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RELATIVE CONTINUITY OF DIRECT SUMS OF M-INJECTIVE MODULES

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Let M be a left R-module and $\mathcal{K}$ be an M-natural class with some additional conditions. It is proved that every direct sum of M-injective left R-modules in $\mathcal{K}$ is $\mathcal{KS}$-continuous ($\mathcal{KS}$-quasi-continuous) if and only if every direct sum of M-injective left R-modules in $\mathcal{K}$ is M-injective.

Let R be a ring with identity. It is well-known that R is left Noetherian if and only if every direct sum of injective left R-modules is injective. Based on this, many characterisations of left Noetherian rings using generalised injectivity of some left R-modules have been obtained. For example, it was shown that R is left Noetherian if and only if every direct sum of injective left R-modules is continuous (or quasi-continuous) (see [**5**]). On the other hand, Albu, Nastasescu, Golan, Goldman, Stenstrom, Teply, Enochs, Ahsan and others have studied the situations when all direct sums of non-singular injective left R-modules are injective, when all direct sums of τ-torsion free injective left R-modules are injective for a hereditary torsion theory τ, and when all direct sums of τ-torsion injective left R-modules are injective for a stable hereditary torsion theory τ. These results are well presented in Golan's book [**4**], and have been generalised in [**12**] by considering when all direct sums of M-injective left R-modules in an M-natural class $\mathcal{K}$ are M-injective. In this paper we consider when all direct sums of M-injective left R-modules in an M-natural class $\mathcal{K}$ are $\mathcal{KS}$-continuous (or $\mathcal{KS}$-quasi-continuous). We shall show that for an M-natural class $\mathcal{K}$, all direct sums of M-injective left R-modules in $\mathcal{K}$ are $\mathcal{KS}$-continuous (or $\mathcal{KS}$-quasi-continuous) if and only if all direct sums of M-injective left R-modules in $\mathcal{K}$ are M-injective.

Throughout this note we write $A \leq_e B$ ($A \mid B$) to denote that A is an essential submodule (a direct summand) of B.

Let M be a left R-module. We say that a left R-module N is subgenerated by M, or that M is a subgenerator for N, if N is isomorphic to a submodule of an M-generated module. Following [**11**], we denote by $\sigma[M]$ the full subcategory of R-Mod whose objects are all R-modules subgenerated by M. By [**11**, 17.9], every module N

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in $\sigma[M]$ has an injective hull $I(N)$ in $\sigma[M]$, which is also called an M-injective hull of N. It is known that the M-injective hulls of a left R-module in $\sigma[M]$ are unique up to isomorphism. In the following, we always denote by $I(N)$ the M-injective hull of N for any left R-module $N \in \sigma[M]$.

According to [**12**], a subclass $\mathcal{K}$ in $\sigma[M]$ which is closed under submodules, direct sums, isomorphic copies, and M-injective hulls is called an M-natural class. There exist a large number of examples of M-natural classes. Among them are $\sigma[M]$ and all natural classes in the sense of [**9**]. In particular, hereditary torsionfree classes, stable hereditary torsion classes, and saturated classes in the sense of Dauns (see [**1**]) are examples of M-natural classes.

For an M-natural class $\mathcal{K}$ and a left R-module N, we denote by $H_{\mathcal{K}}(N)$ the set $\{L \le N \mid N/L \in \mathcal{K}\}$.

Let M, N be left R-modules. Define the family

$$\mathcal{A}(N,M) = \{A \subseteq M \mid \exists X \subseteq N, \exists f \in \operatorname{Hom}(X,M),\ f(X) \le_e A\}.$$

Consider the properties

$\mathcal{A}(N,M)$-(C_1): For all $A \in \mathcal{A}(N,M)$, $\exists A^* \mid M$, such that $A \le_e A^*$.

$\mathcal{A}(N,M)$-(C_2): For all $A \in \mathcal{A}(N,M)$, if $X \mid M$ is such that $A \cong X$, then $A \mid M$.

$\mathcal{A}(N,M)$-(C_3): For all $A \in \mathcal{A}(N,M)$ and $X \mid M$, if $A \mid M$ and $A \cap X = 0$ then $A \oplus X \mid M$.

According to [**7**], M is said to be N-extending, N-quasi-continuous or N-continuous, respectively, if M satisfies $\mathcal{A}(N,M)$-(C_1), $\mathcal{A}(N,M)$-(C_1) and $\mathcal{A}(N,M)$-(C_3), $\mathcal{A}(N,M)$-(C_1) and $\mathcal{A}(N,M)$-(C_2).

Lemma 1. [**7**, Proposition 2.4] *A left R-module M is (quasi-)continuous (see* [**2**]*) if and only if M is M-(quasi-)continuous if and only if M is N-(quasi-)continuous for every left R-module N.*

Given an M-natural class $\mathcal{K}$, a left R-module N is called $\mathcal{K}$-cocritical if $N \in \mathcal{K}$ and $N/P \notin \mathcal{K}$ for any $0 \ne P \subset N$.

Definition 2: Let $\mathcal{K}$ be an M-natural class. A left R-module M is said to be $\mathcal{KS}$-extending, $\mathcal{KS}$-quasi-continuous or $\mathcal{KS}$-continuous, respectively, if for any direct sum $C = \bigoplus_{i \in I} C_i$ of $\mathcal{K}$-cocritical modules C_i $(i \in I)$, M is C-extending, C-quasi-continuous or C-continuous.

Clearly (quasi-)continuous modules are $\mathcal{KS}$-(quasi-)continuous. But the following example shows that the converse is not true.

Example 3. (See [**6**].) Let R be a left Noetherian V-ring which is not Artinian semisimple (see, for example, [**3**]). Then, by [**7**, Corollary 3.7], every left R-module is N-continuous for every semisimple left R-module N. Thus every left R-module is $\mathcal{KS}$-continuous, where $\mathcal{K} = R$-Mod. If all left R-modules are quasi-continuous, then for every left R-module M, $M \oplus E(M)$ is quasi-continuous, and so M is injective by [**8**, Lemma C], where $E(M)$ denotes the injective hull of M. Thus R is Artinian semisimple, a contradiction. Hence there exists a left R-module M which is not quasi-continuous.

LEMMA 4. *Any direct summand of a $\mathcal{KS}$-continuous ($\mathcal{KS}$-quasi-continuous) left R-module is $\mathcal{KS}$-continuous ($\mathcal{KS}$-quasi-continuous).*

PROOF: This follows from the fact that condition $\mathcal{A}(N,M)$-(C_i), $(i = 1,2,3)$ is inherited by direct summands of M [**7**, Proposition 2.4]. ∎

LEMMA 5. [**7**] *If M is N-(quasi-)continuous and $A \in \mathcal{A}(N,M)$ is a direct summand of M then A is indeed (quasi-)continuous.*

Let c be any cardinal. A left R-module M is called c-limited provided every direct sum of non-zero submodules of M contains at most c direct summands [**10**].

We say an M-natural class $\mathcal{K}$ satisfies (*) (see [**12**]), if for any cyclic submodule N of M, and every ascending chain $N_1 \leq N_2 \leq \ldots\ldots$ with each $N_i \in H_{\mathcal{K}}(N)$, the union $\bigcup_i N_i$ belongs to $H_{\mathcal{K}}(N)$.

THEOREM 6. *The following conditions are equivalent for an M-natural class $\mathcal{K}$ with (*).*

(1) *$H_{\mathcal{K}}(A)$ has ACC for any cyclic (or finitely generated) submodule A of M.*
(2) *Every direct sum of M-injective left R-modules in $\mathcal{K}$ is M-injective.*
(3) *Every direct sum of M-injective left R-modules in $\mathcal{K}$ is $\mathcal{KS}$-continuous.*
(4) *Every direct sum of M-injective left R-modules in $\mathcal{K}$ is $\mathcal{KS}$-quasi-continuous.*
(5) *There exists a cardinal c such that every direct sum of M-injective left R-modules in $\mathcal{K}$ is the direct sum of a c-limited module and a $\mathcal{KS}$-continuous module.*
(6) *There exists a cardinal c such that every direct sum of M-injective left R-modules in $\mathcal{K}$ is the direct sum of a c-limited module and a $\mathcal{KS}$-quasi-continuous module.*

PROOF: (1)$\Longleftrightarrow$(2). This follows from [**12**, Theorem 2.4].

(2)$\Longrightarrow$(3). Suppose that $N = \bigoplus_{i \in I} N_i$ is the direct sum of M-injective left R-modules $N_i \in \mathcal{K}$, $i \in I$. Then N is M-injective by (2). On the other hand, N is in $\mathcal{K}$, and so $N \in \sigma[M]$. Thus N is quasi-injective. Now clearly N is $\mathcal{KS}$-continuous by Lemma 1.

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(3) $\Longrightarrow$ (4) is clear.

(4) $\Longrightarrow$ (1). By [**12**, Theorem 2.5], it is sufficient to show that every direct sum of M-injective hulls of $\mathcal{K}$-cocritical left R-modules is M-injective.

Let C_i, $i \in I$, be $\mathcal{K}$-cocritical left R-modules. Then $C_i \in \mathcal{K}$, $i \in I$. Set

$$N = \Big(\bigoplus_{i\in I} I(C_i)\Big) \oplus I\Big(\bigoplus_{i\in I} I(C_i)\Big),$$

$$L = N \oplus I(N).$$

Then clearly L is a direct sum of M-injective left R-modules. Since $\mathcal{K}$ is closed under direct sums and M-injective hulls, it follows that L is a direct sum of M-injective left R-modules in $\mathcal{K}$. Thus L is $\mathcal{K}\mathcal{S}$-quasi-continuous. Denote

$$S = \Big(\bigoplus_{i\in I} C_i\Big) \bigoplus \Big(\bigoplus_{i\in I} C_i\Big).$$

Then L is S-quasi-continuous. For the submodule $A = N \bigoplus 0$ of L, define an R-homomorphism $f : S \longrightarrow L$ as the induced R-homomorphism

$$S = \Big(\bigoplus_{i\in I} C_i\Big) \bigoplus \Big(\bigoplus_{i\in I} C_i\Big) \longrightarrow \Big(\bigoplus_{i\in I} I(C_i)\Big) \oplus I\Big(\bigoplus_{i\in I} I(C_i)\Big) \oplus 0$$

(by the natural maps $C_i \longrightarrow I(C_i)$ and $\bigoplus_{i\in I} C_i \longrightarrow I\Big(\bigoplus_{i\in I} I(C_i)\Big)$). Since $C_i \leq_e I(C_i)$, we have

$$\bigoplus_{i\in I} C_i \leq_e \bigoplus_{i\in I} I(C_i) \leq_e I\Big(\bigoplus_{i\in I} I(C_i)\Big).$$

Thus

$$\begin{aligned} f(S) &= \Big(\Big(\bigoplus_{i\in I} C_i\Big) \oplus \Big(\bigoplus_{i\in I} C_i\Big)\Big) \oplus 0 \\ &\leq_e \Big(\Big(\bigoplus_{i\in I} I(C_i)\Big) \oplus I\Big(\bigoplus_{i\in I} I(C_i)\Big)\Big) \bigoplus 0 = A. \end{aligned}$$

This means that $A \in \mathcal{A}(S, L)$. By Lemma 5, it follows that A is quasi-continuous. Thus N is quasi-continuous. By [**8**, Lemma C], $\bigoplus_{i\in I} I(C_i)$ is $I\Big(\bigoplus_{i\in I} I(C_i)\Big)$-injective. Hence $\bigoplus_{i\in I} I(C_i)$ is M-injective.

The implications (3) $\Longrightarrow$ (5) $\Longrightarrow$ (6) are clear.

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$(6) \Longrightarrow (4)$. Note that, by Lemma 4, any direct summand of a $\mathcal{KS}$-quasi-continuous left R-module is $\mathcal{KS}$-quasi-continuous. By analogy with the proof of [**12**, Theorem 2.6], we can complete the proof. ∎

We denote by $\mathcal{S}^2$ the class of all semisimple left R-modules in $\sigma[M]$.

COROLLARY 7. *The following conditions are equivalent for a left R-module M.*

(1) *M is a locally Noetherian module (that is, every finitely generated submodule of M is Noetherian).*
(2) *Every direct sum of M-injective left R-modules in $\sigma[M]$ is M-injective.*
(3) *Every direct sum of M-injective left R-modules in $\sigma[M]$ is $\mathcal{S}^2$-continuous.*
(4) *Every direct sum of M-injective left R-modules in $\sigma[M]$ is $\mathcal{S}^2$-quasi-continuous.*
(5) *There exists a cardinal c such that every direct sum of M-injective left R-modules in $\sigma[M]$ is the direct sum of a c-limited module and an $\mathcal{S}^2$-continuous module.*
(6) *There exists a cardinal c such that every direct sum of M-injective left R-modules in $\sigma[M]$ is the direct sum of a c-limited module and an $\mathcal{S}^2$-quasi-continuous module.*

COROLLARY 8. *Let $\mathcal{S}^2$ be the class of all semisimple left R-modules. The following conditions are equivalent.*

(1) *R is a left Noetherian ring.*
(2) *Every direct sum of injective left R-modules is $\mathcal{S}^2$-continuous ($\mathcal{S}^2$-quasi-continuous).*
(3) *There exists a cardinal c such that every direct sum of injective left R-modules is the direct sum of a c-limited module and an $\mathcal{S}^2$-continuous ($\mathcal{S}^2$-quasi-continuous) module.*

Given a stable hereditary torsion theory τ on R-Mod, many equivalent conditions were presented in [**9**] and [**12**] to characterise rings which have ACC on τ-dense left ideals. Here we have

COROLLARY 9. *Let τ be a stable hereditary torsion theory on R-Mod and $\mathcal{TS}$ be the class of all τ-torsion semisimple left R-modules. Then the following conditions are equivalent.*

(1) *R has ACC on τ-dense left ideals.*
(2) *Every direct sum of τ-torsion injective left R-modules is injective.*
(3) *Every direct sum of τ-torsion injective left R-modules is $\mathcal{TS}$-continuous.*

(4) *Every direct sum of* τ*-torsion injective left* R*-modules is* $\mathcal{TS}$-quasi-*continuous.*

(5) *There exists a cardinal* c *such that every direct sum of* τ*-torsion injective left* R*-modules is the direct sum of a* c*-limited module and a* $\mathcal{TS}$*-continuous module.*

(6) *There exists a cardinal* c *such that every direct sum of* τ*-torsion injective left* R*-modules is the direct sum of a* c*-limited module and a* $\mathcal{TS}$-quasi-*continuous module.*

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