Can. J. Math., Vol. XXVII, No. 2, 1975, pp. 446–458

HOLOMORPHIC MAPPINGS OF THE HYPERBOLIC SPACE INTO THE COMPLEX EUCLIDEAN SPACE AND THE BLOCH THEOREM

KYONG T. HAHN

1. Introduction. This paper is to study various properties of holomorphic mappings defined on the unit ball B in the complex euclidean space $\mathbf{C}^n$ with ranges in the space $\mathbf{C}^m$. Furnishing B with the standard invariant Kähler metric and $\mathbf{C}^m$ with the ordinary euclidean metric, we define, for each holomorphic mapping $f: B \rightarrow \mathbf{C}^m$, a pair of non-negative continuous functions q_f and Q_f on B; see § 2 for the definition.

Let $\mathscr{B}(\Omega)$, $\Omega > 0$, be the family of holomorphic mappings $f: B \rightarrow \mathbf{C}^n$ such that $Q_f(z) \leqq \Omega$ for all $z \in B$. $\mathscr{B}(\Omega)$ contains the family $\mathscr{H}(M)$ of bounded holomorphic mappings as a proper subfamily for a suitable $M > 0$.

There arises the question whether or not $\mathscr{B}(\Omega)$, subject to some normalization at $z = 0$, carries a positive Bloch constant.

In [**5**] we have studied this question for the family of holomorphic mappings of B into the complex projective space $\mathbf{P}_n(\mathbf{C})$ furnished with the usual Fubini-Study metric and found a positive lower bound for the Bloch constant of the family. It is, however, not likely to be true for the family $\mathscr{B}(\Omega)$, for $n > 1$.

In § 3, we consider the subfamily $\mathscr{H}(M)$ and obtain a positive lower bound for the Bloch constant of $\mathscr{H}(M)$, subject to the normalization $q_f(0) \geqq \alpha > 0$. We then study the univalent mappings on B in § 4, giving a higher dimensional generalization of the Koebe-Faber distortion theorem (Theorem 3) and lower bounds for the Koebe constants of the families $\mathscr{S}_0(M)$ and $\mathscr{S}(M)$. The notion of normal functions has been a useful tool in the study of boundary behaviour of holomorphic functions of one variable. We extend this to holomorphic mappings in the higher dimensional spaces in § 5 and obtain some interesting results (Theorem 4) for the family of normal mappings of finite order. Theorem 4 generalizes some of the results in [**1**].

2. Preliminaries. Let $w = f(z)$ be a holomorphic mapping of the unit ball $B = \{z \in \mathbf{C}^n: |z| < 1\}$, $|z|^2 = \sum_{\alpha=1}^n z_\alpha \bar{z}_\alpha$, in the complex vector space $\mathbf{C}^n$ into the space $\mathbf{C}^m$ with the ordinary euclidean metric:

$$(1) \qquad d\sigma^2(w) = \sum_{\alpha=1}^{m} dw_\alpha \overline{dw_\alpha}, \qquad w \in C^m.$$

Received October 25, 1973 and in revised form January 21, 1974.

https://doi.org/10.4153/CJM-1975-053-0 Published online by Cambridge University Press

The mapping f pulls the metric (1) back to B inducing the pseudo-metric in B:

$$(2) \qquad d\sigma_f^2(z) \equiv d\sigma^2(f(z)) \equiv dz^* \left(\frac{df}{dz}\right)^* \left(\frac{df}{dz}\right) dz, \qquad z \in B,$$

where $(df/dz)^*$ denotes the conjugate transposed of the Jacobian matrix (df/dz) of f.

We furnish B with the standard Kähler metric:

$$(3) \qquad ds_B^2(z) = \frac{(1 - |z|^2)|dz|^2 + |dz^*z|^2}{(1 - |z|^2)^2};$$

see [**6**, p. 162] for example.

The metric (3) is invariant under any holomorphic automorphisms of B, while it is distance-decreasing under holomorphic mappings of B into itself. Namely, if $w = f(z)$ is a holomorphic automorphism of B, then

$$(4) \qquad ds_B^2(z) = ds_B^2(f(z)), \qquad z \in B,$$

and if $w = f(z)$ is a holomorphic mapping of B into B, then

$$(5) \qquad ds_B^2(f(z)) \leqq ds_B^2(z), \qquad z \in B.$$

Inequality (5) is a higher dimensional generalization of the classical Schwarz-Pick lemma. See [**7**] and [**10**] for more details.

We call the unit ball $B \subset \mathbf{C}^n$ furnished with the metric (2) the *hyperbolic space* of dimension n.

The hyperbolic space concerned in this paper is always of fixed dimension, say n, unless stated otherwise.

For each holomorphic mapping f: $B \to \mathbf{C}^m$, we define

$$(6) \qquad q_f(z) = \inf(d\sigma_f/ds_B)(z, x)$$

and

$$(7) \qquad Q_f(z) = \sup(d\sigma_f/ds_B)(z, x), \qquad z \in B,$$

where inf and sup run over all the unit tangent vectors x at z in B.

From the definitions of q_f and Q_f, and the invariant property of ds_B (see (4)), we have

LEMMA 1. *If* $\zeta = S(z)$ *is a holomorphic automorphism of* B, *then*

$$(8) \qquad q_{f\cdot S}(z) = q_f(S(z))$$

and

$$(9) \qquad Q_{f\cdot L}(z) = Q_f(S(z)), \qquad z \in B.$$

See [**5**, Lemma 1] for the proof.

Observing that $0 \leqq |dz^*z| \leqq |dz|\,|z|$, the following inequalities follow from (3):

$$(10)\quad \frac{|dz|^2}{1-|z|^2} \leqq ds_B{}^2(z) \leqq \frac{|dz|^2}{(1-|z|^2)^2}, \qquad z \in B.$$

From (6), (7) and inequalities (10), we have

LEMMA 2. *If $f: B \to \mathbf{C}^m$ is a holomorphic mapping, then*

$$(11)\quad (1-|z|^2)\lambda_f(z) \leqq q_f(z) \leqq (1-|z|^2)^{1/2}\lambda_f(z)$$

and

$$(12)\quad (1-|z|^2)^{1/2}\Lambda_f(z) \geqq Q_f(z) \geqq (1-|z|^2)\Lambda_f(z).$$

Here λ_f and Λ_f are the positive square roots of the smallest and largest characteristic values, respectively, of $(df/dz)^*(df/dz)$.

We remark that the inequalities in Lemma 2 are sharp, equality being held at $z = 0$. The second inequalities of (11) and (12) may be replaced by the following inequalities:

$$(11)\quad q_f(z) \leqq (1-|z|^2)\Lambda_f(z)$$

and

$$(12)\quad Q_f(z) \geqq (1-|z|^2)^{1/2}\lambda_f(z),$$

respectively.

Finally, we review briefly the generalized notions of the Bloch and Koebe constants. See [**4**] for details.

Let $w = f(z)$ be a holomorphic mapping defined on the ball B in the space $\mathbf{C}^n$ into $\mathbf{C}^n$. On B, we define a non-negative continuous function by

$$d_f(z) = \sup\{r > 0: \text{there exists a subdomain } G \subset B \text{ in which } f \text{ is univalent and } B(f(z), r) \subset f(G)\}$$

if $J_f(z) \neq 0$, and $d_f(z) = 0$ if $J_f(z) = 0$. Here $J_f(z)$ denotes the Jacobian of f and $B(z, r) = \{\zeta: |\zeta - z| < r\}$.

The Bloch constant of B relative to the family $\mathscr{H}$ of holomorphic mappings $f: B \to \mathbf{C}^n$ is defined formally by

$$(13)\quad \beta(\mathscr{H}) = \inf\{b(f): f \in \mathscr{H}\},$$

where

$$(14)\quad b(f) = \sup\{d_f(z): z \in B\}.$$

Let $\mathscr{S}_0$ (or $\mathscr{S}$) be the family of univalent holomorphic mappings $f: B \to \mathbf{C}^n$ such that

$$(15)\quad f(0) = 0$$

https://doi.org/10.4153/CJM-1975-053-0 Published online by Cambridge University Press

and

(16) $(df/dz)(0) = I_n$ (or $J_f(0) = 1$),

where $J_f = \det(df/dz)$ and I_n denotes the identity matrix of order n.

The Koebe constant of B relative to $\mathscr{S}_0$ (or $\mathscr{S}$) is formally defined by

(17) $\kappa_0 \equiv \kappa(\mathscr{S}_0) = \inf\{d_f(0) | f \in \mathscr{S}_0\}$
(or $\kappa \equiv \kappa(\mathscr{S}) = \inf\{d_f(0) | f \in \mathscr{S}\}$).

As we have remarked in [**4**], the Koebe constant of B relative to $\mathscr{S}$ fails to be positive for $n > 1$. We therefore consider the sub-family $\mathscr{S}_0(M)$ (or $\mathscr{S}(M)$) of bounded holomorphic mappings

$$f: B \to B_M, \quad B_M = \{Mz | z \in B\}, \quad M > 0.$$

The Koebe constant $\kappa_0(M)$ (or $\kappa(M)$) relative to $\mathscr{S}_0(M)$ (or $\mathscr{S}(M)$) is shown to be positive by a simple normal family argument.

3. Bounded holomorphic mappings and Bloch theorem.

LEMMA 3. *Let $w = f(z)$ be a holomorphic mapping of the ball $B_R \subset C^n$ into the ball $B_M \subset \mathbf{C}^n$, where $B_R = \{Rz: z \in B\}$. Then the positive square root Λ_f of the largest characteristic value of the hermitian matrix $(df/dz)^*(df/dz)$ satisfies the following inequality:*

(1) $\Lambda_f(z) \leqq R(M^2 - |f(z)|^2)^{1/2}/(R^2 - |z|^2)$

for $z \in B_R$. In particular,

(2) $\Lambda_f(z) \leqq RM/(R^2 - |z|^2), \quad z \in B_R.$

Proof. From the Schwarz-Pick lemma and (10) of § 2 we obtain

(3) $$dz^*\left(\frac{df}{dz}\right)^*\left(\frac{df}{dz}\right)dz \leqq \frac{R^2(M^2 - |f(z)|^2)}{(R^2 - |z|^2)^2}|dz|^2$$

from which (1) follows. See also [**5**].

COROLLARY 1. *If $f: B_R \to B_M$ is a holomorphic mapping, then*

(4) $Q_f(z) \leqq [M^2 - |f(z)|^2]^{1/2}/R, \quad z \in B_R.$

In particular,

(5) $Q_f(z) \leqq M/R, \quad z \in B_R.$

Proof. Let $z \in B$ and let $z = S(\zeta)$ be the holomorphic automorphism of B which maps $0 \in B$ to z. The mapping

$$\phi(\zeta) = f(S(\zeta))$$

maps B_R into B_M such that $\phi(0) = f(z)$. By Lemma 3,

$$\Lambda_\phi(\zeta) \leqq R(M^2 - |\phi(\zeta)|^2)^{1/2}/(R^2 - |\zeta|^2).$$

In particular, at $\zeta = 0$,

$$\Lambda_\phi(0) \leqq (M^2 - |f(z)|^2)^{1/2}/R.$$

Since $\Lambda_\phi(0) = Q_\phi(0) = Q_{f.S}(0) = Q_f(S(0)) = Q_f(z)$, we have (4) and (5).

Remark. As shown in Corollary 1, if $f: B \to \mathbf{C}^n$ is bounded, then $Q_f(z)$ is uniformly bounded in B. However, the converse to this fact is obviously false, as the unbounded mapping

$$f(z) = (f_1(z_1), f_2(z_2)) \quad \text{with}$$

$$f_i(z_i) = \frac{1}{2}\log\frac{1+z_i}{1-z_i}, \qquad i = 1, 2,$$

holomorphic in the open unit ball $B \subset \mathbf{C}^2$, satisfies

$$Q_f(z) = \frac{(1-|z|^2)}{2}\,[E + (E^2 - 4F)^{1/2}] < 2,$$

where

$$E = \frac{1-|z_1|^2}{|1-z_1^2|^2} + \frac{1-|z_2|^2}{|1-z_2^2|^2}, \qquad F = \frac{1-|z|^2}{|(1-z_1^2)(1-z_2^2)|^2}$$

and $|z|^2 = |z_1|^2 + |z_2|^2$.

The following higher dimensional analogue of the classical result of Landau [**8**] plays an essential role in the rest of this paper. See [**5**, Theorem 2] for the proof.

LEMMA 4. *Let $w = f(z)$ be a holomorphic mapping of the ball $B_R \subset \mathbf{C}^n$ into $B_M \subset \mathbf{C}^n$. Let $\lambda_f(z)$ denote the square root of the smallest characteristic value of the matrix $(df/dz)^*(df/dz)$ at $z \in B_R$. If $\lambda_f(0) \neq 0$, then the following hold:*

(a) *$w = f(z)$ is univalent (one-to-one) in the ball B_{r_0} with*

$$r_0 = 3^{1/2}R^2\lambda_f(0)/9M;$$

(b) *$w = f(z)$ maps B_{r_0} onto a domain which contains $B(f(0), \gamma_0\lambda_f(0)/2)$, the ball of radius $r_0\lambda_f(0)/2$ centered at $f(0)$.*

By $\mathscr{H}(M)$ we denote the family of bounded holomorphic mappings $f: \bar{B} \to B_M \subset \mathbf{C}^n$.

THEOREM 1. *The Bloch constant β of the family $\mathscr{H}(M)$ with the condition:*

(6) $\quad q_f(0) \geqq \alpha$ *for some* $\alpha > 0$

satisfies the following inequalities:

(7) $\quad \beta \geqq 3^{1/2}\alpha^2/18M$

and

(8) $\quad \beta \geqq 3^{1/2}\alpha^2\kappa_0(N)/9M$

with

(9) $N = \frac{M}{2\alpha\rho_0}\log(1+\rho_0)/(1-\rho_0), \qquad \rho_0 = 3^{1/2}\alpha/9M,$

where $\kappa_0(N)$ *denotes the Koebe constant of the family* $\mathscr{S}_0(N)$ (*see* (17), § 2).

Proof. We define

(10) $g(\zeta) = [\rho_0 A_f(0)]^{-1}[f(\rho_0\zeta) - f(0)],$

where $A_f \equiv (df/dz)$ and $\rho_0 = 3^{1/2}\alpha/9M$. Clearly, $g(0) = 0$ and $(dg/d\zeta)_0 = I_n$. By Lemma 4, $w = g(\zeta)$ is univalent in B. Moreover, for $\zeta \in B$,

(11) $|g(\zeta)| \leqq \rho_0^{-1}\|A_f^{-1}(0)\|\,|f(\rho_0\zeta) - f(0)| \leqq |f(\rho_0\zeta) - f(0)|/\rho_0\alpha.$

By (2) of [**4**, § 3],

(12) $f(\rho_0\zeta) - f(0) = \int_0^1 A_f(s\rho_0\zeta)\rho_0\zeta ds.$

Hence, by Lemma 3,

$$|f(\rho_0\zeta) - f(0)| \leqq \rho_0 \int_0^1 \Lambda_f(s\rho_0\zeta)\, ds \leqq \rho_0 \int_0^1 \frac{M}{1 - s^2\rho_0^2}\, ds = \frac{M}{2}\log\frac{1+\rho_0}{1-\rho_0}.$$

From (11) and (13), we have $|g(\zeta)| \leqq N$ for all $\zeta \in B$ with N given as (9). Thus, $g(B) \subset B_N$. By definition, the Koebe constant satisfies the following inequality:

(14) $\kappa_0 = \kappa_0(N) \leqq \min_{|\zeta|=1} |g(\zeta)|$

or

(15) $\kappa_0\rho_0\alpha \leqq \min_{|\zeta|=1} |f(\rho_0\zeta) - f(0)|.$

This implies that Bloch constant β must satisfy:

$$\beta \geqq \kappa_0\rho_0\alpha \geqq 3^{1/2}\alpha^2\kappa_0(N)/9M,$$

which is (8). Inequality (7) follows from Lemma 4.

COROLLARY 3. *The Bloch constant* β *of the family* $\mathscr{H}(M)$ *of holomorphic mappings* $f\colon \bar{B} \to B_M$ *such that*

(16) $|J_f(0)| = 1$

satisfies the following inequalities:

(17) $\beta \geqq 3^{1/2}/18M^{2n-1}$

https://doi.org/10.4153/CJM-1975-053-0 Published online by Cambridge University Press

and

(18) $\beta \geqq 3^{1/2}\kappa_0(N)/9M^{2n-1}$

with

(19) $N = \dfrac{M}{2\rho_0}\log(1+\rho_0)/(1-\rho_0), \qquad \rho_0 = 3^{1/2}/9M^n.$

Proof. From Lemma 3, if $f \in \mathscr{H}(M)$, then $\Lambda_f(0) \leqq M$. From (16),

$$1 = |J_f(0)| \leqq \lambda_f(0)\Lambda_f^{n-1}(0) \leqq \lambda_f(0)M^{n-1}.$$

Thus, $M^{1-n} \leqq \lambda_f(0)$. From Theorem 1 with $\alpha = M^{1-n}$, the corollary follows.

COROLLARY 4. The Bloch constant β of $\mathscr{H}(M)$ satisfying

(20) $(df/dz)_0 = I_n$

for $f \in \mathscr{H}(M)$, has the following lower bounds:

(21) $\beta \geqq 3^{1/2}/18M$

(22) $\beta \geqq 3^{1/2}\kappa_0(N)/9M$

(23) $N = \dfrac{M}{2\rho_0}\log(1+\rho_0)/(1-\rho_0), \qquad \rho_0 = 3^{1/2}/9M.$

4. Distortion theorems of univalent mappings.

THEOREM 2. *Let $w = f(z)$ be a holomorphic mapping of the unit ball $B \subset \mathbf{C}^n$ into $\mathbf{C}^n$. Then*

(1) $d_f(z) \leqq q_f(z).$

In particular,

(2) $d_f(z) \leqq (1 - |z|^2)\Lambda_f(z)$

and

(3) $d_f(z) \leqq (1 - |z|^2)^{1/2}\lambda_f(z).$

The equalities in (1), (2) *and* (3) *hold for the holomorphic automorphism which maps z to* 0.

Proof. If $J_f(z^0) = 0$ at $z^0 \in B$, then $d_f(z^0) = 0$. So inequality (1) trivially holds. If $J_f(z^0) \neq 0$, then there exists a neighborhood U of z^0 in which $w = f(z)$ is univalent. Let $z = f^{-1}(w)$ be the inverse mapping defined on the ball $B[f(z^0), d_f(z^0)]$. Then

(4) $\eta = h(\xi) = f^{-1}(w^0 + d_f(z^0)\xi)$

is a univalent mapping of B into itself with $h(0) = f^{-1}(w^0) = z^0$. Moreover,

$$(dh/d\xi) = (df^{-1}/d\zeta)(d\zeta/d\xi),$$

where $\zeta = w^0 + d_f(z^0)\xi$. From $d\zeta/d\xi = d_f(z^0)I_n$, we have

(5) $(df/dz)_{z^0} = d_f(z^0)(dh/d\xi)_0^{-1} = d_f(z^0)(d\xi/d\eta)_{z^0}$.

Since $\eta = h(\xi)$ maps B into itself, we have

$$ds_B{}^2(\eta) \leqq ds_B{}^2(\xi), \qquad \eta = h(\xi),$$

by (3), § 2. For $\xi = 0$, $\eta = h(0) = z^0$, and h is one-to-one in B. Hence,

(6) $$\frac{1 - |z^0|^2 \sin^2 \varphi}{(1 - |z^0|^2)^2} I_n \leqq \left(\frac{d\xi}{d\eta}\right)^*_{z^0}\left(\frac{d\xi}{d\eta}\right)_{z^0} = \left(\frac{df}{dz}\right)^*_{z^0}\left(\frac{df}{dz}\right)_{z^0}/d_f{}^2(z_0),$$

where

$$\cos \varphi = |d\eta^* \cdot \eta|/|\eta|\ |d\eta|,$$

evaluated at $\xi = 0$. From (6), we have

(7) $$d_f(z^0) \leqq \left[\frac{d\sigma_f}{ds_B}\right](z^0, x)$$

for all tangent vectors x at z^0 in B, from which (1) follows. Inequalities (2) and (3) are immediate from Lemma 2 and the subsequent remark. In order to complete the proof, we let $w = S(z)$ be the holomorphic automorphism of B which maps $z^0 \in B$ into the origin $0 \in B$. Then, by definition $d_S(z^0) = 1$. Moreover, by (4), § 2,

(8) $(dS/dz)_{z^0}{}^*(dS/dz)_{z^0} = (1 - |z^0|^2 \sin^2 \varphi)I_n/(1 - |z^0|^2)^2$.

Thus,

(9) $q_f(z^0) = Q_f(z^0) = 1$,

(10) $\Lambda^2{}_S(z^0) = 1/(1 - |z^0|^2)^2$

and

(11) $\lambda^2{}_S(z^0) = 1/(1 - |z^0|^2)$.

This shows that equality holds in (1), (2) and (3) for S.

We remark that if $\mathscr{H}$ is a family of holomorphic mappings $f: B \to \mathbf{C}^n$ for which $\beta(\mathscr{H})$ is positive, $\sup_{z\in B} q_f(z)$, $f \in \mathscr{H}$, gives an upper bound for $\beta(\mathscr{H})$ and $\inf_{f\in\mathscr{H}} \sup_{z\in B} q_f(z)$ gives the least of such upper bounds.

We now prove the following higher dimensional analogue of the Koebe-Faber distortion theorem:

THEOREM 3. *Let $w = f(z)$ be a holomorphic mapping of the unit ball B into $\mathbf{C}^n$. If f is univalent in B, then*

(12) $\kappa_0(N_f)q_f(z) \leqq d_f(z), \qquad z \in B,$

https://doi.org/10.4153/CJM-1975-053-0 Published online by Cambridge University Press

where

(13) $N_f = \operatorname{diam} f(B)/\inf_{z\in B} q_f(z)$.

In particular, if $w = f(z)$ *is bounded, i.e.,* $f(B) \subset B_M$ *for some* $M > 0$, *then*

$$(14)\quad q_f(z) \leqq 6\left(\frac{Md_f(z)}{\sqrt{3}}\right)^{1/2}, \qquad z \in B.$$

Proof. Let $\eta = S(\xi)$ be the holomorphic automorphism of B which maps 0 to $z \in B$. We define

$$(15)\quad \varphi(\xi) = A_{f.S}^{-1}(0)[f(S(\xi)) - f(z)],$$

where $A_{f.S}(0)$ denotes the Jacobian matrix of $f \cdot S$ at $z = 0$. Then $w = \varphi(\xi)$ is a univalent holomorphic mapping on B and satisfies $\varphi(0) = 0$, $(d\varphi/d\xi)(0) = I_n$ and

$$(16)\quad |\varphi(\xi)| \leqq ||A_{f\cdot S}^{-1}(0)||\,|f(S(\xi)) - f(z)|, \qquad \xi \in B.$$

Clearly,

$$(17)\quad ||A_{f.S}^{-1}(0)|| = 1/\lambda_{f.S}(0),$$

and by Lemma 1,

$$(18)\quad \lambda_{f.S}(0) = q_{f.S}(0) = q_f(S(0)) = q_f(z).$$

Hence, from (16),

$$(19)\quad \inf_{|\xi|=1} |\varphi(\xi)| \leqq \inf_{|\xi|=1} |f(S(\xi)) - f(z)|/q_f(z)$$

$$(20)\quad \sup_{|\xi|=1} |\varphi(\xi)| \leqq N_f,$$

with N_f given as in (13). By Lemma 4, (b), $w = \varphi(\xi)$ maps a sub-domain of B univalently onto an open ball of radius $\sqrt{3}/18N_f$. Thus,

$$(21)\quad \inf_{|\xi|=1} |\varphi(\xi)| \geqq \sqrt{3}/18N_f.$$

Since f is univalent in B,

$$(22)\quad d_f(z) = \inf_{|\xi|=1} |f(S(\xi)) - f(z)|$$

and

$$(23)\quad \kappa_0(N_f) \leqq \inf_{|\xi|=1} |\varphi(\xi)| = d_\varphi(0).$$

Inequality (19), together with (22) and (23), now implies (12). Inequality (14) follows from (19), (21), and (22) when we observe that $\operatorname{diam} f(B) \leqq 2M$.

It is well-known that for $n = 1$ the Koebe constant for the family of holomorphic functions f defined on the unit disc Δ such that $f(0) = 0$ and $f'(0) = 1$

is precisely 1/4. Therefore, inequality (19) immediately implies the classical distortion theorem of Koebe-Faber [**12**, p. 147]:

(12′) $|f'(z)| \leqq 4d_f(z)/(1 - |z|^2), \quad z \in \Delta,$

when we observe that $q_f(z) = (1 - |z|^2)|f'(z)| = Q_f(z)$.

Corollary 1. *The Koebe constant* $\kappa_0(M)$ *of* B *relative to the family* $\mathcal{S}_0(M)$ (*see* § 2), *satisfies the inequalities*

(24) $3^{1/2}/36M \leqq \kappa_0(M) \leqq 1/4.$

Proof. The first inequality of (24) follows from inequality (14) when we set $z = 0$ and $q_f(0) = 1$. The second inequality was shown in (12) of [**4**, § 3].

We remark that if $f \in \mathcal{S}_0(M)$, then the image of B under $w = f(z)$ can not be contained completely in the ball B_M with $M < 1$. It follows from Lemma 3 as applied to $f \in \mathcal{S}_0(M)$ with $R = 1$ and $z = 0$.

Corollary 2. *The Koebe constant* $\kappa(M)$ *of* B *relative to the family* $\mathcal{S}(M)$ (*see* § 2), *satisfies*

(25) $3^{1/2}/36M^{2n-1} \leqq \kappa(M) \leqq 1/4.$

Proof. By (14), if $z = 0$, then

(26) $\lambda_f^2(0) \leqq 36Md_f(0)/3^{1/2}.$

By Lemma 3, if $f \in \mathcal{S}(M)$, then $\Lambda_f(0) \leqq M$. Hence, from (18) and (16) of § 3, we have

$$3^{1/2}/36M^{2n-1} \leqq d_f(0)$$

for all $f \in \mathcal{S}(M)$, which gives the first inequality of (25). The second inequality follows from (24) when we observe $\kappa(M) \leqq \kappa_0(M)$.

From Theorems 2 and 3 we also have

Corollary 3. *Let* $w = f(z)$ *be a bounded univalent mapping of* $B \subset \mathbf{C}^n$ *into* $\mathbf{C}^n$. *For any sequence* $\{z^{(n)}\}$ *of points in* B,

$$\lim_{n\to\infty} d_f(z^{(n)}) = 0 \quad \textit{if and only if} \quad \lim_{n\to\infty} q_f(z^{(n)}) = 0.$$

In particular, if $\{z^{(n)}\}$ *is a sequence of points in* B *such that* $\lim_{n\to\infty} |z^{(n)}| = 1$, *then* $\lim_{n\to\infty} q_f(z^{(n)}) = 0$.

We note that Corollary 3 is not valid without f being bounded. For example, the univalent mapping $w = f(z)$ defined on $B \subset \mathbf{C}^n$ by

(27) $w_i = f_i(z_i) = \dfrac{1 + z_i}{1 - z_i}, \quad i = 1, 2,$

is unbounded on B. A formal computation shows that

$$(28)\quad q_f(z) = \frac{(1-|z|^2)}{2}\,[E - (E^2 - 4F)^{1/2}],$$

$$(29)\quad Q_f(z) = \frac{(1-|z|^2)}{2}\,[E + (E^2 - 4F)^{1/2}]$$

where

$$(30)\quad E = \frac{4(1-|z_1|^2)}{|1-z_1|^4} + \frac{4(1-|z_2|^2)}{|1-z_2|^4}$$

$$F = \frac{16(1-|z|^2)}{|(1-z_1)(1-z_2)|^4}.$$

Letting $z^{(n)} = (1 - 1/n, 0)$, n positive integer, we get

$$q_f(z^{(n)}) \to \infty, \quad \text{while } d_f(z^{(n)}) \to 0$$

as $n \to \infty$.

The univalency of f, too, seems to be essential for Corollary 3 to be true as the second assertion of Corollary 3 fails to hold for the bounded holomorphic mapping

$$(31)\quad f(z) = e^{\frac{z+1}{z-1}}$$

on the unit disc $\Delta \subset \mathbf{C}^1$. A simple calculation shows that

$$(32)\quad q_f(z) = (1-|z|^2)|f'(z)| = 2/e$$

along the curve: $r = \cos\theta$, where $z = re^{i\theta}$. Therefore, for any sequence $\{z^{(n)}\}$ of points along the curve which tends to $z = 1$ $q_f(z^{(n)}) = 2/\mathrm{e} > 0$. (See [**11**, p. 151].)

5. Concluding remarks. Let $\mathscr{B}(\Omega)$ be the family of all holomorphic mappings $f\colon B \subset \mathbf{C}^n \to \mathbf{C}^m$ such that

$$(1)\quad N_f = \sup_{z\in B} Q_f(z) \leqq \Omega$$

for a positive constant Ω. Clearly, if $\Omega_1 \leqq \Omega_2$, then $\mathscr{B}(\Omega_1) \subset \mathscr{B}(\Omega_2)$. Set

$$(2)\quad \mathscr{B} = \bigcap_{\Omega>0} \mathscr{B}(\Omega).$$

For $n = m = 1$, $\mathscr{B}$ coincides with the class of Bloch functions considered in [**1**]. Since the family $\mathscr{B}(\Omega)$ is equicontinious and every closed bounded subset of $\mathbf{C}^m$ is compact, by [**14**, Lemma 1.1], $\mathscr{B}(\Omega)$ is a normal family (in the sense of Wu).

Following Lehto and Virtanen [**9**], we call a holomorphic mapping $f\colon B \to \mathbf{C}^m$ *normal* if the family $\{f \cdot S\}$, $S \in G$, the group of holomorphic automorphisms of B, forms a normal family. This notion may be extended to a holomorphic mapping of a hermitian manifold into another if the first is homogeneous.

If $f \in \mathscr{B}(\Omega)$, by Lemma 1, § 2,

$$\sup_{z\in B} Q_{f\cdot S}(z) = \sup_{z\in B} Q_f(S(z)) = \sup_{\zeta\in B} Q_f(\zeta) \leqq \Omega$$

for all $S \in G$. Thus, each $f \in \mathscr{B}$ is a normal mapping.

If $N_f < \infty$, we call f a *normal mapping of finite order* N_f. Therefore, all normal mappings of finite order constitute $\mathscr{B}$, while there are normal mappings which are not in $\mathscr{B}$, as the following example shows:

For each $b = (b_1, b_2)$ with $|b_1|^2 + |b_2|^2 = 1$,

$$(3) \qquad f_b(z) = \frac{1 + (\bar{b}_1 z_1 + \bar{b}_2 z_2)}{1 - (\bar{b}_1 z_1 - \bar{b}_2 z_2)}$$

is a holomorphic function defined on the unit ball $B \subset \mathbf{C}^n$ with $\operatorname{Re} f_b > 0$ and $f_b(z) \to \infty$ as $z \to b$, $z \in B$. It is easy to see that f_b is normal. In fact, any holomorphic function f defined on the unit disc $\Delta \subset \mathbf{C}^1$ with $\operatorname{Re} f > 0$ is normal by the classical theorem of Montel. Since f_b is a holomorphic function defined on B with the same range, i.e., $\operatorname{Re} f_b > 0$, by a result due to T. Barth [**2**], f_b is also normal on B. For $z = bt$, $|t| < 1$, $t \in \mathbf{C}^1$,

$$(4) \qquad Q_{f_b}(z) \geqq (1 - |z|^2)\, \Lambda_{f_b}(z) = \frac{2(1 - |t|^2)}{|1 - t|^2} \max (|b_1|, |b_2|),$$

and hence, $\sup_{z\in B} Q_{f_b}(z) = \infty$.

It follows from definition that the sum of two normal mappings is normal if either one of these mappings is bounded. It is, however, not true in general that the sum of any two normal mappings is normal. See [**1**] and the literature given there. On the other hand, the class $\mathscr{B}$ of normal mappings of finite order provides an interesting subclass. In fact, we have the following result.

THEOREM 4. *The class $\mathscr{B}$ forms a Banach space with respect to the norm:*

$$(5) \qquad \|f\|_{\mathscr{B}} = |f(0)| + N_f.$$

Furthermore, let $\mathscr{B}_0$ be the subclass of $\mathscr{B}$ such that $Q_f(z) \to 0$ as $|z| \to 1$, $z \in B$. Then $\mathscr{B}_0$ is a separable closed subspace of $\mathscr{B}$ which is the closure of the polynomials with respect to the norm $\| \;\; \|_{\mathscr{B}}$.

The proof of Theorem 4 can be carried out by following the procedure used in [**1**].

A further study of the class $\mathscr{B}$ and the details of the proof of Theorem 4 will be given in the forthcoming paper.

REFERENCES

1. J. M. Anderson, J. Clunie, and Ch. Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew Math. *270* (1974), 12–37.

2. T. J. Barth, *Taut and tight complex manifolds*, Proc. Amer. Math. Soc. *24* (1970), 429–431.

3. K. T. Hahn, *The non-euclidean Pythagorean theorem with respect to the Bergman metric*, Duke Math. J. *33* (1966), 523–534.

4. ——— *Higher dimensional generalization of the Bloch constant and their lower bounds*, Trans. Amer. Math. Soc. *179* (1973), 263–274.

5. ——— *Quantitative Bloch's theorem for certain classes of holomorphic mappings of the ball into* $P_n(C)$ (to appear).

6. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. 2 (Interscience, New York, 1969).

7. S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings* (Marcel Dekker, New York, 1970).

8. E. Landau, *Über die Blochsche Konstante und zwei verwandte Weltkonstanten*, Math. Z. *30* (1920), 608–634.

9. O. Lehto and K. J. Virtanen, *Boundary behaviour and normal meromorphic functions*, Acta Math. *97* (1957), 47–65.

10. K. H. Look, *Schwarz lemma and analytic invariants*, Sci. Sinica *7* (1958), 453–504.

11. W. Seidel and J. L. Walsh, *On the derivative of functions analytic in the unit circle and their radii of univalence and of p-valence*, Trans. Amer. Math. Soc. *52* (1942), 129–216.

12. W. A. Veech, *A second course in complex analysis* (Benjamin, New York, 1967).

13. H. Wu, *Normal families of holomorphic mappings*, Acta Math. *119* (1967), 193–233.

The Pennsylvania State University,
University Park, Pennsylvania