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THE SPECTRA OF THE LAPLACIANS OF FRACTAL GRAPHS NOT SATISFYING SPECTRAL DECIMATION

JONATHAN JORDAN

Department of Probability and Statistics, University of Sheffield, Hounsfield Road, Sheffield S3 7RH, UK (jonathan.jordan@shef.ac.uk)

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Abstract We consider the spectra of the Laplacians of two sequences of fractal graphs in the context of the general theory introduced by Sabot in 2003. For the sequence of graphs associated with the pentagasket, we give a description of the eigenvalues in terms of the iteration of a map from $(\mathbb{C}^2)^3$ to itself. For the sequence of graphs introduced in a previous paper by the author, we show that the results found therein can be related to Sabot's theory.

Keywords: fractal graphs; Laplacian; spectral decimation; pentagasket

2010 *Mathematics subject classification:* Primary 28A80

1. Introduction

Many fractals, and related self-similar graphs, display a property known as *spectral decimation*: the spectrum of the Laplacian can be described in terms of the iteration of a rational function f. Eigenvalues λ of the Laplacian at a given stage of the construction are related to eigenvalues μ of the Laplacian at the following stage of the construction by a relationship

$$\lambda = f(\mu), \tag{1.1}$$

where f is a rational function on $\mathbb{R}$, unless μ is a member of a small *exceptional set*, $\mathcal{E}$. This was first observed for the specific case of the Sierpiński gasket graph by Rammal and Toulouse in [**8**], and this was given a rigorous mathematical treatment in [**4**, **12**, **13**].

A generalization of spectral decimation to a much larger class of self-similar graphs, including the Vicsek set graph, is developed by Malozemov and Teplyaev in [**7**], in which a symmetry condition is developed which, if satisfied, ensures that spectral decimation applies to the graph. Each self-similar graph in this class has a function f and exceptional set $\mathcal{E}$ associated with it. Further examples of calculations for examples satisfying this symmetry condition are found in [**2**].

In [**10**], Sabot developed a more general theory which does not require the symmetry condition of [**7**]. This involves a rational map on a projective variety rather than on $\mathbb{R}$, and the derivation of the spectral decimation phenomenon for the Sierpiński gasket from the general theory is covered in detail in [**10**, § 5].

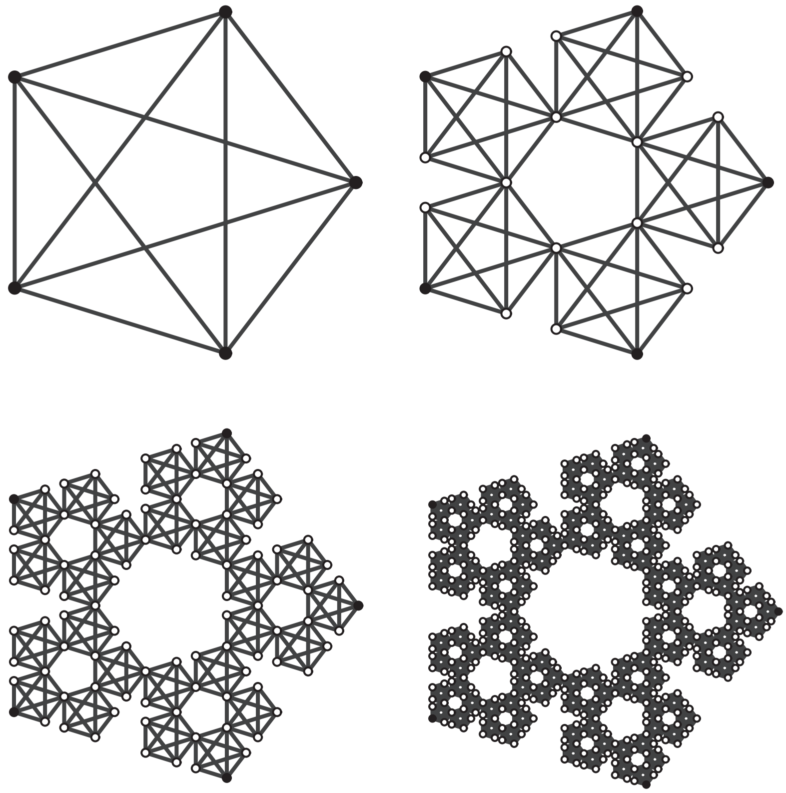

Figure 1. The first few graphs, $\Gamma^{(0)}$, $\Gamma^{(1)}$, $\Gamma^{(2)}$ and $\Gamma^{(3)}$, in the sequence associated with the pentagasket. The filled-in vertices are the boundary vertices.

We investigate two examples of the spectral theory of the Laplacians of fractal graphs in the context of the general theory developed by Sabot. One example is the pentagasket, where the related problem of the spectral theory of the Laplacian on the fractal itself is investigated in [**1**], and the other is related to the variant on spectral decimation found in [**5**]. Although the graphs defined in [**5**] do not quite fit the definitions in [**10**, § 1.1.1], we will see that much of the theory does apply.

2. The framework

The notation here is based on that in [**10**].

We work with a sequence of graphs $(\Gamma^{(n)})_{n\in\mathbb{N}}$, which will approximate a limiting self-similar graph as $n \to \infty$. This sequence is obtained by starting with $\Gamma^{(0)}$ a complete graph on N_0 vertices, $\mathcal{R}$ an equivalence relation on $\{1, 2, \dots, N\} \times \{1, 2, \dots, N_0\}$ (for a constant $N \geqslant N_0$) and $\beta\colon \{1, 2, \dots, N_0\} \to \{1, 2, \dots, N_0\}$ a function which will determine the boundary vertices.

Then if $\Gamma^{(n)}$ is the level-n graph, with a set of N_0 vertices identified as its boundary, $\partial\Gamma^{(n)}$, and the remaining vertices its interior $\mathring{\Gamma}^{(n)}$, we form $\Gamma^{(n+1)}$ by taking N copies

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of $\Gamma^{(n)}$ and identifying boundary vertex j_1 of copy k_1 with boundary vertex j_2 of copy k_2 if and only if $(k_1, j_1)\mathcal{R}(k_2, j_2)$. We then let boundary vertex j of $\Gamma^{(n+1)}$ be boundary vertex $\beta(j)$ of copy j of $\Gamma^{(n-1)}$. Also, define a set of scaling factors for each copy of $\Gamma^{(n-1)}$, α_i, $1 \leqslant i \leqslant N$.

We will refer to each copy of the complete graph on N_0 vertices within $\Gamma^{(n)}$ as a cell. Then N is the number of cells in $\Gamma^{(1)}$. Let Sym^G be the set of symmetric $N_0 \times N_0$ matrices invariant under a symmetry group G acting on $\{1, \dots, N\}$ keeping $\{1, \dots, N_0\}$ invariant, which in the cases of interest is thought of as the symmetry group of the related fractal.

For example, for the pentagasket $N_0 = N = 5$, the equivalence relation $\mathcal{R}$ is given by $(1,3)\mathcal{R}(2,5)$, $(2,4)\mathcal{R}(3,1)$, $(3,5)\mathcal{R}(4,2)$, $(4,1)\mathcal{R}(5,3)$ and $(5,2)\mathcal{R}(1,4)$, the α_i are all equal and the function β is simply $\beta(j) = j$. The first few graphs in the resulting sequence are shown in Figure 1.

3. The Sabot theory

In this section we give an introduction to the theory developed by Sabot in [**10**], showing how the iteration of a rational map defined on a Grassmann algebra can be used to describe the spectra of Laplacian operators on self-similar graphs fitting into the framework described in § 2.

3.1. Construction of the Laplacian

The construction of a Laplacian on the self-similar graph is described in [**10**, § 1.2].

If Q is an $N_0 \times N_0$ matrix, form an $|V(\Gamma^{(n)})| \times |V(\Gamma^{(n)})|$ matrix $Q^{(n)}$ as follows. Let $Q^{(0)} = Q$, and define $Q^{(n)}$ by taking copies of $Q^{(n-1)}$ on each of the copies of $\Gamma^{(n-1)}$, multiplying the one on copy i by $\alpha_1\alpha_i^{-1}$ and adding them together.

The construction of a Laplacian operator on the self-similar graph proceeds by starting with a G-invariant difference operator A (which we will take to be the graph Laplacian of $\Gamma^{(0)}$) on $V(\Gamma^{(0)})$ and a G-invariant positive measure b. Then the above gives an operator $A^{(n)}$ on $\mathbb{R}^{V(\Gamma^{(n)})}$, and we similarly define a sequence of measures $(b^{(n)})_{n\in\mathbb{N}}$ by letting $b^{(0)} = b$ and taking copies of $b^{(n-1)}$ on each copy of $\Gamma^{(n-1)}$, multiplying the one on copy i by $\alpha_1\alpha_i^{-1}$ and adding them together. A Laplacian $L^{(n)}$ can then be defined by

$$\langle A^{(n)} f, g\rangle = \int L^{(n)} f g \,\mathrm{d}b^{(n)} \quad \text{for } f, g \in \mathbb{R}^{|V(\Gamma^{(n)})|},$$

with the Laplacian on the infinite self-similar graph being defined as an extension of this. This definition ensures that in the case where all α_i are equal and b is uniform the eigenvalues are the same as those for the graph Laplacian defined in [**3**].

3.2. The iteration on the Grassmann algebra

The underlying iteration used in [**10**] to describe the spectrum takes place on a Grassmann algebra $\mathcal{A}$, defined in [**10**, Chapter 2]. The space Sym^G is embedded in $\mathcal{A}$ via a map $\zeta\colon \mathrm{Sym}^G \to \mathcal{A}$, and a linear operator $\mathcal{A} \to \mathbb{C}$ (which we will call D) is defined such that $D(\zeta(Q)) = \det Q$.

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We will need the definition of the trace of a matrix on a subset from [**10**]: let Q be an $F \times F$ matrix with F a finite set. If $F' \subseteq F$, then let $Q|_{F'}$ be the restriction of Q to F', and define the trace of Q on F', $Q_{F'}$, by $Q_{F'} = ((Q^{-1})|_{F'})^{-1}$. Then the argument in [**10**, Proposition 2.2] shows that

$$\det Q = \det(Q_{F'}) \det(Q|_{F \setminus F'}).$$

Using this definition, § 3.1 of [**10**] defines $T\colon \mathrm{Sym}^G \to \mathrm{Sym}^G$ by $T(Q) = (Q^{(1)})_{\partial \Gamma^{(1)}}$ and then shows that $T^n(Q) = (Q^{(n)})_{\partial \Gamma^{(n)}}$ [**10**, Equation (47)].

The iteration uses a map $R\colon \mathcal{A} \to \mathcal{A}$, defined so that, for $Q \in \mathrm{Sym}^G$,

$$R(\zeta(Q)) = C \det((Q^{(1)})|_{\mathring{\Gamma}^{(1)}}) \zeta(T(Q)) \tag{3.1}$$

and [**10**, Equation (46)]

$$R^n(\zeta(Q)) = C^{(n)} \det((Q^{(n)})|_{\mathring{\Gamma}^{(n)}}) \zeta(T^n(Q)), \tag{3.2}$$

where C and $C^{(n)}$ are constants depending on the scaling factors α_i. Proposition 3.1 of [**10**] states that R is homogeneous of degree N.

To find the eigenvalues of the Laplacian of $\Gamma^{(n)}$, we define $Q_\lambda \in \mathrm{Sym}^G$ by $Q_\lambda = L - \lambda I$, where L is the Laplacian of the initial graph G_0. The theory in [**10**] tells us that the eigenvalues of the level-n Laplacian can be found as the roots of $D(R^n(\zeta(Q_\lambda))) = 0$; our aim will be to describe these roots.

In the case of the nested fractals defined by Lindstrøm in [**6**], which include the pentagasket and also the example of the Sierpiński gasket considered in [**10**, § 5.1], it is possible to consider the map R as operating on $(\mathbb{C}^2)^k$ for some k (in the Sierpiński gasket case $k = 2$ and in the pentagasket case $k = 3$) instead of working on the Grassmann algebra $\mathcal{A}$.

4. The pentagasket

We consider the methods of [**10**] applied to the pentagasket, an example of a fractal structure satisfying the conditions of [**10**] but for which spectral decimation does not apply. Some results on the spectrum of the Laplacian on the pentagasket, together with some numerical computations, are found in [**1**].

4.1. The iteration

We follow the method used for the Sierpiński gasket in [**10**, § 5.1]. We decompose $\mathbb{C}^5$ as a direct sum of three orthogonal subspaces $W_0 \oplus W_1 \oplus W_2$, each of which is preserved by the symmetry group of the pentagasket. The space W_0 consists of constant vectors, W_1 has orthogonal basis vectors

$$\left(0, \frac{1-\sqrt{5}}{2}, 1, -1, \frac{\sqrt{5}-1}{2}\right) \quad \text{and} \quad \left(1, -\frac{1+\sqrt{5}}{4}, \frac{\sqrt{5}-1}{4}, \frac{\sqrt{5}-1}{4}, -\frac{1+\sqrt{5}}{4}\right)$$

and W_2 has orthogonal basis vectors

$$\left(0, 1, \frac{\sqrt{5}-1}{2}, \frac{1-\sqrt{5}}{2}, -1\right) \quad \text{and} \quad \left(1, \frac{\sqrt{5}-1}{4}, -\frac{1+\sqrt{5}}{4}, -\frac{1+\sqrt{5}}{4}, \frac{\sqrt{5}-1}{4}\right).$$

Then we let M_0, M_1 and M_2 be matrices which fix W_0, W_1 and W_2, respectively:

$$M_0 = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix},$$

$$M_1 = \begin{pmatrix} \dfrac{2}{5} & -\dfrac{1+\sqrt{5}}{10} & \dfrac{\sqrt{5}-1}{10} & \dfrac{\sqrt{5}-1}{10} & -\dfrac{1+\sqrt{5}}{10} \\ -\dfrac{1+\sqrt{5}}{10} & \dfrac{2}{5} & -\dfrac{1+\sqrt{5}}{10} & \dfrac{\sqrt{5}-1}{10} & \dfrac{\sqrt{5}-1}{10} \\ \dfrac{\sqrt{5}-1}{10} & -\dfrac{1+\sqrt{5}}{10} & \dfrac{2}{5} & -\dfrac{1+\sqrt{5}}{10} & \dfrac{\sqrt{5}-1}{10} \\ \dfrac{\sqrt{5}-1}{10} & \dfrac{\sqrt{5}-1}{10} & -\dfrac{1+\sqrt{5}}{10} & \dfrac{2}{5} & -\dfrac{1+\sqrt{5}}{10} \\ -\dfrac{1+\sqrt{5}}{10} & \dfrac{\sqrt{5}-1}{10} & \dfrac{\sqrt{5}-1}{10} & -\dfrac{1+\sqrt{5}}{10} & \dfrac{2}{5} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \dfrac{2}{5} & \dfrac{\sqrt{5}-1}{10} & -\dfrac{1+\sqrt{5}}{10} & -\dfrac{1+\sqrt{5}}{10} & \dfrac{\sqrt{5}-1}{10} \\ \dfrac{\sqrt{5}-1}{10} & \dfrac{2}{5} & \dfrac{\sqrt{5}-1}{10} & -\dfrac{1+\sqrt{5}}{10} & -\dfrac{1+\sqrt{5}}{10} \\ -\dfrac{1+\sqrt{5}}{10} & \dfrac{\sqrt{5}-1}{10} & \dfrac{2}{5} & \dfrac{\sqrt{5}-1}{10} & -\dfrac{1+\sqrt{5}}{10} \\ -\dfrac{1+\sqrt{5}}{10} & -\dfrac{1+\sqrt{5}}{10} & \dfrac{\sqrt{5}-1}{10} & \dfrac{2}{5} & \dfrac{\sqrt{5}-1}{10} \\ \dfrac{\sqrt{5}-1}{10} & -\dfrac{1+\sqrt{5}}{10} & -\dfrac{1+\sqrt{5}}{10} & \dfrac{\sqrt{5}-1}{10} & \dfrac{2}{5} \end{pmatrix}.$$

The space Sym^G of complex symmetric 5×5 matrices invariant under the symmetry group of the pentagasket consists of matrices of the form $Q = aM_0 + bM_1 + cM_2$, so we can represent an element of Sym^G by an element $(a, b, c) \in \mathbb{C}^3$, and we denote this element by $Q(a, b, c)$. The coordinates a, b and c correspond to irreducible representations of the symmetry group of the pentagasket described in [**1**]: a to the trivial representation and b and c to the two-dimensional representations. The determinant of a matrix $Q = aM_0 + bM_1 + cM_2$ is ab^2c^2.

We now calculate the map T. As a map from $\mathbb{C}^3$ to $\mathbb{C}^3$ we have

$$T(a, b, c) = (T_0(a, b, c), T_1(a, b, c), T_2(a, b, c)),$$

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where

$$T_0(a,b,c) = \frac{5abc}{bc+2ab+2ac},$$
$$T_1(a,b,c) = \frac{(2\sqrt{5}+5)(10ab+(5-\sqrt{5})ac+(5+\sqrt{5})bc)bc}{2ab^2+(\sqrt{5}+3)ac^2+(9+3\sqrt{5})bc^2+(46+20\sqrt{5})b^2c+(40+16\sqrt{5})abc},$$
$$T_2(a,b,c) = \frac{5(2ab+(5-\sqrt{5})ac+(3-\sqrt{5})bc)bc}{2ab^2+(3+\sqrt{5})ac^2+6b^2c+(19-7\sqrt{5})bc^2+(20-4\sqrt{5})abc}.$$

We now follow the method used for the Sierpiński gasket in [**10**, Chapter 5] to calculate a representation of the map R as a map from $(\mathbb{C}^2)^3$ to itself. This uses a function $s\colon (\mathbb{C}^2)^3 \to \mathcal{A}$, constructed in the same way as the corresponding function for the Sierpiński gasket, such that $s((a,1),(b,1),(c,1)) = \zeta(Q(a,b,c))$ and that s is $(1,2,2)$-homogeneous.

We know from (3.2) that

$$R(\zeta(Q)) = \det((Q^{(1)})|_{\mathring{\Gamma}^{(1)}})\zeta(T(Q)),$$

and we can calculate that if $Q = Q(a,b,c)$, then

$$\det((Q^{(1)})|_{\mathring{\Gamma}^{(1)}}) = \frac{(25-11\sqrt{5})e_0(a,b,c)(e_1(a,b,c))^2(e_2(a,b,c))^2}{12\,500\,000},$$

where

$$e_0(a,b,c) = (bc+2ab+2ac)(2b+(3+\sqrt{5})c),$$
$$e_1(a,b,c) = 2ab^2+(\sqrt{5}+3)ac^2+(9+3\sqrt{5})bc^2+(46+20\sqrt{5})b^2c+(40+16\sqrt{5})abc,$$
$$e_2(a,b,c) = 2ab^2+(3+\sqrt{5})ac^2+6b^2c+(19-7\sqrt{5})bc^2+(20-4\sqrt{5})abc.$$

The homogeneity of R and s implies that

$$R(s((u_0,v_0),(u_1,v_1),(u_2,v_2))) = (v_0v_1^2v_2^2)^5 R\biggl(s\biggl(\biggl(\frac{u_0}{v_0},1\biggr),\biggl(\frac{u_1}{v_1},1\biggr),\biggl(\frac{u_2}{v_2},1\biggr)\biggr)\biggr).$$

Putting these together,

$$\begin{aligned} &R(s((u_0,v_0),(u_1,v_1),(u_2,v_2))) \\ &\quad = (v_0v_1^2v_2^2)^5\frac{25-11\sqrt{5}}{12\,500\,000}e_0\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)\biggl(e_1\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)\biggr)^2\biggl(e_2\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)\biggr)^2 \\ &\qquad \times s\biggl(\biggl(T_0\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr),1\biggr),\biggl(T_1\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr),1\biggr),\biggl(T_2\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr),1\biggr)\biggr), \end{aligned}$$

and using the homogeneity of s we have

$$\begin{aligned}
&R(s((u_0,v_0),(u_1,v_1),(u_2,v_2)))\\
&\quad = s\Biggl(\biggl(\frac{25-11\sqrt{5}}{12\,500\,000}v_0v_1^2v_2^2T_0\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)e_0\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \frac{25-11\sqrt{5}}{12\,500\,000}v_0v_1^2v_2^2e_0\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)\biggr),\\
&\qquad\quad \biggl(v_0v_1^2v_2^2T_1\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)e_1\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr),v_0v_1^2v_2^2e_1\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)\biggr),\\
&\qquad\quad \biggl(v_0v_1^2v_2^2T_2\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)e_2\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr),v_0v_1^2v_2^2e_2\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)\biggr)\Biggr),
\end{aligned}$$

so the representation of R as a map from $(\mathbb{C}^2)^3$ to itself can be written

$$\begin{aligned}
&R((u_0,v_0),(u_1,v_1),(u_2,v_2))\\
&\quad = \Biggl(\biggl(\frac{25-11\sqrt{5}}{12\,500\,000}v_0v_1^2v_2^2T_0\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)e_0\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \frac{25-11\sqrt{5}}{12\,500\,000}v_0v_1^2v_2^2e_0\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)\biggr),\\
&\qquad\quad \biggl(v_0v_1^2v_2^2T_1\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)e_1\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr),v_0v_1^2v_2^2e_1\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)\biggr),\\
&\qquad\quad \biggl(v_0v_1^2v_2^2T_2\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)e_2\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr),v_0v_1^2v_2^2e_2\biggl(\frac{u_0}{v_0},\frac{u_1}{v_1},\frac{u_2}{v_2}\biggr)\biggr)\Biggr).
\end{aligned}$$

Hence we have

$$R((u_0,v_0),(u_1,v_1),(u_2,v_2)) = ((R_{00},R_{01}),(R_{10},R_{11}),(R_{20},R_{21})),$$

where

$$\begin{aligned}
R_{00} &= (25-11\sqrt{5})((\sqrt{5}-3)u_1v_2-2u_2v_1)u_2u_0u_1/2\,500\,000,\\
R_{01} &= (25-11\sqrt{5})(u_1u_2v_0+2u_0u_1v_2+2u_0u_2v_1)((\sqrt{5}-3)u_1v_2-2u_2v_1)/12\,500\,000,\\
R_{10} &= (2\sqrt{5}+5)(10u_0u_1v_2+(5-\sqrt{5})u_0u_2v_1+(5+\sqrt{5})u_1u_2v_0)u_2u_1,\\
R_{11} &= 2u_0u_1^2v_2^2+(3+\sqrt{5})u_0u_2^2v_1^2+(9+3\sqrt{5})u_1u_2^2v_0v_1+(46+20\sqrt{5})u_1^2u_2v_0v_2\\
&\qquad + (40+16\sqrt{5})u_0u_1u_2v_1v_2,\\
R_{20} &= 5(2u_0u_1v_2+(5-\sqrt{5})u_0u_2v_1+(3-\sqrt{5})u_1u_2v_0)u_2u_1,\\
R_{21} &= 2u_0u_1^2v_2^2+(3+\sqrt{5})u_0u_2^2v_1^2+6u_1^2u_2v_0v_2\\
&\qquad + (19-7\sqrt{5})u_1u_2^2v_0v_1+(20-4\sqrt{5})u_0u_1u_2v_1v_2.
\end{aligned}$$

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For a potential eigenvalue of the Laplacian λ, we start with an initial matrix

$$Q_\lambda = \begin{pmatrix} 1-\lambda & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 1-\lambda & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1-\lambda & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1-\lambda & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1-\lambda \end{pmatrix},$$

which corresponds to

$$\left(-\lambda, \frac{5-4\lambda}{4}, \frac{5-4\lambda}{4}\right) \in \mathbb{C}^3,$$

so let

$$u_0^{(0)} = -\lambda, \qquad u_1^{(0)} = \frac{5-4\lambda}{4}, \qquad u_2^{(0)} = \frac{5-4\lambda}{4}, \qquad v_0^{(0)} = v_1^{(0)} = v_2^{(0)} = 1$$

and let

$$\begin{aligned}((u_0^{(n)}, v_0^{(n)}), (u_1^{(n)}, v_1^{(n)}), (u_2^{(n)}, v_2^{(n)})) \\ = R((u_0^{(n-1)}, v_0^{(n-1)}), (u_1^{(n-1)}, v_1^{(n-1)}), (u_2^{(n-1)}, v_2^{(n-1)})).\end{aligned}$$

Because the operator D is linear, using the homogeneity of s we have

$$\begin{aligned} D(s((u_0, v_0), (u_1, v_1), (u_2, v_2))) &= (v_0 v_1^2 v_2^2) D\left(s\left(\left(\frac{u_0}{v_0}, 1\right), \left(\frac{u_1}{v_1}, 1\right), \left(\frac{u_2}{v_2}, 1\right)\right)\right) \\ &= (v_0 v_1^2 v_2^2)\left(\frac{u_0 u_1^2 u_2^2}{v_0 v_1^2 v_2^2}\right) \\ &= u_0 u_1^2 u_2^2. \end{aligned}$$

Hence, the eigenvalues of the n-level matrix are the roots of

$$u_0^{(n)} (u_1^{(n)})^2 (u_2^{(n)})^2 = 0.$$

Each $u_i^{(n)}$ and $v_i^{(n)}$ can be expressed as a polynomial in λ. If we let d_n be the degree of $u_i^{(n)}$ and d'_n be the degree of $v_i^{(n)}$, then we have $d_n = 4d_{n-1} + d'_{n-1}$ and $d'_n = 3d_{n-1} + 2d'_{n-1}$, with $d_0 = 1$ and $d'_0 = 0$. Hence, $d_n = \frac{1}{4}(3 \cdot 5^n + 1)$ and $d'_n = d_n - 1$. (The total number of eigenvalues, which is the number of vertices in the level-n graph, is $5d_n = \frac{1}{4}(3 \cdot 5^{n+1} + 5)$.)

Similarly, let $a^{(0)} = -\lambda$, $b^{(0)} = c^{(0)} = \frac{1}{4}(5 - 4\lambda)$ and write the iterates of the map T as $(a^{(n)}, b^{(n)}, c^{(n)}) = T(a^{(n-1)}, b^{(n-1)}, c^{(n-1)})$.

4.2. Eigenvalues that first arise at level n

We consider the ways that components of the iterates of R can become zero. In each case we assume that the components not mentioned are non-zero at level n.

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(1) Let

$$F_1^{(n)} = (\sqrt{5}-3)u_1^{(n-1)}v_2^{(n-1)} - 2u_2^{(n-1)}v_1^{(n-1)}.$$

Then $F_1^{(n)}$ is a factor of both $u_0^{(n)}$ and $v_0^{(n)}$ with multiplicity 1, so roots of $F_1^{(n)} = 0$ give eigenvalues with multiplicity 1 at level n. Iterating R shows that $F_1^{(n)}$ is a factor of all the components at level $n+1$ with multiplicity 1 and of all components at level $n+m$ ($m \geqslant 1$) with multiplicity 5^{m-1}, so the eigenvalue has multiplicity 5^m at level $n+m$. As the total number of eigenvalues at level $n+m$ is $\frac{1}{4}(3 \cdot 5^{n+m+1})$, the limiting spectral measure of an eigenvalue which appears as a type-1 eigenvalue at level n is $\frac{4}{3}(\frac{1}{5})^{n+1}$. Because $F_1^{(n)}$ is a factor of $u_i^{(m)}$ and $v_i^{(m)}$ with the same multiplicity, these eigenvalues do not appear as zeros of the iterates of T.

(2) Let

$$F_2^{(n)} = (5+\sqrt{5})u_0^{(n-1)}u_1^{(n-1)}v_2^{(n-1)} + 2u_0^{(n-1)}u_2^{(n-1)}v_1^{(n-1)} + (\sqrt{5}+3)u_1^{(n-1)}u_2^{(n-1)}v_0^{(n-1)}.$$

Then $F_2^{(n)}$ is a factor of $u_1^{(n)}$ with multiplicity 1, so roots of $F_2^{(n)} = 0$ give eigenvalues with multiplicity 2 at level n. In this case $F_2^{(n)}$ is a factor of each of $u_0^{(n+1)}$, $u_1^{(n+1)}$ and $u_2^{(n+1)}$ with multiplicity 1 (and is not a factor of $v_0^{(n+1)}$, $v_1^{(n+1)}$ or $v_2^{(n+1)}$) so the eigenvalue has multiplicity 5 at level $n+1$. Inductively iterating R, for $m \geqslant 1$, $F_2^{(n)}$ is a factor of each of $u_0^{(n+m)}$, $u_1^{(n+m)}$ and $u_2^{(n+m)}$ with multiplicity $\frac{1}{4}(3 \cdot 5^{m-1}+1)$ and of $v_0^{(n+m)}$, $v_1^{(n+m)}$ and $v_2^{(n+m)}$ with multiplicity $\frac{1}{4}(3 \cdot 5^{m-1}-3)$, so the eigenvalue has multiplicity $\frac{1}{4}(3 \cdot 5^m + 5)$ at level $n+m$. (The sequence of multiplicities starts $2, 5, 20, 95, 470, \dots$.) The limiting spectral measure of an eigenvalue which appears as a type-2 eigenvalue at level n is $(\frac{1}{5})^{n+1}$. These eigenvalues appear as zeros with multiplicity 1 of $b^{(n)}$ and of each of $a^{(m)}$, $b^{(m)}$ and $c^{(m)}$ for $m > n$.

(3) Let

$$F_3^{(n)} = (5+\sqrt{5})u_0^{(n-1)}u_1^{(n-1)}v_2^{(n-1)} + 10u_0^{(n-1)}u_2^{(n-1)}v_1^{(n-1)} + (5-\sqrt{5})u_1^{(n-1)}u_2^{(n-1)}v_0^{(n-1)}.$$

Then $F_3^{(n)}$ is a factor of $u_2^{(n)}$ with multiplicity 1, so roots of $F_3^{(n)} = 0$ give eigenvalues with multiplicity 2 at level n. The behaviour of the multiplicities in this case is the same as for type 2. These eigenvalues appear as zeros with multiplicity 1 of $c^{(n)}$ and of each of $a^{(m)}$, $b^{(m)}$ and $c^{(m)}$ for $m > n$.

(4) The value $\lambda = \frac{5}{4}$ is a special case, because $F_4 = (5-4\lambda)$ is a factor of both $u_1^{(0)}$ and $u_2^{(0)}$. Hence, this eigenvalue has multiplicity 4 at level 0. Iterating R, F_4 is a factor of each $u_i^{(1)}$ with multiplicity 3 and of each $v_i^{(1)}$ with multiplicity 2, and, again by induction, F_4 is a factor of each $u_i^{(m)}$ with multiplicity $\frac{1}{4}(11 \cdot 5^{m-1}+1)$ and of each $v_i^{(m)}$ with multiplicity $\frac{1}{4}(11 \cdot 5^{m-1}-3)$, so the eigenvalue has multiplicity $\frac{1}{4}(11 \cdot 5^m + 5)$ at level m. (The sequence of multiplicities starts $4, 15, 70, 345, \dots$.) The limiting spectral measure of $\frac{5}{4}$ is $\frac{11}{3}(\frac{1}{5}) = \frac{11}{15}$. This eigenvalue appears as a zero with multiplicity 1 of $b^{(0)}$ and $c^{(0)}$ and of each of $a^{(m)}$, $b^{(m)}$ and $c^{(m)}$ for $m > 0$.

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(5) The value $\lambda = 0$ is also a special case, as λ is a factor of $u_0^{(0)}$ (but not of $v_0^{(0)}$, so the behaviour is different from that of type-1 eigenvalues). Iterating R, λ is a factor of $u_0^{(n)}$ for all n but not of any of the other components, producing a zero eigenvalue with multiplicity 1. This eigenvalue appears as a zero with multiplicity 1 of $a^{(m)}$ for all m.

Type-1 eigenvalues correspond to the alternating one-dimensional irreducible representation of the symmetry group, and types 2 and 3 correspond to the two two-dimensional irreducible representations. These types of eigenvalues, and the single type-5 eigenvalue (which corresponds to the trivial representation), thus correspond to the types of eigenvalues found for the Laplacian on the continuous pentagasket in [**1**]. The multiplicities of eigenvalues at levels $m > n$ found above by factorizing components of R also match those found by geometric arguments in [**1**].

The type-4 eigenvalue does not correspond to any eigenvalue on the continuous pentagasket as, when the scaling factor $(5/r)^n$ is applied to the level-n spectrum (where $r = \frac{1}{8}(\sqrt{161} - 9)$ as in [**1**]), we obtain

$$\frac{5}{4}\left(\frac{5}{r}\right)^n \to \infty.$$

4.3. Numbers of eigenvalues of different types

We show by induction that for each $n \geqslant 1$ there are 3^{n-1} eigenvalues each of type 2 and type 3 appearing at level n and $3^{n-1} - 1$ eigenvalues of type 1.

Assuming that this holds for all $m < n$, we analyse the degrees of the polynomials $F_i^{(n)}$. The degree of $F_1^{(n)}$ is

$$d_{n-1} + d'_{n-1} = \tfrac{1}{2}(3 \cdot 5^{n-1} - 1)$$

and the degrees of $F_2^{(n)}$ and $F_3^{(n)}$ are each

$$2d_{n-1} + d'_{n-1} = \tfrac{1}{4}(9 \cdot 5^{n-1} - 1).$$

Now the structure of $F_1^{(n)}$ and the factorization of $u_i^{(m)}$ and $v_i^{(m)}$ show that, for $m < n - 1$, $F_1^{(m)}$ appears as a factor in $F_1^{(n)}$ with multiplicity $2 \cdot 5^{n-m-2}$, $F_2^{(m)}$ and $F_3^{(m)}$ appear as factors in $F_1^{(n)}$ each with multiplicity $\frac{1}{2}(3 \cdot 5^{n-m-2} - 1)$ and F_4 appears as a factor in $F_1^{(n)}$ with multiplicity $\frac{1}{2}(11 \cdot 5^{n-2} - 1)$ (if $n \geqslant 2$, it is a factor of $F_1^{(1)}$ with multiplicity 1). Hence (assuming the induction hypothesis), eigenvalues from levels $m < n$ account for

$$\sum_{m=1}^{n-2} (2(3^{m-1} - 1)5^{n-m-2} + 2(3^{m-1})(\tfrac{3}{2} \cdot 5^{n-m-2} - \tfrac{1}{2})) + \tfrac{1}{2}(11 \cdot 5^{n-2} - 1) \\ = \tfrac{1}{2}(5^{n-1} + 1) - 3^{n-1}$$

roots of $F_1^{(n)}$, which leaves

$$\tfrac{1}{2}(3 \cdot 5^{n-1} - 1) - (\tfrac{1}{2}(5^{n-1} + 1) - 3^{n-1}) = 3^{n-1} - 1$$

roots, giving type-1 eigenvalues at level n.

Similarly, the structure of $F_2^{(n)}$ and $F_3^{(n)}$ and the factorization of $u_i^{(m)}$ and $v_i^{(m)}$ show that, for $m < n-1$, $F_1^{(m)}$ appears as a factor in each of $F_2^{(n)}$ and $F_3^{(n)}$ with multiplicity $3 \cdot 5^{n-m-2}$, $F_2^{(m)}$ and $F_3^{(m)}$ each appear as factors in both $F_2^{(n)}$ and $F_3^{(n)}$ with multiplicity $\frac{1}{4}(9 \cdot 5^{n-m-2} - 1)$, and F_4 appears as a factor in both $F_2^{(n)}$ and $F_3^{(n)}$ with multiplicity $\frac{1}{4}(33 \cdot 5^{n-2} - 1)$ (if $n \geqslant 2$, it is a factor of $F_2^{(1)}$ and $F_3^{(1)}$ with multiplicity 1). Additionally, $F_1^{(n-1)}$ occurs as a factor in each of $F_2^{(n)}$ and $F_3^{(n)}$ with multiplicity 1. Hence (assuming the induction hypothesis), eigenvalues from levels $m < n$ account for

$$\sum_{m=1}^{n-2} (3(3^{m-1} - 1)5^{n-m-2} + 2(3^{m-1})(\tfrac{9}{4} \cdot 5^{n-m-2} - \tfrac{1}{4})) + \tfrac{1}{4}(33 \cdot 5^{n-2} - 1) = \tfrac{1}{4}(9 \cdot 5^{n-1} - 1) - 3^{n-1}$$

roots of both $F_2^{(n)}$ and $F_3^{(n)}$, which leaves

$$\tfrac{1}{4}(9 \cdot 5^{n-1} - 1) - (\tfrac{1}{4}(9 \cdot 5^{n-1} - 1) - 3^{n-1}) = 3^{n-1}$$

roots of each, giving type-2 and type-3 eigenvalues at level n.

Define

$$\hat{F}_i^{(1)} = F_i^{(1)}/F_4,$$

and then for $n \geqslant 2$ define

$$\hat{F}_1^{(n)} = F_1^{(n)} \biggl(\prod_{m=1}^{n-2} ((\hat{F}_1^{(m)})^{2 \cdot 5^{n-m-2}} (\hat{F}_2^{(m)})^{(3 \cdot 5^{n-m-2} - 1)/2} (\hat{F}_3^{(m)})^{(3 \cdot 5^{n-m-2} - 1)/2}) \times (F_4)^{(11 \cdot 5^{n-2} - 1)/2} \biggr)^{-1},$$

$$\hat{F}_2^{(n)} = F_2^{(n)} \biggl(\prod_{m=1}^{n-2} ((\hat{F}_1^{(m)})^{3 \cdot 5^{n-m-2}} (\hat{F}_2^{(m)})^{(9 \cdot 5^{n-m-2} - 1)/4} (\hat{F}_3^{(m)})^{(9 \cdot 5^{n-m-2} - 1)/4}) \times (F_4)^{(33 \cdot 5^{n-2} - 1)/4} (\hat{F}_1^{(n-1)}) \biggr)^{-1},$$

$$\hat{F}_3^{(n)} = F_3^{(n)} \biggl(\prod_{m=1}^{n-2} ((\hat{F}_1^{(m)})^{3 \cdot 5^{n-m-2}} (\hat{F}_2^{(m)})^{(9 \cdot 5^{n-m-2} - 1)/4} (\hat{F}_3^{(m)})^{(9 \cdot 5^{n-m-2} - 1)/4}) \times (F_4)^{(33 \cdot 5^{n-2} - 1)/4} (\hat{F}_1^{(n-1)}) \biggr)^{-1}.$$

Let $\lambda_i^{(n)}$, $1 \leqslant i \leqslant 3^{n-1} - 1$, be the $3^{n-1} - 1$ roots of $\hat{F}_1^{(n)} = 0$ and $\mu_i^{(n)}$, $1 \leqslant i \leqslant 3^{n-1}$, be the 3^{n-1} roots of $\hat{F}_2^{(n)} = 0$ and $\nu_i^{(n)}$, $1 \leqslant i \leqslant 3^{n-1}$, be the 3^{n-1} roots of $\hat{F}_3^{(n)} = 0$. Then the $\lambda_i^{(n)}$ are type-1 eigenvalues at level n, the $\mu_i^{(n)}$ are type-2 eigenvalues at level n and the $\nu_i^{(n)}$ are type-3 eigenvalues at level n.

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4.4. Spectral measure

The calculations above show that the spectral measure at level n is

$$\frac{4}{3\cdot 5^{n+1}+5}\biggl(\delta_0+\Bigl(\frac{11}{4}5^n+\frac{5}{4}\Bigr)\delta_{5/4}+\sum_{m=2}^{n-1}5^{n-m}\sum_{i=1}^{3^{m-1}-1}\delta_{\lambda_i^{(m)}} + \sum_{m=1}^{n-1}\Bigl(\frac{3}{4}5^{n-m}+\frac{5}{4}\Bigr)\sum_{i=1}^{3^{m-1}}(\delta_{\mu_i^{(m)}}+\delta_{\nu_i^{(m)}})\biggr).$$

The limiting spectral measure is then

$$\frac{11}{15}\delta_{5/4}+\sum_{m=2}^{\infty}\frac{4}{3}\Bigl(\frac{1}{5}\Bigr)^{m+1}\sum_{i=1}^{3^{m-1}-1}\delta_{\lambda_i^{(m)}}+\sum_{m=1}^{\infty}\Bigl(\frac{1}{5}\Bigr)^{m+1}\sum_{i=1}^{3^{m-1}}(\delta_{\mu_i^{(m)}}+\delta_{\nu_i^{(m)}}).$$

The limiting spectral measure of the set of the $\lambda_i^{(n)}$ eigenvalues is

$$\sum_{m=1}^{\infty}(3^{m-1}-1)\tfrac{4}{3}(\tfrac{1}{5})^{m+1}=\tfrac{1}{15}$$

and the limiting spectral measures of the sets of $\mu_i^{(n)}$ and $\nu_i^{(n)}$ eigenvalues are each

$$\sum_{m=1}^{\infty}(3^{m-1})(\tfrac{1}{5})^{m+1}=\tfrac{1}{10}.$$

4.5. Numerical computation of eigenvalues

Using numerical solution of the equations obtained by the above factorizations of the components of R, we calculate (see Table 1) the eigenvalues that appear in the first three levels, their multiplicity in the spectrum of the Laplacian of $\Gamma^{(3)}$ and their limiting spectral measure.

5. The self-similar unit interval with a reflection map

In [**9**, **11**], and in [**10**, § 5.2], the self-similar structure on the unit interval with respect to the maps $\Psi_1(x)=\alpha x$ and $\Psi_2(x)=1+(1-\alpha)(x-1)$ is considered.

We consider a similar self-similar structure, but with the second map altered to reflect and contract the interval, i.e. we will take $\Psi_2(x)=1-(1-\alpha)x$, with Ψ_1 as above. Here $N=N_0=2$, the equivalence relation is given by $(1,2)\mathcal{R}(2,2)$, the function β is given by $\beta(1)=\beta(2)=1$ and $\alpha_1=\alpha$, $\alpha_2=1-\alpha$. If $\alpha=\frac{1}{3}$, this is closely related to the fractal graph studied in [**5**]; the double edges in that graph correspond to the shorter edges here. If $\alpha=\frac{2}{3}$, it is similarly closely related to the graph obtained by reversing the orientation of the model graph mentioned at the end of [**5**].

The symmetry group G is trivial and there are two boundary points. Hence, the symmetric matrices Q are of the form

$$\begin{pmatrix} a & q\\ q & d\end{pmatrix},$$

Table 1. *Calculations using the numerical solution of the equations obtained by factorizations of the components of R.*

eigenvalue	level	type	level-3 multiplicity	spectral measure
0	0	5	1	0
0.00168338	3	3	2	1/625
0.00419185	3	2	2	1/625
0.01843319	2	3	5	1/125
0.02226818	3	1	1	4/1875
0.02464238	3	3	2	1/625
0.03227973	3	2	2	1/625
0.04400310	2	2	5	1/125
0.05954335	3	2	2	1/625
0.07854993	3	3	2	1/625
0.08951707	3	1	1	4/1875
0.17274575	1	3	20	1/25
0.18550404	3	1	1	4/1875
0.19513683	3	3	2	1/625
0.20677282	3	2	2	1/625
0.21215304	2	1	5	4/375
0.23593551	2	3	5	1/125
0.24270214	3	1	1	4/1875
0.24721715	3	3	2	1/625
0.26124041	3	2	2	1/625
0.30573224	2	2	5	1/125
0.31924348	3	2	2	1/625
0.34161493	3	3	2	1/625
0.35271477	3	1	1	4/1875
0.45225424	1	2	20	1/25
0.50602804	3	1	1	4/1875
0.51203514	3	3	2	1/625
0.52157728	3	2	2	1/625
0.52526466	2	2	5	1/125
0.59549976	3	2	2	1/625
0.60014028	3	3	2	1/625
0.60279371	3	1	1	4/1875
0.62063130	2	3	5	1/125
0.62347205	3	1	1	4/1875
0.62397999	3	3	2	1/625
0.62465131	3	2	2	1/625
0.66284695	2	1	5	4/375
5/4	0	4	345	11/15

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and the relationship of these symmetric matrices to the Grassmann algebra is exactly the same as that for the interval without reflection in [**10**, § 5.2]. The Grassmann algebra is generated by $\{\bar\eta_0, \eta_0, \bar\eta_1, \eta_1\}$, where $\{\eta_0, \eta_1\}$ and $\{\bar\eta_0, \bar\eta_1\}$ are canonical bases of two copies of $\mathbb{C}^2$, and, using the same notation as in [**10**],

$$\exp(\bar\eta Q\eta) = 1 + a\bar\eta_0\eta_0 + d\bar\eta_1\eta_1 + q(\bar\eta_0\eta_1 + \bar\eta_1\eta_0) + (ad - q^2)\bar\eta_0\eta_0\bar\eta_1\eta_1,$$

and the map R will act on elements of the form

$$Z + a\bar\eta_0\eta_0 + d\bar\eta_1\eta_1 + q(\bar\eta_0\eta_1 + \bar\eta_1\eta_0) + D\bar\eta_0\eta_0\bar\eta_1\eta_1$$

with $ad - q^2 = DZ$.

Letting $\delta = \alpha/(1-\alpha)$, the matrix $Q^{(1)}$ formed by adding scaled copies of Q in each cell of $\Gamma^{(1)}$ is

$$\begin{pmatrix} a & q & 0 \\ q & d(1+\delta) & \delta q \\ 0 & \delta q & \delta a \end{pmatrix},$$

and hence the matrix $T(Q)$ is

$$\frac{1}{d(1+\delta)}\begin{pmatrix} ad(1+\delta) - q^2 & -q^2\delta \\ -q^2\delta & ad(\delta+\delta^2) - q^2\delta \end{pmatrix},$$

so that the map T can be represented as

$$T(a,d,q) = \frac{1}{d(1+\delta)}(ad(1+\delta) - q^2, ad(\delta+\delta^2) - \delta^2 q^2, -q^2\delta). \tag{5.1}$$

Using the relationship between the maps T and R from [**10**], we can now calculate the map R as

$$\begin{aligned} R\biggl(Z + a\bar\eta_0\eta_0 + d\bar\eta_1\eta_1 + q(\bar\eta_0\eta_1 + \bar\eta_1\eta_0) + \biggl(\frac{ad-q^2}{Z}\biggr)\bar\eta_0\eta_0\bar\eta_1\eta_1\biggr) \\ = \tilde Z + \tilde a\bar\eta_0\eta_0 + \tilde d\bar\eta_1\eta_1 + \tilde q(\bar\eta_0\eta_1 + \bar\eta_1\eta_0) + \tilde D\bar\eta_0\eta_0\bar\eta_1\eta_1, \end{aligned}$$

with

$$\begin{aligned} \tilde Z &= Zd(1+\delta) \\ \tilde a &= ad(1+\delta) - q^2 \\ \tilde d &= ad\delta(1+\delta) - \delta^2 q^2 \\ \tilde q &= -\delta q^2 \\ \tilde D &= \frac{\tilde a\tilde d - \tilde q^2}{\tilde Z}. \end{aligned}$$

Hence, we can follow the evolution of $y = ad$ and $v = q^2$ by considering the two-dimensional map

$$h(y,v) = ((1+\delta)^2\delta y^2 - (1+\delta)^2\delta yv + \delta^2 v^2, \delta^2 v^2),$$

and if we let $u = y/v$, we can obtain a map $\hat{h}$ on $\mathbb{P}^1$:

$$\hat{h}(u) = \frac{(1+\delta)^2}{\delta}u^2 - \frac{(1+\delta)^2}{\delta}u + 1.$$

Let $a^{(0)} = d^{(0)} = 1 - \lambda$ and $q^{(0)} = -1$. Also let $Z^{(0)} = 1$; then an eigenvalue λ is mapped into the Grassmann algebra as

$$\phi(\lambda) = Z^{(0)} + a^{(0)}\bar{\eta}_0\eta_0 + d^{(0)}\bar{\eta}_1\eta_1 + q^{(0)}(\bar{\eta}_0\eta_1 + \bar{\eta}_1\eta_0) + \left(\frac{a^{(0)}d^{(0)} - (q^{(0)})^2}{Z^{(0)}}\right)\bar{\eta}_0\eta_0\bar{\eta}_1\eta_1.$$

Now define $a^{(n)}$, $d^{(n)}$, $q^{(n)}$ and $Z^{(n)}$ by

$$\begin{aligned} &Z^{(n)} + a^{(n)}\bar{\eta}_0\eta_0 + d^{(n)}\bar{\eta}_1\eta_1 + q^{(n)}(\bar{\eta}_0\eta_1 + \bar{\eta}_1\eta_0) + \frac{a^{(n)}d^{(n)} - (q^{(n)})^2}{Z^{(n)}}\bar{\eta}_0\eta_0\bar{\eta}_1\eta_1 \\ &\quad = R^n\left(Z^{(0)} + a^{(0)}\bar{\eta}_0\eta_0 + d^{(0)}\bar{\eta}_1\eta_1 + q^{(0)}(\bar{\eta}_0\eta_1 + \bar{\eta}_1\eta_0) + \left(\frac{a^{(0)}d^{(0)} - (q^{(0)})^2}{Z^{(0)}}\right)\bar{\eta}_0\eta_0\bar{\eta}_1\eta_1\right) \end{aligned}$$

and let $y^{(n)} = a^{(n)}d^{(n)}$, $v^{(n)} = (q^{(n)})^2$, $u^{(n)} = y^{(n)}/v^{(n)}$.

Eigenvalues of the level-n Laplacian are values where $(a^{(n)}d^{(n)} - (q^{(n)})^2)/Z^{(n)} = 0$, which implies that $u^{(n)} = 1$, i.e. that $\hat{h}^n((1-\lambda)^2) = 1$.

Now $u^{(n)} = 1$ if and only if $u^{(0)} = 1$ or $u^{(m)} = 0$ for some $m < n$. The former case gives eigenvalues 0 and 2. The latter case happens if either $a^{(m)} = 0$ (if $m \geqslant 1$, this implies that $u^{(m-1)} = 1/(1+\delta)$) or $d^{(m)} = 0$ (if $m \geqslant 1$, this implies that $u^{(m-1)} = \delta/(1+\delta)$). However, the case where $d^{(m)} = 0$ does not produce eigenvalues of the Laplacian because in this case $Z^{(n)} = 0$. So the eigenvalues of the Laplacian at level n are 0, 1, 2 and values λ such that $u^{(m)} = 1/(1+\delta)$ for some $m < n - 1$.

To see the link between the theory in [**10**] and the results in [**5**], note that in the case where $\delta = \frac{1}{2}$ or $\delta = 2$ it can be seen that $\hat{h}((1-\lambda)^2)$ is the quartic polynomial in [**5**]. However, in the one-dimensional setting we consider here, the Dirichlet–Neumann eigenvalues found in [**5**] do not appear. They occur at the values mentioned above where $d^{(m)} = 0$ for some $m < n$.

References

1. B. Adams, S. A. Smith, R. S. Strichartz and A. Teplyaev, The spectrum of the Laplacian on the pentagasket, in *Fractals in Graz 2001*, Trends in Mathematics, pp. 1–24 (Birkhäuser, Basel, 2003).
2. N. Bajorin, T. Chen, A. Dagan, C. Emmons, M. Hussein, M. Khalil, P. Mody, B. Steinhurst and A. Teplyaev, Vibration nodes of $3n$-gaskets and other fractals, *J. Phys.* A **41** (2008), 015101.
3. F. R. K. Chung, *Spectral graph theory*, CBMS Regional Conference Series, Volume 92 (American Mathematical Society, Providence, RI, 1997).
4. M. Fukushima and T. Shima, On a spectral analysis for the Sierpiński gasket, *Potent. Analysis* **1** (1992), 1–35.
5. J. H. Jordan, Spectrum of the Laplacian of an asymmetric fractal graph, *Proc. Edinb. Math. Soc.* **49** (2006), 101–113.

6. T. Lindstrøm, *Brownian motion on nested fractals*, Memoirs of the American Mathematical Society, Volume 420 (American Mathematical Society, Providence, RI, 1990).
7. L. Malozemov and A. Teplyaev, Self-similarity, operators and dynamics, *Math. Phys. Analysis Geom.* **6** (2003), 201–218.
8. R. Rammal and G. Toulouse, Random walks on fractal structures and percolation clusters, *J. Phys. Lett.* **44** (1983), L13–L22.
9. C. Sabot, Integrated density of states and self-similar Sturm–Liouville operators and holomorphic dynamics in higher dimension, *Annales Inst. H. Poincaré* B **37** (2001), 275–311.
10. C. Sabot, *Spectral properties of self-similar lattices and iteration of rational maps*, Mémoires de la Société Mathématique de France, Volume 92 (Société Mathématique de France, Paris, 2003).
11. C. Sabot, Spectral analysis of a self-similar Sturm–Liouville operator, *Indiana Univ. Math. J.* **54** (2005), 645–668.
12. T. Shima, On eigenvalue problems for the random walks on the Sierpiński pre-gaskets, *Jpn. J. Ind. Appl. Math.* **8** (1991), 127–141.
13. T. Shima, On eigenvalue problems for Laplacians on PCF self-similar sets, *Jpn. J. Ind. Appl. Math.* **13** (1996), 1–23.

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