bar{x}'_t, \bar{y}_{t-1}, \ldots, \bar{x}'_2, \bar{y}_1]$ for some $x'_i \neq x_i$.

Similarly, reflect A so that $x_1 \mapsto \bar{x}_{(t/2)+1}$; we obtain a new pointed $2t$-gon

$$[\bar{x}_{(t/2)+1}, \bar{y}_{t/2}, \bar{x}_{t/2}, \ldots, \bar{y}_1, \bar{x}_1, \bar{y}_t, \bar{x}_t, \ldots, \bar{x}_{(t/2)+2}, \bar{y}_{(t/2)+1}]. \tag{7-1}$$

We say that a pointed $2t$-gon $B \in \mathcal{S}_t$ is U-adjacent to A if B is of the form (7-1), but with all elements y_i replaced with some $y'_i \neq y_i$ (Figure 9).

We define the U- and V-adjacency matrices, M_U and M_V, respectively, to be the $2t\alpha\beta \times 2t\alpha\beta$ matrices with ABth entry 1 if A and B are U-adjacent (respectively, V-adjacent), and 0 otherwise.

PROPOSITION 7.3. *Let t be even, and $(\kappa, U, V, \mathcal{S}_t(\kappa))$ be a $2t$-gon system with adjacency matrices M_U, M_V. Then these matrices commute, and $(\kappa, U, V, \mathcal{S}_t(\kappa))$ has the UCE property. Hence, $(\kappa, U, V, \mathcal{S}_t(\kappa))$ has a 2-rank graph structure.*

PROOF. Consider the pointed $2t$-gon $A = [u_i^1, v_j^1, \ldots, u_i^t, v_j^t] \in \mathcal{S}_t(\kappa)$; those $2t$-gons corresponding to its reflections and rotations are treated similarly. Then a pointed $2t$-gon B is V-adjacent to A if and only if $B = [\bar{u}_k^1, \bar{v}_j^t, \ldots, \bar{u}_k^2, \bar{v}_j^1]$ for some $k \neq i$. Suppose B is such a $2t$-gon V-adjacent to A; then a pointed $2t$-gon D is U-adjacent to B if and only if

$$D = [u_k^{(t/2)+1}, v_l^{(t/2)+1}, \ldots, u_k^t, v_l^t, u_k^1, v_k^1, \ldots, u_k^{t/2}, v_l^{t/2}] \tag{7-2}$$

for some $l \neq j$. Likewise, C is U-adjacent to A if and only if

$$C = [\bar{u}_i^{(t/2)+1}, \bar{v}_l^{t/2}, \ldots, \bar{u}_i^1, \bar{v}_l^t, \ldots, \bar{u}_i^{(t/2)+2}, \bar{v}_l^{(t/2)+1}]$$

for some $l \neq j$. Clearly, if C is such a $2t$-gon, then D is V-adjacent to C if and only if it is of the form (7-2). Exactly one such D exists in $\mathcal{S}_t(\kappa)$; hence, M_U and M_V commute. Then $(\kappa, U, V, \mathcal{S}_t(\kappa))$ has the UCE property, and the 2-rank graph structure follows from [6, Section 6]. □

Recall the 2-rank graph $\Lambda(\kappa)$ induced from a tile system and its adjacency matrices M_1, M_2 in Section 3, and recall its associated universal C^*-algebra $C^\star(\Lambda)$ from Definition 3.7. We similarly write $\Lambda_t(\kappa)$ for the 2-rank graph induced from the U- and V-adjacency matrices M_U and M_V, and observe that $\Lambda_t(\kappa)$ is row-finite, with a finite vertex set and no sources. Hence, from Theorem 3.8 we can deduce the following theorem.

THEOREM 7.4 (K-groups for pointed $2t$-gon systems, t even). *Let $\alpha, \beta \geq 2$, let $t \geq 2$ be even, and let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on α white and β black vertices. Then*

$$K_\epsilon(C^\star(\Lambda_t(\kappa))) \cong (K_\epsilon(C^\star(\Lambda(\kappa))))^{t/2}$$

for $\epsilon = 0, 1$.

PROOF. Fix t and assume, without loss of generality, that $\alpha \leq \beta$. Analogously to the proof of Theorem 3.10, we denote the pointed $2t$-gons in $\mathcal{S}_t(\kappa)$ as follows:

- $(A_r)_{ij} = [u_i^r, v_j^r, \ldots, u_i^t, v_j^t, u_i^1, v_j^1, \ldots, u_i^{r-1}, v_j^{r-1}]$;
- $(B_r)_{ij} = [\bar{u}_i^r, \bar{v}_j^{r-1}, \ldots, \bar{u}_i^1, \bar{v}_j^t, \ldots, \bar{u}_i^{r+1}, \bar{v}_j^r]$;
- $(C_r)_{ij} = [\bar{u}_i^{(t/2)+r}, \bar{v}_j^{(t/2)+r-1}, \ldots, \bar{u}_i^1, \bar{v}_j^t, \ldots, \bar{u}_i^{(t/2)+r+1}, \bar{v}_j^{(t/2)+r}]$;
- $(D_r)_{ij} = [u_i^{(t/2)+r}, v_j^{(t/2)+r}, \ldots, u_i^t, v_j^t, u_i^1, v_j^1, \ldots, u_i^{(t/2)+r-1}, v_j^{(t/2)+r-1}]$

for $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$, $1 \leq r \leq t/2$. Here, the addition in the subscript indices is defined modulo t. Note that each $S \in \mathcal{S}_t(\kappa)$ takes one of the above forms. Then

$$\mathrm{coker}(\mathbf{1} - M_U^T, \mathbf{1} - M_V^T) = \Big\langle (A_r)_{pq}, (B_r)_{pq}, (C_r)_{pq}, (D_r)_{pq} \Big|$$

$$(A_r)_{pq} = \sum_{i \neq p} (B_r)_{iq} = \sum_{j \neq q} (C_r)_{pj},$$

$$(B_r)_{pq} = \sum_{i \neq p} (A_r)_{iq} = \sum_{j \neq q} (D_r)_{pj},$$

$$(C_r)_{pq} = \sum_{i \neq p} (D_r)_{iq} = \sum_{j \neq q} (A_r)_{pj},$$

$$(D_r)_{pq} = \sum_{i \neq p} (C_r)_{iq} = \sum_{j \neq q} (B_r)_{pj},$$

$$\text{for } 1 \leq p \leq \alpha, 1 \leq q \leq \beta, \text{ and } 1 \leq r \leq t/2 \Big\rangle.$$

But, comparing this to (3-1), we see this is precisely a presentation for the direct sum of $t/2$ copies of $\mathrm{coker}(I - M_1^T, I - M_2^T)$ as in Theorem 3.10, and the result follows. □

THEOREM 7.5. *Let $\alpha, \beta \geq 3$, let $t \geq 2$ be even, and let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on α white and β black vertices. Then the order of the class of the identity $[\mathbf{1}]$ in $K_0(C^\star(\Lambda_t(\kappa)))$ is equal to $g := \gcd(\alpha - 2, \beta - 2)$.*

Furthermore, the isomorphism class of $C^\star(\Lambda_t(\kappa))$ is completely determined by the K-groups from Theorem 7.4 and the order of $[\mathbf{1}]$ in K_0.

PROOF. The result follows from Theorems 3.12 and 7.4, and considerations similar to those in the proof of Proposition 4.6. □

If we extend the concept of U- and V-adjacency from Definition 7.2 in the obvious way, we can obtain a generalisation of Section 5 for unpointed $2t$-gon systems of complete bipartite graphs. Write $\Lambda'_t(\kappa)$ for the induced 2-rank graph. We realise that the proof of Theorem 5.3 does not depend on the number of sides $2t$ of the $2t$-gons; nor do the K-groups associated with $\Lambda'_t(\kappa)$.

COROLLARY 7.6 (to Theorem 5.3: K-groups for unpointed $2t$-gon systems). *Let $\alpha, \beta \geq 2$, and let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on α white and β black vertices. Then*

$$K_\epsilon(C^\star(\Lambda'_t(\kappa))) \cong K_\epsilon(C^\star(\Lambda'(\kappa)))$$

for $\epsilon = 0, 1$, and all $t \geq 1$.

PROPOSITION 7.7. *Let $\alpha, \beta \geq 3$, let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph, and write $g := \gcd(\alpha - 2, \beta - 2)$. Then, for all $t \geq 1$, the order of the class of the identity $[\mathbf{1}]$ in $K_0(C^\star(\Lambda'_t(\kappa)))$ is equal to g if g is odd, and $g/2$ if g is even.*

Furthermore, the isomorphism class of $C^\star(\Lambda'_t(\kappa))$ is completely determined by the K-groups in Corollary 7.6 and the order of $[\mathbf{1}]$ in K_0.

COROLLARY 7.8 (to Theorem 6.1). *Let $(\kappa, U, V, \mathcal{S}'_t(\kappa))$ be an unpointed $2t$-gon system, and let $P(\kappa)$ be its associated $2t$-polyhedron. Then the homology groups of $P(\kappa)$ do not depend on t, that is,*

$$H_n(P(\kappa)) \cong \begin{cases} 0 & \text{for } n = 0, \\ \mathbb{Z}^{\alpha+\beta-2} & \text{for } n = 1, \\ \mathbb{Z}^{(\alpha-1)(\beta-1)} & \text{for } n = 2, \\ 0 & \text{for } n \geq 3. \end{cases}$$

7.1. Questions on canonicality. Corollary 7.6 gives us a collection of K-groups corresponding to systems of $2t$-gons with an arbitrary even number of sides $2t$, whereas in the pointed case, Theorem 7.4 insists on $2t$ being divisible by 4. This is due to how we define adjacency in each instance: in the $2t$-polyhedron $P(\kappa)$, each face is adjacent to every other, and since the number of faces is not dependent on t, nor are the U- and V-adjacency matrices in an unpointed $2t$-gon system.

Adjacency in the pointed case is more difficult to define canonically. When $t = 2$, and we are dealing with tiles, there is an obvious pair of adjacency functions. We extend these in Definition 7.2, thinking of two $2t$-gons as adjacent if we can reflect one horizontally or vertically in order to obtain the form of the other. This works since horizontal and vertical reflections commute, and so the $2t$-gon system has the UCE property. If t is not even, then there are no two distinct reflections of $2t$-gons which commute, and preserve the structure of pointed $2t$-gons. We must pick the same two reflections for both adjacency functions, or else some combination of rotations and identity transformations. None of these options is a direct extension of our horizontal and vertical adjacency functions from Section 3, and so there is no natural choice.

We suggest that the following definitions of U- and V-adjacency for pointed $2t$-gons are the most intuitive for $t \geq 3$, based on the idea that adjacent $2t$-gons should have opposite orientations. They do not, however, generalise the tile systems from Sections 2–5, themselves being the most natural constructions when $t = 2$. Because of this, the previous constructions are the main focus of this paper.

DEFINITION 7.9. Let $t \geq 1$ be a fixed arbitrary integer, let $(G, U, V, \mathcal{S}_t)$ be a $2t$-gon system, and let $A = [x_1, y_1, \ldots, x_t, y_t] \in \mathcal{S}_t$ be a pointed $2t$-gon.

A pointed $2t$-gon $B \in \mathcal{S}_t$ is *V^*-adjacent* to A if and only if $B = [\bar{x}'_1, \bar{y}_t, \ldots, \bar{x}'_2, \bar{y}_1]$ for some $x'_i \neq x_i$.

Similarly, we say that a pointed $2t$-gon $C \in \mathcal{S}_t$ is *U^*-adjacent* to A if and only if $C = [\bar{x}_1, \bar{y}'_t, \ldots, \bar{x}_2, \bar{y}'_1]$ for some $y'_i \neq y_i$. We define the *U^*- and V^*-adjacency matrices* M^*_U and M^*_V, respectively, as above.

The proof of the following proposition is almost identical to that of Proposition 7.3, together with Proposition 3.5. From this, along with Theorem 3.8, we can deduce Theorem 7.11.

PROPOSITION 7.10. *Let $(\kappa, U, V, \mathcal{S}_t(\kappa))$ be a 2t-gon system with adjacency matrices M_U^*, M_V^*. Then $(\kappa, U, V, \mathcal{S}_t(\kappa))$ induces a 2-rank graph $\Lambda_t^*(\kappa)$, which is row-finite, with a finite vertex set and no sources.*

THEOREM 7.11 (K-groups for pointed 2t-gon systems, t arbitrary). *Let $a, b \geq 0$, let $t \geq 1$, and let $\kappa = \kappa(a + 2, b + 2)$ be the complete bipartite graph on $a + 2$ white and $b + 2$ black vertices. Without loss of generality, we assume that $a \leq b$. Then, for $\epsilon = 0, 1$, we have the following assertions.*

(1)   *If $a = b = 0$, then $K_\epsilon(C^\star(\Lambda_t^*(\kappa))) \cong \mathbb{Z}^{4t}$.*
(2)   *If $b \geq 1$ and a, b are coprime, then $K_\epsilon(C^\star(\Lambda_t^*(\kappa))) \cong \mathbb{Z}^{2t(a+1)(b+1)}$.*
(3)   *If $b \geq 1$ and a, b are not coprime, then*

$$K_\epsilon(C^\star(\Lambda_t^*(\kappa))) \cong \mathbb{Z}^{2t(a+1)(b+1)} \oplus (\mathbb{Z}/g)^t,$$

where $g := \gcd(a, b)$.

PROOF. The proof follows the same lines as those of Theorems 3.10, 5.3, and 7.4. Write $\alpha := a + 2$, $\beta := b + 2$, and let $\beta \geq 3$. We denote the pointed 2t-gons in $\mathcal{S}_t(\kappa)$ by

- $(A_r)_{ij} := [u_i^r, v_j^r, \ldots, u_i^t, v_j^t, u_i^1, v_j^1, \ldots, u_i^{r-1}, v_j^{r-1}]$,
- $(B_r)_{ij} := [\bar{u}_i^r, \bar{v}_j^{r-1}, \ldots, \bar{u}_i^1, \bar{v}_j^t, \ldots, \bar{u}_i^{r+1}, \bar{v}_j^r]$

for $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$, $1 \leq r \leq t$, and with addition in superscript indices defined modulo t. Observe that each $S \in \mathcal{S}_t(\kappa)$ is either of the form $(A_r)_{ij}$ or $(B_r)_{ij}$. Then

$$\text{coker} = \left\langle (A_r)_{pq}, (B_r)_{pq} \right|$$
$$(A_r)_{pq} = \sum_{i \neq p} (B_r)_{iq} = \sum_{j \neq q} (B_r)_{pj},$$
$$(B_r)_{pq} = \sum_{i \neq p} (A_r)_{iq} = \sum_{j \neq q} (A_r)_{pj},$$
$$\text{for } 1 \leq p \leq \alpha, 1 \leq q \leq \beta, \text{ and } 1 \leq r \leq t \Big\rangle,$$

where $\text{coker} := \text{coker}(\mathbf{1} - (M_U^*)^T, \mathbf{1} - (M_V^*)^T)$. As in the proof of Theorem 3.10, define $(J_r)_q := \sum_{i=1}^\alpha (A_r)_{iq}$, and $(I_r)_p := \sum_{j=1}^\beta (A_r)_{pj}$. Through a sequence of Tietze transformations, and using some observations from previous proofs, we see that the above presentation, and hence coker, is equivalent to

$$\left\langle (A_r)_{pq} \right| (A_r)_{pq} = \sum_{i \neq p} \sum_{k \neq i} (A_r)_{kq} = \sum_{j \neq q} \sum_{l \neq j} (A_r)_{pl}, \sum_{i \neq p} (A_r)_{iq} = \sum_{j \neq q} (A_r)_{pj} \right\rangle$$
$$= \left\langle (A_r)_{pq} \right| (\alpha - 2)(J_r)_q = (\beta - 2)(I_r)_p = 0, \sum_{i \neq p} (A_r)_{iq} = \sum_{j \neq q} (A_r)_{pj} \right\rangle$$
$$= \left\langle (A_r)_{pq} \right| (\alpha - 2)(J_r)_q = (\beta - 2)(I_r)_p = 0, (J_r)_q = (I_r)_p \text{ for all } p, q \right\rangle.$$

We can rewrite each $(A_r)_{i1}$ and $(A_r)_{1j}$ in terms of the other $(A_r)_{ij}$, the $(J_r)_q$, and the $(I_r)_p$, and, hence, remove them from the list of generators. Then, since $(J_r)_q = (I_r)_p$ for all $1 \le p \le \alpha$, $1 \le q \le \beta$, we can remove all but one of these from the list of generators as well, leaving

$$\text{coker} = \langle (A_r)_{pq}, (J_r)_1 \mid (\alpha - 2)(J_r)_1 = (\beta - 2)(J_r)_1 = 0,$$
$$\text{for } 2 \le p \le \alpha, 2 \le q \le \beta, \text{ and } 1 \le r \le t \rangle. \tag{7-3}$$

We substitute $a = \alpha - 2$, $b = \beta - 2$, and write $g := \gcd(a, b)$. Then (7-3) is a presentation of $\mathbb{Z}^{t(a+1)(b+1)} \oplus (\mathbb{Z}/g)^t$ if $g > 1$, and $\mathbb{Z}^{t(a+1)(b+1)}$ otherwise. If $\alpha = \beta = 2$, then (7-3) gives a presentation of $\mathbb{Z}^2$. Together with Theorem 3.8, this gives the desired result.   □

PROPOSITION 7.12. *Let $\alpha, \beta \ge 3$, and let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on α white and β black vertices. Then, for all $t \ge 1$, the order of the class of the identity $[\mathbf{1}]$ in $K_0(C^\star(\Lambda_t^*(\kappa)))$ is equal to $g := \gcd(\alpha - 2, \beta - 2)$.*

Furthermore, the isomorphism class of $C^\star(\Lambda_t^*(\kappa))$ is completely determined by the K-groups from Theorem 7.11 and the order of $[\mathbf{1}]$ in K_0.

Acknowledgements

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