NUMERICAL EFFICIENCY OF THE ELLIPTIC FUNCTION EXPANSIONS OF THE FIRST-ORDER INTERMEDIARY FOR GENERAL PLANETARY THEORY

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Abstract. We compare numerical efficiency of the two kinds of series for the first-order intermediate orbit for general planetary theory: (1) the classical expansion involving mean longitudes of the planets; (2) an expansion resulting from the theory of elliptic functions. We conclude that mutual perturbations of close couples of planets (the ratio of major semi-axes ~ 1) can be represented in more compact form with the aid of the second kind of series.

1. Two kinds of series for the first-order intermediary

In spite of significant progress of numerical approaches to construct the theories of motion of the major planets, analytical theory of planetary motion remains to be a challenging and interesting scientific task. Recently the interest in constructing analytical planetary theories has been revived by the idea to use an elliptic function of time (instead of time itself) as independent variable in order to make the resulting series more compact (see, Brumberg (1996) and references therein). Our aim is to compare the numerical efficiency of the classical series involving mean longitudes of planets and the series resulting from the application of elliptic functions for the particular case of the first-order intermediate orbit for general planetary theory (Brumberg, 1994).

We consider two kind of series for the first order intermediary $\mu \underset{1}{T}_j^{(i)}+$

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defined by Eq. (5.19) of Brumberg (1994). The ρ-series are

$$\mu \underset{1}{T}{}_{j}^{(i)+} = \sum_{k=-\infty}^{\infty} A_k^{ij} \rho_{ij}^k, \tag{1}$$

$$\rho_{ij} = \exp\left[\overset{\circ}{\imath}\, l_{ij}\right], \quad l_{ij} = 2\varphi_{ij}, \quad \varphi_{ij} = \frac{1}{2}\left(\pi - (\lambda_i - \lambda_j)\right), \tag{2}$$

λ_i is the mean longitude of the i^{th} planet. The τ-series read

$$\mu \underset{1}{T}{}_{j}^{(i)+} = \sum_{k=-\infty}^{\infty} B_k^{ij} \tau_{ij}^k, \tag{3}$$

$$\tau_{ij} = \exp\left[\overset{\circ}{\imath}\, w_{ij}\right], \ w_{ij} = \frac{\pi}{K(k_{ij})} F(\varphi_{ij}, k_{ij}), \ k_{ij}^2 = \frac{4a_i a_j}{(a_i + a_j)^2}, \tag{4}$$

where a_i is major semi-axis of the i^{th} planet, $F(\varphi, k)$ and $K(k)$ are elliptic integral and complete elliptic integral of the first kind. Our aim is to compute numerically the coefficients A_k^{ij} and B_k^{ij}.

2. Numerical techniques used to compute A_k^{ij} and B_k^{ij}

In order to evaluate numerically Eq.(5.19) of Brumberg (1994) defining $\mu \underset{1}{T}{}_{j}^{(i)+}$, we have to be able to evaluate numerically functions $\Phi_3(\varphi, k, s, \alpha)$ for $s = -2, 0, 2$ and $\alpha = \pm 2m_{ij}$ defined as

$$\Phi_n(\varphi, k, s, \alpha) = \exp(\overset{\circ}{\imath}\, \alpha\varphi) \left(C + \int_0^{\varphi} \frac{1}{\delta^n(\varphi')} \exp\left(\overset{\circ}{\imath}\, (s - \alpha)\varphi'\right) d\varphi'\right), \tag{5}$$

where $\delta(\varphi) = (1 - k^2 \sin^2 \varphi)^{1/2}$, and C is a constant of integration. Due to the symmetry (see Eq.(4.62) of Brumberg (1994)) $\Phi_n(\varphi, k, -s, -\alpha) = \overline{\Phi}_n(\varphi, k, s, \alpha)$, it is sufficient to compute $\Phi_3(\varphi, \ldots)$ only for $\alpha = 2m_{ij}$. For any even s, $\Phi_3(\varphi, \ldots)$ has a period π. Therefore, both $\Phi_3(l, \ldots)$ and $\Phi_3(w, \ldots)$ are periodic functions with a period 2π. Since the real part of $\Phi_3(\varphi, \ldots)$ is an even function and the imaginary part of $\Phi_3(\varphi, \ldots)$ is an odd function, it is sufficient to know $\Phi_3(\varphi, \ldots)$ for $\varphi \in [0, \pi/2]$.

It is easy to see that the condition $\Phi_n(0, k, s, \alpha) = \Phi_n(\pi, k, s, \alpha)$ allows one to determine C uniquely:

$$\Phi_n(0, k, s, \alpha) = C = \frac{1}{\exp[-\overset{\circ}{\imath}\, \pi\alpha] - 1} \int_0^{\pi} \frac{\exp \overset{\circ}{\imath}\, (s - \alpha)\varphi}{\delta^n} d\varphi. \tag{6}$$

The integral in (6) can be evaluated numerically for any given k, s and α. One can check that it is the initial value (6) which results in Φ_3 having the secular part defined by Eq. (4.59) of Brumberg (1994). Numerical integration of (5) and (6) for $\varphi \in [0, \pi/2]$ allows us to compute $\Phi_n(l, \ldots)$ for any l. $\Phi_3(w, \ldots)$ can be computed as $\Phi_3(\varphi(w), \ldots)$ where $\varphi(w)$ can be computed from (4).

For each couple of planets, we compute both $\mu \underset{1}{T}{}_j^{(i)+}(l_{ij})$ and $\mu \underset{1}{T}{}_j^{(i)+}(w_{ij})$ numerically in a sufficiently large number of points uniformly distributed on the interval $[0, 2\pi[$ and then apply Fast Fourier Transform to get numerical values of the coefficients A_k^{ij} and B_k^{ij} respectively. We checked that the errors of numerical integration of (5) and (6), the errors of discretization, and the round-off errors do not influence the values shown in the Tables below.

3. Results

In Table I, for the couple Venus-Earth, we give the number of harmonics in the ρ-series (N_ρ) and τ-series (N_τ) whose magnitude is higher than $\varepsilon/2$ for several ε. This couple of planets is the most difficult one since $a_2/a_3 \sim 0.723$. Δ_ρ and Δ_τ are actual maximal errors of the ρ-series and the τ-series, respectively, provided that the specified number of harmonics are retained.

Table II shows the number of harmonics of the ρ-series and the τ-series with magnitudes higher than $5 \cdot 10^{-14}$. This cut-off level is sufficient to represent the first-order intermediate motion of the major planets with errors negligible in comparison with contemporary accuracy of observations.

Table III gives the total number of harmonics whose magnitudes are higher than $\varepsilon/2$ for the whole system of 8 planets. Δ_ρ and Δ_ε are maximal absolute errors of the corresponding series representing $\mu \underset{1}{T}{}_j^{(i)+}$ among all the couples.

The Tables show that the τ-series are more efficient as compared to the ρ-series when we need to represent mutual perturbations of close couples of planets (when the ratio of the major semi-axes $a_i/a_j \sim 1$) with relatively high accuracy. Since it is mutual perturbations of close couples (primarily Venus-Earth and Jupiter-Saturn) which present the most difficult task, we hope that the τ-series facilitate the problem. However, it is easy to see that for the couples with $\min(a_i/a_j, a_j/a_i) \ll 1$ or if relatively low accuracy is required, the ρ-series may be more preferable.

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TABLE 1. Number of harmonics for the couple Venus-Earth.

ε	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	10^{-13}	10^{-14}
N_ρ	11	17	24	33	41	52	61	72	86
N_τ	17	23	26	29	33	37	41	44	47
N_ρ/N_τ	0.65	0.74	0.92	1.14	1.24	1.41	1.49	1.64	1.83
Δ_ρ/ε	0.66	0.90	1.3	1.2	1.7	1.1	0.99	2.7	2.9
Δ_τ/ε	0.87	0.31	0.42	0.81	0.65	0.48	0.34	0.35	0.60

TABLE 2. Number of harmonics of the ρ-series (left values) and the τ-series (right values) with magnitudes higher than $5 \cdot 10^{-14}$.

$i \setminus j$	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
Mercury		38/29	25/22	15/15	13/14	10/11	7/ 9	7/ 8
Venus	29/20		72/44	28/23	17/17	12/13	8/10	8/ 9
Earth	18/17	74/37		48/35	21/19	14/15	10/11	8/10
Mars	11/13	29/22	56/31		28/23	17/17	12/12	10/11
Jupiter	4/ 8	8/12	12/14	13/15		50/35	22/20	17/16
Saturn	2/ 6	6/10	8/10	8/11	54/30		40/29	25/22
Uranus	2/ 6	4/ 8	5/ 8	5/ 8	23/21	44/28		60/42
Neptune	2/ 5	2/ 7	4/ 8	4/ 8	16/17	25/21	63/33	

TABLE 3. Total number of harmonics for the system of 8 major planets

ε	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	10^{-13}	10^{-14}
N_ρ	153	267	388	524	667	826	994	1173	1361
N_τ	255	364	472	576	680	780	880	975	1069
N_ρ/N_τ	0.60	0.73	0.82	0.91	0.98	1.06	1.13	1.20	1.27
Δ_ρ/ε	1.0	1.2	1.4	3.6	2.3	2.1	2.5	3.0	2.9
Δ_ρ/ε	0.87	0.63	0.93	0.81	0.68	0.55	0.58	0.60	0.67

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