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PRICING PERPETUAL TIMER OPTION UNDER THE STOCHASTIC VOLATILITY MODEL OF HULL–WHITE

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Abstract

The valuation of perpetual timer options under the Hull–White stochastic volatility model is discussed here. By exploring the connection between the Hull–White model and the Bessel process and using time-change techniques, the triple joint distribution for the instantaneous volatility, the cumulative reciprocal volatility and the cumulative realized variance is obtained. An explicit analytical solution for the price of perpetual timer call options is derived as a Black–Scholes–Merton-type formula.

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1. Introduction

A timer option is an exotic option with random expiration date which depends on the realized volatility of the underlying asset. In particular, a timer option is exercised at the first time when the accumulated variance of the underlying asset reaches a predetermined level. This type of product offers better control of volatility risk with higher flexibility for investors. Timer options were first studied by Bick [2] many years ago, but were not traded on the market until April 2007 by Société Générale Corporate and Investment Banking [5]. Since then there have been renewed interests in timer options. Li [11] studied in detail the Monte Carlo simulation of the price of timer options. Bernard and Cui [1] simplified the pricing problem into a one-dimensional problem by using the time-change technique. They proposed a fast and accurate almost-exact simulation technique coupled with a powerful control variate for the Hull–White and Heston stochastic volatility models [7, 8]. Liang et al. [14] derived a multi-dimensional numerical integration expression using the path-integral technique developed from quantum field theory.

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However, the Monte Carlo simulations are very time consuming, while the multi-dimensional numerical integration method is complicated since it involves complex Fourier inversion. As a result, it is advantageous to develop analytical or semi-analytical methods for the evaluation of timer options. Carr and Lee [4] investigated the pricing and hedging of variance options when realized variance reaches a fixed level and derived a semi-analytical formula for the option price when the asset process was a continuous semi-martingale. Saunders [20] obtained an asymptotic approximation of the price of timer options under fast mean-reverting stochastic volatility. Although the approximation is in closed form, it requires an extremely large mean-reversion coefficient to obtain satisfactory numerical accuracy. Based on Saunders's work, Li and Mercurio [13] developed an asymptotic expansion technique around small volatility of variance and obtained closed-form approximation formulas for the Heston model and the (3/2) model, when the mean-reverting coefficient is zero. Li [12] characterized the joint distribution of the stopping time of the realized variance and corresponding variance, and obtained a Black–Scholes–Merton-type formula for pricing timer options by exploring the connection between the Heston model and the Bessel process with drift.

In this paper, we present an explicit Black–Scholes–Merton-type formula for the price of perpetual timer options under the stochastic volatility model of Hull–White, using properties of Bessel processes. We start with dynamic hedging of timer options under the Hull–White stochastic volatility model. The price is represented as a risk-neutral expectation by the Feynman–Kac formula (see the book by Karatzas and Shreve [10]). By conditioning with respect to quantities related to the volatility process, the conditional Black–Scholes formula is derived. Using time-change techniques, by exploring the connection between the Bessel process and the stochastic volatility, the triple joint distribution is expressed in terms of modified Bessel functions. Thus, the conditional Black–Scholes formula is made further explicit, which results in a closed-form Black–Scholes–Merton-type formula for pricing perpetual timer options under the stochastic volatility model of Hull–White.

The organization of the rest of this paper is as follows. In Section 2, we first introduce the Hull–White volatility model and some basic notations; then we derive the governing partial differential equation for pricing perpetual timer options. In Section 3, we obtain a closed-form solution of the price of the perpetual timer call option under the stochastic volatility model of Hull–White. Section 4 provides some concluding remarks.

2. Timer option under Hull–White model

Consider a derivative asset with a price that depends on the underlying asset price, S_t, with return rate μ and its instantaneous variance v_t. Under the Hull–White model, S_t and v_t are assumed to satisfy the following stochastic differential equations:

$$\begin{aligned} dS_t &= \mu S_t\,dt + \sqrt{v_t} S_t\,dZ_t^1, \quad S_0 > 0, \\ dv_t &= kv_t\,dt + \sigma v_t\,dZ_t^2, \quad v_0 > 0, \end{aligned} \tag{2.1}$$

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where k and σ are two positive parameters and (Z_t^1, Z_t^2) are standard Brownian motions with correlation coefficient ρ. Let

$$Z_t^3 = \frac{1}{\sqrt{1-\rho^2}} Z_t^1 - \frac{\rho}{\sqrt{1-\rho^2}} Z_t^2.$$

It is easy to verify that Z_t^3 is independent of Z_t^2 and the price process of the underlying asset can be rewritten as

$$dS_t = \mu S_t\,dt + \sqrt{v_t} S_t\Big(\rho\, dZ_t^2 + \sqrt{1-\rho^2}\, dZ_t^3\Big), \quad S_0 > 0. \tag{2.2}$$

A timer option is characterized by its random expiry date, which is linked to the accumulated variance of S_t. An investor sets a target variance budget, B. Then the first time that the realized variance reaches the pre-determined variance budget B is the stopping time or the random maturity time, denoted by τ_B. In the continuous-time framework,

$$\tau_B = \inf\Big\{t > 0 \;\Big|\; \int_0^t v_s\, ds = B\Big\}.$$

The payoff of a timer call is $H(S_{\tau_B \wedge T}) = \max(S_{\tau_B \wedge T} - K, 0)$, where K is the strike price and T is the maturity. When $T \to \infty$, it corresponds to a perpetual timer option.

We define the accumulated variance at the valuation date t by

$$I_t = \int_0^t v_s\, ds, \tag{2.3}$$

which is often referred to as a business clock or a stochastic variance clock. If I_t exceeds the variance budget B, the option would expire; otherwise, the price of the timer option will be calculated at time t, $0 < t < \tau_B$.

We are now ready to derive the governing partial differential equation for the price of a perpetual timer option. There are two sources of randomness denoted by Z_t^2 and Z_t^3, respectively, in equations (2.1) and (2.2). The hedging can be done by using two timer options with different fixed maturities, T and T'. The self-financing portfolio Π_t contains a timer option $C(t)$, a quantity $-\Delta_1$ of the auxiliary option $G(t)$ and a quantity $-\Delta_2$ of the underlying asset S_t at time t.

Proposition 2.1. *Let $\Pi_t = C(t) - \Delta_1 G(t) - \Delta_2 S_t$ be the portfolio illustrated above. Assume that the pricing functions $C(t, s, v, I) \in \mathbb{C}^{1,2,2,1}$ and $G(t, s, v, I) \in \mathbb{C}^{1,2,2,1}$. Then, under the arbitrage-free principle,*

$$C_t + \tfrac{1}{2} v s^2 C_{ss} + v C_I + \tfrac{1}{2}\sigma^2 v^2 C_{vv} + \rho\sigma v^{3/2} s C_{vs} + r(sC_s - C) + [kv - \sigma\Delta] C_v = 0,$$

where $\Delta_1 = C_v/G_v$, $\Delta_2 = C_s - \Delta_1 G_s$ and Δ is related to the market price of volatility risk.

Note that the subscripts in C and G indicate their corresponding partial derivatives.

PROOF. Applying the Itô–Doeblin formula [9] to Π_t,

$$\begin{aligned} d\Pi_t = {} & C_t\,dt + C_v\,dv_t + C_s\,dS_t + vC_I\,dt + \tfrac{1}{2}[C_{vv}\,dv_t\,dv_t + C_{ss}\,dS_t\,dS_t + 2C_{sv}\,dS_t\,dv_t] \\ & - \Delta_1(G_t\,dt + G_v\,dv_t + vG_I\,dt + G_s\,dS_t + \tfrac{1}{2}G_{vv}\,dv_t\,dv_t + C_{sv}\,dS_t\,dv_t \\ & + \tfrac{1}{2}C_{ss}\,dS_t\,dS_t) - \Delta_2\,dS_t, \end{aligned}$$

where v_t, S_t and I_t satisfy equations (2.1), (2.2) and (2.3), respectively. Using these equations, we obtain the coefficient of dZ_t^2 in $d\Pi_t$ as

$$\sigma v C_v + \rho\sqrt{v}sC_s - \Delta_1\sigma vG_v - \Delta_1\rho\sqrt{v}sG_s - \Delta_2\rho\sqrt{v}s$$

and the coefficient of dZ_t^3 as

$$\sqrt{1-\rho^2}\,\sqrt{v}sC_s - \Delta_1\sqrt{1-\rho^2}\,\sqrt{v}sG_s - \Delta_2\sqrt{1-\rho^2}\,\sqrt{v}s.$$

In order to make the portfolio instantaneously risk-free in the sense that the randomness introduced by Brownian motions Z_t^2, Z_t^3 vanishes, assuming that C_v and G_v are nonzero, we choose

$$\Delta_1 = \frac{C_v}{G_v}, \quad \Delta_2 = C_s - \Delta_1 G_s.$$

Applying the arbitrage-free principle, $d\Pi_t = r\Pi_t\,dt$,

$$\begin{aligned} & C_t + kvC_v + rsC_s + \tfrac{1}{2}vs^2C_{ss} + vC_I + \tfrac{1}{2}\sigma^2v^2C_{vv} + \rho\sigma v^{3/2}sC_{vs} - rC \\ & \quad = \Delta_1G_t + \Delta_1kvG_v + \Delta_1vG_I + \Delta_1rsG_s + \Delta_1\tfrac{1}{2}vs^2G_{ss} + \Delta_1\tfrac{1}{2}\sigma^2v^2G_{vv} \\ & \qquad + \Delta_1\rho\sigma v^{3/2}sG_{vs} - r\Delta_1G. \end{aligned}$$

Alternatively,

$$\begin{aligned} & \frac{1}{C_v}(C_t + vC_I + rsC_s + \tfrac{1}{2}vs^2C_{ss} + \tfrac{1}{2}\sigma^2v^2C_{vv} + \rho\sigma v^{3/2}sC_{vs} - rC) \\ & \quad = \frac{1}{G_v}(G_t + vG_I + rsG_s + \tfrac{1}{2}vs^2G_{ss} + \tfrac{1}{2}\sigma^2v^2G_{vv} + \rho\sigma v^{3/2}sG_{vs} - rG). \end{aligned} \tag{2.4}$$

Equation (2.4) holds if and only if both sides of the equation equal some function ψ independent of the expiry dates. Using the results of Fouque et al. [6, Ch. 2.4],

$$\psi = -\Big[kv - \sigma v\Big(\rho\frac{\mu - r}{\sqrt{v}} + \phi(t, s, v, I)\sqrt{1-\rho^2}\Big)\Big],$$

such that

$$C_t + kvC_v + rsC_s + \tfrac{1}{2}vs^2C_{ss} + vC_I + \tfrac{1}{2}\sigma^2v^2C_{vv} + \rho\sigma v^{3/2}sC_{vs} - rC = \sigma\Delta C_v,$$

where $\Delta = \rho(\mu - r)\sqrt{v} + v\phi(t, s, v, I)\sqrt{1-\rho^2}$ and $\phi(t, s, v, I)$ is the market price of the volatility risk. □

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In this study, the market is assumed to be complete by trading auxiliary volatility derivatives, which are priced in a risk-neutral probability measure $\mathbb{Q}$ with $\Delta = 0$.

Now let $x_t = \ln(S_t/S_0)$. Using the Itô–Doeblin formula,

$$dx_t = \Big(r - \frac{v_t}{2}\Big)dt + \sqrt{v_t}\Big(\rho\, dZ_t{}^2 + \sqrt{1-\rho^2}\, dZ_t{}^3\Big), \quad x_0 = 0. \tag{2.5}$$

COROLLARY 2.2. *Assume that $C(I_t, x_t, v_t) \in C^{1,2,2}$; then the function C, as the price of a perpetual timer call option, satisfies the following partial differential equation:*

$$vC_I + \frac{1}{2}vC_{xx} + \rho\sigma v^{3/2}C_{xv} + \frac{1}{2}\sigma^2 v^2 C_{vv} + \Big(r - \frac{v}{2}\Big)C_x - rC + kvC_v = 0, \tag{2.6}$$

with the boundary condition

$$C(B, x_{\tau_B}, v_{\tau_B}) = h(x_{\tau_B}) = (S_0 e^{x_{\tau_B}} - K)^+ \tag{2.7}$$

in the region $0 \le I_t \le B$, $x > 0$ and $v > 0$.

REMARK 2.3. Unlike pricing perpetual American options where a degenerate elliptic equation is obtained, the partial differential equation (PDE) for the price of perpetual timer options is still a Black–Scholes–Merton-type parabolic equation [16] with the new time clock being the variance clock I_t which satisfies equation (2.3), rather than the natural time clock t,

$$C_I + \Big(\frac{r}{v} - \frac{1}{2}\Big)C_x + \frac{1}{2}C_{xx} + \rho\sigma\sqrt{v}C_{xv} + \frac{1}{2}\sigma^2 vC_{vv} + kC_v - \frac{r}{v}C = 0.$$

THEOREM 2.4. *The price of a perpetual timer call satisfying equation (2.6) with variance budget B and payoff function $h(x_{\tau_B})$ as defined in equation (2.7) is given by*

$$C(\xi, x, v) = \mathrm{E}^{\mathbb{Q}}[e^{-r(\tau_B - \tau_\xi)}h(x_{\tau_B}) | x_t = x, v_t = v, I_t = \xi],$$

where I_t, x_t and v_t satisfy the equations (2.3), (2.5) and (2.1), respectively.

PROOF. Let $C(I_t, x_t, v_t)$ be the solution of PDE (2.6). Recall that $\tau_{I_t} = t$. Applying the Itô–Doeblin formula to $e^{-r\tau_{I_t}}C(I_t, x_t, v_t)$,

$$\begin{aligned}
&de^{-r\tau_{I_t}}C(I_t, x_t, v_t)\\
&\quad= e^{-r\tau_{I_t}}[-rC\,dt + C_I\,dI_t + C_x\,dx + C_v\,dv + \tfrac{1}{2}(C_{xx}\,dx\,dx + C_{vv}\,dv\,dv) + C_{xv}\,dx\,dv]\\
&\quad= e^{-r\tau_{I_t}}\Big[vC_I + \frac{1}{2}vC_{xx} + \rho\sigma v^{3/2}C_{xv} + \frac{1}{2}\sigma^2 v^2 C_{vv} + \Big(r - \frac{v}{2}\Big)C_x - rC + kvC_v\Big]dt\\
&\qquad + e^{-r\tau_{I_t}}\sqrt{v}C_x\Big(\sqrt{1-\rho^2}\,dZ_t^3 + \rho\,dZ_t^2\Big) + e^{-r\tau_{I_t}}C_v\sigma v\,dZ^2\\
&\quad= e^{-r\tau_{I_t}}\sqrt{v}C_x\Big(\sqrt{1-\rho^2}\,dZ_t^3 + \rho\,dZ_t^2\Big) + e^{-r\tau_{I_t}}C_v\sigma v\,dZ_t^2.
\end{aligned}$$

Integrating the above equation from τ_ξ to τ_B,

$$\begin{aligned}
&e^{-r\tau_B}C(B, x_{\tau_B}, v_{\tau_B}) - e^{-r\tau_{I_t}}C(I_t, x_t, v_t)\\
&\quad= \int_{\tau_\xi}^{\tau_B} e^{-r\tau_{I_s}}\sqrt{v}C_x\Big(\sqrt{1-\rho^2}\,dZ_s^3 + \rho\,dZ_s^2\Big) + \int_{\tau_\xi}^{\tau_B} e^{-r\tau_{I_s}}C_v\sigma v\,dZ_s^2.
\end{aligned}$$

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Taking the conditional expectation with respect to $x_t = x, v_t = v, I_t = \xi$, and using the boundary condition (2.7),

$$C(\xi, x, v) = \mathrm{E}^{\mathbb{Q}}[e^{-r(\tau_B-\tau_\xi)}C(B, x_{\tau_B}, v_{\tau_B})|x_t = x, v_t = v, I_t = \xi]$$

and this completes the proof. □

3. The solution of perpetual timer options

Recall that the variance clock I_t is defined by equation (2.3). By the inverse function theorem,

$$\frac{dI^{-1}(t)}{dt} = \frac{1}{v(I^{-1}(t))},$$

so that

$$\tau_B = I^{-1}(B) = \int_0^B \frac{1}{v(I^{-1}(s))}\, ds.$$

For simplicity, $v(I^{-1}(t))$ and $x(I^{-1}(t))$ are denoted as V_t and X_t, respectively. By the Dubins–Dambis–Schwarz theorem [10, p. 174], $\sqrt{I^{-1}(t)}Z^2_{I^{-1}(t)}$ and $\sqrt{I^{-1}(t)}Z^3_{I^{-1}(t)}$ are standard Brownian motions, denoted by W_t^2 and W_t^3, respectively. Thus, the stochastic differential equations on V_t and X_t, $t \in [0, B]$, may be rewritten as

$$dX_t = \left(\frac{r}{V_t} - \frac{1}{2}\right)dt + \left(\rho\, dW_t{}^2 + \sqrt{1-\rho^2}\, dW_t{}^3\right), \tag{3.1}$$

$$dV_t = k\, dt + \sigma\sqrt{V_t}\, dW_t{}^2. \tag{3.2}$$

Define

$$z_t = \frac{2}{\sigma}\sqrt{V_t}, \quad y_t = X_t - \rho z_t.$$

The equations (3.1) and (3.2) are transformed to

$$dy_t = \left[\frac{r}{b(z)} - \frac{1}{2} - \rho\, a(z)\right]dt + \sqrt{1-\rho^2}dW_t{}^3,$$
$$dz_t = a(z)\, dt + dW_t{}^2, \tag{3.3}$$

respectively, where

$$\frac{1}{b(z)} = \frac{4}{\sigma^2}\frac{1}{z^2}, \quad a(z) = \left(\frac{2k}{\sigma^2} - \frac{1}{2}\right)\frac{1}{z}.$$

Definition 3.1 [15, 19]. For any $\delta \geq 2$, the δ-dimensional Bessel process BES^δ is a diffusion process $z(t)$ which is the unique strong solution to the following stochastic differential equation (SDE):

$$dz_t = \frac{\delta-1}{2z_t}\, dt + dW_t^2, \quad z_0 = \frac{2}{\sigma}\sqrt{v_0} > 0,$$

where W_t^2 is a standard Brownian motion.

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For $\delta \geq 2$, BES^{δ} processes will never reach 0 for $t > 0$. For a detailed analysis of the properties of Bessel processes, we refer to the work of Shiga and Watanabe [21], Pitman and Yor [18] and Yor [22].

Note that $z(t)$ defined in equation (3.3) is a Bessel process for any $\delta = 4k/\sigma^2 \geq 2$. Hence, X_t has a solution for $t \in [0, B]$:

$$X_B = X_0 + r\tau_B - \frac{B}{2} + \rho z_B - \rho z_0 - \int_0^B \frac{\rho}{z_t}\Big(\frac{2k}{\sigma^2} - \frac{1}{2}\Big)\,dt + \int_0^B \sqrt{1-\rho^2}\,dW_t^3. \tag{3.4}$$

THEOREM 3.2. *Assume that*

$$\phi_t = \int_0^t \frac{1}{z_s}\,ds < \infty \quad \text{and} \quad \int_0^t \frac{1}{z_s^2}\,ds < \infty.$$

We have the following conditional Black–Scholes–Merton-type pricing formula [17]:

$$\begin{aligned} C_0 &= \mathrm{E}^{\mathbb{Q}}[e^{-r\tau_B}\max(S_0 e^{X_B} - K, 0)] \\ &= \mathrm{E}^{\mathbb{Q}}[S_0 e^{d_0(z_B,\phi_B,\tau_B) - r\tau_B + rB} N(d_2(z_B,\phi_B,\tau_B)) - Ke^{-r\tau_B} N(d_1(z_B,\phi_B,\tau_B))], \end{aligned}$$

where $N(\cdot)$ is the cumulative distribution for the standard normal random variable, and

$$\begin{aligned} d_0(z_B,\phi_B,\tau_B) &= r(\tau_B - B) + \rho(z_B - z_0) - \rho\Big(\frac{2k}{\sigma^2} - \frac{1}{2}\Big)\phi_B - \frac{\rho^2}{2}B, \\ d_1(z_B,\phi_B,\tau_B) &= \frac{\log(S_0/K) + rB + d_0(z_B,\phi_B,\tau_B) - (1-\rho^2)B/2}{\sqrt{(1-\rho^2)B}}, \\ d_2(z_B,\phi_B,\tau_B) &= d_1(z_B,\phi_B,\tau_B) + \sqrt{(1-\rho^2)B}. \end{aligned}$$

Furthermore, let $p_{z_B,\phi_B,\tau_B}(z,h,g)$ be the joint density on (z_B,ϕ_B,τ_B). Then

$$C_0 = \int_{R_B^3} [S_0 e^{d_0(z,h,g) - rg + rB} N(d_2(z,h,g)) - Ke^{-rg} N(d_1(z,h,g))]\, p_{z_B,\phi_B,\tau_B}(z,h,g)\,dz\,dh\,dg,$$

where

$$\begin{aligned} p_{z_B,\phi_B,\tau_B}(z,h,g;B) &= \frac{2e^{\theta B}}{\pi}\int_0^\infty \cos(\lambda B)\mathrm{Re}(\tilde{P}(z,h,g;\theta+i\lambda))\,d\lambda, \\ \tilde{P}(z,h,g;\alpha) &= \frac{\alpha\sqrt{2\alpha}z^{\nu+1}}{8z_0^{\nu}\sinh(h\sqrt{\alpha/2})}\exp\Big(-\frac{\nu^2 g\sigma^2}{8} - (z_0+z)\sqrt{2\alpha}\coth\Big(h\sqrt{\frac{\alpha}{2}}\Big)\Big) i_{g\sigma^2/32} \\ &\quad \times \Big(\frac{2\sqrt{2\alpha z_0 z}}{\sinh(h\sqrt{\alpha/2})}\Big), \quad \text{with } \alpha = \theta + i\lambda \end{aligned}$$

and

$$i_y(\beta) = \frac{\beta e^{\pi^2/4y}}{\pi\sqrt{y\pi}}\int_0^\infty \exp\Big(-\beta\cosh u - \frac{u^2}{4y}\Big)\sinh u \sin\Big(\frac{\pi u}{2y}\Big)\,du.$$

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PROOF. By Theorem 2.4, the price of the timer call option under the risk-neutral measure should be

$$\begin{aligned} C_0 &= \mathrm{E}^{\mathbb{Q}}[e^{-r\tau_B}\max(S_0e^{X_B}-K,\ 0)] \\ &= \mathrm{E}^{\mathbb{Q}}[\mathrm{E}^{\mathbb{Q}}[e^{-r\tau_B}\max(S_0e^{X_B}-K,\ 0)|z_B,\tau_B,\phi_B]]. \end{aligned}$$

With equation (3.4),

$$\begin{aligned} X_B &= rB - \frac{1-\rho^2}{2}B + r\tau_B - \frac{B}{2} - rB + \frac{1-\rho^2}{2}B + \rho z_B - \rho z_0 \\ &\quad - \int_0^B \frac{\rho}{z_t}\Big(\frac{2k}{\sigma^2}-\frac{1}{2}\Big)\,dt + \int_0^B \sqrt{1-\rho^2}\,dW_t^3 \\ &= \Big(r - \frac{1-\rho^2}{2}\Big)B + \int_0^B \sqrt{1-\rho^2}\,dW_t^3 + r(\tau_B - B) - \frac{\rho^2}{2}B \\ &\quad + \rho(z_B - z_0) - \int_0^B \frac{\rho}{z_t}\Big(\frac{2k}{\sigma^2}-\frac{1}{2}\Big)\,dt \\ &= \Big(r - \frac{1-\rho^2}{2}\Big)B + \int_0^B \sqrt{1-\rho^2}\,dW_t^3 + d_0, \end{aligned}$$

where

$$d_0 = r\tau_B - rB + \rho(z_B - z_0) - \int_0^B \frac{\rho}{z_t}\Big(\frac{2k}{\sigma^2}-\frac{1}{2}\Big)\,dt - \frac{\rho^2}{2}B.$$

Let $Y = -W_B^3/\sqrt{B}$; then

$$\begin{aligned} C_0 &= \mathrm{E}^{\mathbb{Q}}\Big[\mathrm{E}^{\mathbb{Q}}\Big[e^{-r\tau_B}\Big(S_0\exp\Big\{\Big(r-\frac{1-\rho^2}{2}\Big)B - \sqrt{(1-\rho^2)B}Y + d_0\Big\} - K\Big)^+\Big| z_B,\tau_B,\phi_B\Big]\Big] \\ &= \int_{R_B^3}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-r\tau_B}\Big(S_0\exp\Big\{\Big(r-\frac{1-\rho^2}{2}\Big)B - \sqrt{(1-\rho^2)B}y + d_0\Big\} - K\Big)^+ \\ &\quad \times e^{-y^2/2}dy\, d\mathcal{H}_{z_B,\phi_B,\tau_B}(z,h,g), \end{aligned}$$

where $\mathcal{H}_{z_B,\phi_B,\tau_B}(z,h,g)$ is the joint distribution function of (z_B,ϕ_B,τ_B), and the joint density function is denoted by $p_{z_B,\phi_B,\tau_B}(z,h,g)$.

Let

$$d_1 = \frac{\log(S_0/K) + rB + d_0(z_B,\phi_B,\tau_B) - (1-\rho^2)B/2}{\sqrt{(1-\rho^2)B}} \quad \text{and}$$
$$d_2 = d_1 + \sqrt{(1-\rho^2)B}.$$

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Then the inner integral in C_0 becomes

$$
\begin{aligned}
\hat{C} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-r\tau_B}\Big(S_0 \exp\Big\{\Big(r - \frac{1-\rho^2}{2}\Big)B + \sqrt{(1-\rho^2)B}y + d_0\Big\} - K\Big)^+ e^{-y^2/2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-r\tau_B}\Big(S_0 \exp\Big\{\Big(r - \frac{1-\rho^2}{2}\Big)B + \sqrt{(1-\rho^2)B}y + d_0\Big\} - K\Big) e^{-y^2/2} dy \\
&= \frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{d_1} \exp\Big\{\Big(r - \frac{1-\rho^2}{2}\Big)B + \sqrt{(1-\rho^2)B}y + d_0\Big\} e^{-y^2/2} dy \\
&\quad - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} K e^{-r\tau_B} e^{-y^2/2} dy \\
&= \frac{S_0}{\sqrt{2\pi}} e^{d_0 - r\tau_B + rB} \int_{-\infty}^{d_1} e^{-(1/2)\left(y + \sqrt{(1-\rho^2)B}\right)^2} dy - K e^{-r\tau_B} N(d_1) \\
&= S_0 e^{d_0 - r\tau_B + rB} N(d_2) - K e^{-r\tau_B} N(d_1),
\end{aligned}
$$

which is the Black–Scholes–Merton formula. Hence, by properties of conditional expectation,

$$C_0 = \mathrm{E}^{\mathbb{Q}}[S_0 e^{d_0(z_B,\phi_B,\tau_B) - r\tau_B + rB} N(d_2(z_B,\phi_B,\tau_B)) - K e^{-r\tau_B} N(d_1(z_B,\phi_B,\tau_B))].$$

In order to establish an explicit expression of the joint density function $p_{z_B,\phi_B,\tau_B}(z,h,g)$, the following lemma is needed.

Lemma 3.3 [3]. *For a Bessel process z_t with $z_0 > 0$, index $\nu = \delta/2 - 1$ and $\nu \geq 0$, suppose that T is any positive real number independent of z_t and $T \sim \mathrm{Exp}(\alpha)$ with intensity α; then the following is true:*

$$
\begin{aligned}
&P\Big(\int_0^T \frac{ds}{z_s^2} \in dg, \int_0^T \frac{ds}{z_s} \in dh, z_T \in dz\Big) \\
&= \frac{\alpha\sqrt{2\alpha} z^{\nu+1}}{8 z_0^{\nu} \sinh(h\sqrt{\alpha/2})} \exp\Big(-\frac{\nu^2 g}{2} - \frac{(z_0 + z)\sqrt{2\alpha}\cosh(h\sqrt{\alpha/2})}{\sinh(h\sqrt{\alpha/2})}\Big) i_{g/8} \\
&\quad \times \Big(\frac{2\sqrt{2\alpha z_0 z}}{\sinh(h\sqrt{\alpha/2})}\Big) dg\, dh\, dz.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&P\Big(\int_0^T \frac{ds}{z_s^2} \in dg, \int_0^T \frac{ds}{z_s} \in dh, z_T \in dz\Big) \\
&\qquad = \alpha \int_0^{\infty} e^{-\alpha t} P\Big(\int_0^t \frac{ds}{z_s^2} \in dg, \int_0^t \frac{ds}{z_s} \in dh, z_t \in dz\Big) dt.
\end{aligned}
$$

By Lemma 3.3 and the Laplace inverse transform, we derive that

$$P\Big(\int_0^B \frac{ds}{z_s^2} \in dg, \int_0^B \frac{ds}{z_s} \in dh, z_B \in dz\Big) = \frac{2e^{\theta B}}{\pi} \int_0^{+\infty} \cos(\lambda B)\mathrm{Re}(P(z,h,g;\theta + i\lambda))\, d\lambda,$$

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where $\alpha = \theta + i\lambda$, $\theta > 0$ and

$$P(z, h, g; \alpha) = \frac{\alpha\sqrt{2\alpha}z^{\nu+1}}{8z_0^\nu \sinh(h\sqrt{\alpha/2})} \exp\Big(-\frac{\nu^2 g}{2} - (z_0 + z)\sqrt{2\alpha}\coth\Big(h\sqrt{\frac{\alpha}{2}}\Big)\Big) i_{g/8}\Big(\frac{2\sqrt{2\alpha}z_0 z}{\sinh(h\sqrt{\alpha/2})}\Big).$$

Hence, we obtain the expression of the triple joint density function on (z_B, ϕ_B, τ_B) as

$$p_{z_B,\phi_B,\tau_B}(z, h, g; B) = \frac{2e^{\theta B}}{\pi}\int_0^{+\infty} \cos(\lambda B)\mathrm{Re}(\tilde{P}(z, h, g; \theta + i\lambda))\, d\lambda,$$

where

$$\tilde{P}(z, h, g; \alpha) = \frac{\alpha\sqrt{2\alpha}z^{\nu+1}}{8z_0^\nu \sinh(h\sqrt{\alpha/2})} \times \exp\Big(-\frac{\nu^2 g\sigma^2}{8} - (z_0 + z)\sqrt{2\alpha}\coth\Big(h\sqrt{\frac{\alpha}{2}}\Big)\Big) i_{g\sigma^2/32}\Big(\frac{2\sqrt{2\alpha}z_0 z}{\sinh(h\sqrt{\alpha/2})}\Big).$$

This completes the proof of Theorem 3.2. □

4. Conclusion

In this paper, a thorough investigation is carried out on the analytical tractability of the price of perpetual timer options under the Hull–White stochastic volatility model. An explicit formula for the triple joint transition density for the instantaneous volatility, the cumulative reciprocal volatility and the cumulative realized variance is derived. Thus, a closed-form Black–Scholes–Merton-type formula is obtained for pricing perpetual timer options under the Hull–White stochastic volatility model. Most of the existing analytical formula for the joint density functions and characteristic functions are in terms of the joint density of the instantaneous variance and the continuous realized variance (quadratic variation). Our result fills the gap in the literature by providing a complete description of the triple joint distribution under the Hull–White stochastic volatility model. The techniques used in this paper could be extended to derive pricing formula for barrier timer options under the Hull–White model.

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