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RIESZ SETS AND THE RADON-NIKODYM PROPERTY

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Abstract

Let X be a complex Banach space, G a compact abelian group and Λ a subset of $\hat{G}$, the dual group of G. Then $L^1_\Lambda(G, X)$ has the Radon-Nikodym property if and only if X has the Radon-Nikodym property and Λ is a Riesz set. In particular, $H^1(\mathbb{T}, X)$ has the Radon-Nikodym property if and only if X has the Radon-Nikodym property. This solves a problem of Hensgen.

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1. Introduction

This note can be considered as a continuation of the work of Sundaresan [8] and an extension of a result of Lust-Piquard [6]. In [8] Sundaresan showed that if (Ω, Σ, μ) is a finite measure space and X is a Banach space then $L^p(\mu, X)$ has the Radon-Nikodym property if and only if $1 < p < \infty$ and X has the Radon-Nikodym property (see also [2] and [9]). On the other hand, Lust-Piquard [6] proved that if G is a compact abelian group and if Λ is a subset of $\hat{G}$, the dual group of G, then $L^1_\Lambda(G)$ has the Radon-Nikodym property if and only if Λ is a Riesz set. In this note we will show that if X is a complex Banach space then $L^1_\Lambda(G, X)$ has the Radon-Nikodym property if and only if X has the Radon-Nikodym property and Λ is a Riesz set. One consequence of this is that $H^1(\mathbb{T}, X)$ has the Radon-Nikodym property if and only if X has the Radon-Nikodym property. This solves a problem of

Hensgen [4,5]. The method of proof we use here is very different from those of Sundaresan and Lust-Piquard. The idea of the proof stems from a method of Costé [1].

2. Preliminaries and results

Throughout this section, G will denote a compact abelian group, $\mathcal{B}(G)$ will denote the σ-algebra of Borel subsets of G and λ will denote normalised Haar measure on G. Let $\Gamma = \hat{G}$ be the dual group of G. If μ is a measure on G and $\gamma \in \Gamma$ we define $\hat{\mu}(\gamma)$ by

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma(y)}\, d\mu(y).$$

A subset Λ of Γ is said to be a Riesz set if every Radon measure μ on G which satisfies $\hat{\mu}(\gamma) = 0$ for all $\gamma \notin \Lambda$ is absolutely continuous with respect to λ. If $G = \mathbb{T}$ then $\Gamma = \mathbb{Z}$ and by the F. and M. Riesz theorem $\mathbb{N}$ is a Riesz subset of $\mathbb{Z}$ (see [7]).

We define for a complex Banach space X,

$$L^1_\Lambda(G, X) = \left\{ f \in L^1(G, X) \colon \hat{f}(\gamma) = \int_G \overline{\gamma(y)} f(y)\, d\lambda(y) = 0 \text{ for all } \gamma \notin \Lambda \right\}.$$

THEOREM. *Let G be a compact abelian group, let Λ be a subset of $\hat{G}$ and let X be a complex Banach space. Then $L^1_\Lambda(G, X)$ has the Radon-Nikodym property if and only if X has the Radon-Nikodym property and Λ is a Riesz set.*

PROOF. Suppose $L^1_\Lambda(G, X)$ has the Radon-Nikodym property. Then so does X and $L^1_\Lambda(G)$, since they are isomorphic to subspaces of $L^1_\Lambda(G, X)$. By the result of Lust-Piquard [6], Λ is a Riesz set because $L^1_\Lambda(G)$ has the Radon-Nikodym property.

Conversely, suppose Λ is a Riesz set and X has the Radon-Nikodym property. To prove that $L^1_\Lambda(G, X)$ has the Radon-Nikodym property it suffices to show that every bounded linear operator from $L^1[0, 1]$ into $L^1_\Lambda(G, X)$ is Bochner representable [2]. To this end, let $T \colon L^1[0, 1] \to L^1_\Lambda(G, X)$ be a bounded linear operator and define

$$F(A \times B) = \int_B T(1_A)(y)\, d\lambda(y)$$

where A is a Lebesgue measurable subset of $[0, 1]$ and $B \in \mathcal{B}(G)$. Let $\mathcal{C}$ denote the algebra generated by the measurable rectangles of $[0, 1] \times G$. It is clear that F is a finitely additive bounded X-valued measure on $\mathcal{C}$.

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For each $E \in \mathcal{C}$, we define $\nu(E) = |F|(E)$, the variation of F on E. An elementary calculation shows that $\nu([0, 1] \times G) \leq \|T\| < \infty$, so ν is finite. Therefore ν is a finitely additive non-negative real-valued measure on $\mathcal{C}$ and F is absolutely continuous with respect to ν. Consequently [2, p. 28, Corollary 3], $F(\mathcal{C})$ is a relatively weakly compact subset of X. Hence, by [2, p. 27, Theorem 2], F has a unique countably additive extension to an X-valued measure, $\overline{F}$, on the σ-algebra generated by $\mathcal{C}$. The σ-algebra generated by $\mathcal{C}$ is the product σ-algebra $\mathcal{A} \times \mathcal{B}(G)$ where $\mathcal{A}$ is the σ-algebra of Lebesgue measurable subsets of $[0, 1]$.

We will show that the measure $\overline{F}: \mathcal{A} \times \mathcal{B}(G) \to X$ is absolutely continuous with respect to $m \times \lambda$, where m is Lebesgue measure on $[0, 1]$. To do this, it suffices to show that if $\{C_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{C}$ such that $(m \times \lambda)(C_n) \to 0$ as $n \to \infty$, then $F(C_n) \to 0$ weakly in X as $n \to \infty$. For $x^* \in X^*$ we define the bounded linear operator $T_{x^*}: L^1[0, 1] \to L^1_\Lambda(G)$ by $(T_{x^*}f)(y) = x^*((Tf)(y))$, where $f \in L^1[0, 1]$ and $y \in G$.

Since Λ is a Riesz set, $L^1_\Lambda(G)$ has the Radon-Nikodym property [6] and so T_{x^*} is Bochner representable. That is, there is a function

$$g_{x^*} \in L^\infty([0, 1], L^1_\Lambda(G))$$

such that

$$T_{x^*}f = \int_{[0,1]} f(t) g_{x^*}(t)\, dm(t)$$

for all $f \in L^1[0, 1]$.

In particular, if $A \in \mathcal{A}$ then

$$T_{x^*}(1_A) = \int_A g_{x^*}(t)\, dm(t).$$

Therefore, for $A \in \mathcal{A}$ and $B \in \mathcal{B}(G)$ we have

$$\begin{aligned}
x^* F(A \times B) &= x^* \left(\int_B T(1_A)(y)\, d\lambda(y) \right) \\
&= \int_B x^*(T(1_A)(y))\, d\lambda(y) \\
&= \int_B (T_{x^*}(1_A))(y)\, d\lambda(y) \\
&= \int_B \left(\int_A g_{x^*}(t)\, dm(t) \right)(y)\, d\lambda(y) \\
&= \int_B \left(\int_A (g_{x^*}(t))(y)\, dm(t) \right) d\lambda(y) \\
&= \int_{A \times B} (g_{x^*}(t))(y)\, d(m \times \lambda)(t, y).
\end{aligned}$$

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Note that,

$$\int_{[0,1]}\int_G |(g_{x^*}(t))(y)|\,d\lambda(y)\,dm(t) = \int_{[0,1]} \|g_{x^*}(t)\|_{L^1_\Lambda(G)}\,dm(t)$$
$$= \|g_{x^*}\|_{L^1([0,1],L^1_\Lambda(G))} < \infty.$$

Hence the function $(g_{x^*}(\cdot))(\cdot) \in L^1([0,1])\times G)$. Therefore, if $(m\times\lambda)(C_n) \to 0$ as $n\to\infty$ then $x^*F(C_n)\to 0$ as $n\to\infty$ and so we have proved that $\overline{F}$ is absolutely continuous with respect to $m\times\lambda$. Also, it is easily seen that $\overline{F}$ is a measure of bounded variation. Consequently, since X has the Radon-Nikodym property there exists a function $g \in L^1([0,1]\times G, X)$ such that

$$\overline{F}(C) = \int_C g(t,y)\,d(m\times\lambda)(t,y)$$

for all $C \in \mathcal{A}\times\mathcal{B}(G)$.

By Fubini's theorem [3, p. 190], the function $g(t,\cdot)$ is an element of $L^1(G,X)$ for m-almost all $t\in[0,1]$. If we set

$$H(t) = \begin{cases} g(t,\cdot) & \text{if } g(t,\cdot)\in L^1(G,X), \\ 0 & \text{otherwise}, \end{cases}$$

then $H\colon [0,1]\to L^1(G,X)$ is an m-measurable function [3, p. 196, Lemma 16(b)] and it is easy to see that $H\in L^1([0,1], L^1(G,X))$. Fix $A\in\mathcal{A}$. Then for all $B\in\mathcal{B}(G)$

$$F(A\times B) = \int_{A\times B} g(t,y)\,d(m\times\lambda)(t,y) = \int_B\left(\int_A g(t,y)\,dm(t)\right)\,d\lambda(y).$$

Therefore

$$\int_B\left(\int_A g(t,y)\,dm(t) - T(1_A)(y)\right)\,d\lambda(y) = 0$$

for all $B\in\mathcal{B}(G)$. Thus

$$T(1_A)(y) = \int_A g(t,y)\,dm(t)$$

for λ-almost all $y\in G$, and so as elements of $L^1(G,X)$

$$T(1_A) = \int_A g(t,\cdot)\,dm(t) = \int_A H(t)\,dm(t).$$

Since T is $L^1_\Lambda(G,X)$-valued so is H, m-almost surely. We define $\tilde{H}\colon [0,1] \to L^1_\Lambda(G,X)$ by

$$\tilde{H}(t) = \begin{cases} H(t) & \text{if } H(t)\in L^1_\Lambda(G,X), \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $A \in \mathcal{A}$, $T(1_A) = \int_A \tilde{H}(t)\,dm(t)$ and $\tilde{H} \in L^1([0, 1], L^1_\Lambda(G, X))$. Hence, by [2, p. 62, Lemma 4] T is Bochner representable. This completes the proof.

If we let $G = \mathbb{T}$ then $\hat{G} = \mathbb{Z}$. We define the Hardy space $H^1(\mathbb{T}, X)$, for a complex Banach space X, by

$$H^1(\mathbb{T}, X) = \{f \in L^1(\mathbb{T}, X) : \hat{f}(n) = 0 \quad \text{for all } n < 0\}.$$

In [4] Hensgen proves that if X is a separable dual space then $H^1(\mathbb{T}, X)$ is also a separable dual and so has the Radon-Nikodym property, and asks the question; If X has the Radon-Nikodym property does $H^1(\mathbb{T}, X)$ have the Radon-Nikodym property?

We give a positive answer to this question.

COROLLARY. *If X is a complex Banach space with the Radon-Nikodym property then $H^1(\mathbb{T}, X)$ has the Radon-Nikodym property.*

PROOF. $H^1(\mathbb{T}, X) = L^1_\Lambda(G, X)$ where $\Lambda = \{n \in \mathbb{Z}; n \geq 0\}$. By the F. and M. Riesz theorem [7] Λ is a Riesz subset of $\mathbb{Z}$. An application of the theorem finishes the proof.

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