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TOPOLOGICALLY SIMPLE BANACH ALGEBRAS WITH DERIVATION

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It is not known if a commutative, topologically simple, radical Banach algebra exists. If, however, every derivation on such an algebra is continuous, this yields the automatic continuity of all derivations on commutative, semiprime Banach algebras. Utilising techniques used by Thomas in his proof of the Singer-Wermer conjecture, we show that, if A is a commutative, topologically simple Banach algebra with a non-zero derivation on it, then a quotient of a certain localisation of A has a power series structure. A pivotal rôle is played by what we call ample sets of denominators.

INTRODUCTION

In [**10**], Johnson proved that derivations on commutative, semisimple Banach algebras are automatically continuous and thus, by the classical Singer-Wermer theorem [**12**], equal to zero. This result confirmed a long open conjecture by Kaplansky. In [**6**], Dixon observed that the class of semiprime Banach algebras was a natural extension of the class of semisimple Banach algebras and asked if the automatic continuity results known to hold for semisimple Banach algebras extend to this larger class. The problem of whether every derivation on a (commutative) semiprime Banach algebra is automatically continuous has been open ever since.

Partial results — all positive — are given, for instance, in [**8**], [**11**], and [**9**]. All these results utilise the connections of the automatic continuity question for derivations on commutative, semiprime Banach algebras with the perhaps deepest open problem in general Banach algebra theory, the *closed ideal problem*, which were first discovered by Cusack [**5**].

Recall that a Banach algebra A with $A^2 \neq \{0\}$ is called *topologically simple* if it has no two-sided closed ideals other than $\{0\}$ and A. A topologically simple Banach algebra is necessarily primitive or radical. The closed ideal problem is the question of

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whether there is a commutative, topologically simple Banach algebra other than $\mathbb{C}$. If a commutative, topologically simple Banach algebra other than $\mathbb{C}$ exists, it must be radical, and it is also not difficult to see that it must also be an integral domain. Let A be a commutative Banach algebra which is an integral domain — by [8], the automatic continuity of derivations on commutative, semiprime Banach algebras follows once it has been established for integral domains, and let I denote the intersection of the non-zero closed ideals of A. It is easy to see that I is either $\{0\}$ or a topologically simple, radical Banach algebra. If $I = \{0\}$, that is, if $\{0\}$ is accessible in the terminology of [**4**], then every derivation on A is continuous as a simple consequence of [**4**, 1.1 Lemma]. It follows that every derivation on A is continuous if $\mathbb{C}$ is the only commutative, topologically simple Banach algebra. The results from [**8, 11, 9**] all give, in one way or another, a necessary condition for I to be zero.

In this note we do not even attempt to make any step towards a resolution of the closed ideal problem. Instead, we attack the mythological beasts directly: We consider derivations $D : A \to A$, where $A \neq \mathbb{C}$ is a commutative, topologically simple Banach algebra. We are keenly aware of the fact that our approach might ultimately turn out to be a contribution to the theory of the empty set. However, since the closed ideal problem has been open since the early days of Banach algebra theory, and no solution even seems to be in sight, we feel justified to proceed this way in view of [**11**, Theorem 6]: If all derivations on commutative, topologically simple Banach algebras are automatically continuous, then all derivations on commutative, semiprime Banach algebras are continuous. Certainly, the ideas contained in [**8, 11, 9**] do not help us any further. Instead, we shall borrow heavily from the first part of Thomas' proof of the Singer–Wermer conjecture [**15**], and show that, under certain additional conditions on D, a certain quotient of a localisation of A has a power series structure.

1. Localisations of Banach algebras

In [**10**], Johson reduced the Singer–Wermer conjecture to the case of commutative, local Banach algebras, that is, commutative radical Banach algebras with identities adjoined. The Singer–Wermer conjecture was thus equivalent to the question of whether a derivation on a commutative, local Banach algebra could attain invertible values. If A is a commutative, local Banch algebra, $D : A \to A$ is a derivation, and $z \in \operatorname{rad}(A)$ is such that $Dz \in \operatorname{Inv}(A)$, then $A \ni a \mapsto (Dz)^{-1}D$ is a derivation attaining the value 1 at z. Under these hypotheses, Thomas showed in [**15**] that $A/\bigcap_{n=1}^{\infty} z^n A$ has a power series structure (and, secondly, that this is irreconcilable with A being a commutative, local Banach algebra).

We wish to obtain a similar power series structure result to that in [**15**], when A is a commutative, topologically simple Banach algebra, and $D : A \to A$ is a non-zero derivation. In this context, however, we have no invertible elements at our disposal.

What we shall do therefore, is to make certain elements of A artificially invertible by localising the algebra.

Let A be a commutative algebra, and let S be a multiplicative subsemigroup of A containing no divisors of zero, that is, a *set of denominators*. Define an equivalence relation $\sim$ on $A \times S$ through

$$(a, s) \sim (a', s') \quad :\Longleftrightarrow \quad as' = a's \qquad (a, a' \in A,\ s, s' \in S).$$

and let $S^{-1}A$ denote the set of the corresponding equivalence classes; write a/s or as^{-1} for the equivalence class of $(a, s) \in A \times S$. With the usual rules for calculating with fractions, $S^{-1}A$ becomes a commutative, unital algebra, in which every element of S becomes invertible. It is called the *localisation of A with respect to S.*

REMARKS 1. Unlike most standard texts on commutative algebra, such as [**3**], we do not assume A to be unital. If $s \in S$ is an arbitrary element, the map $A \ni a \mapsto as/s$ is an embedding which does not depend on the choice of s; we shall therefore simply view A as a subalgebra of $S^{-1}A$. If A is non-unital with unitisation $A^{\#}$, the canonical embedding $A \hookrightarrow S^{-1}A$ extends (canonically) to $A^{\#}$.

2. In general, if A is a Banach algebra, $S^{-1}A$ — except under very restrictive conditions (see [**13**] and [**14**]) — is *not* a Banach algebra.

Let $D\colon A \to A$ be a derivation. Then D has a unique extension, likewise denoted by D, to $S^{-1}A$ defined through

$$D\left(\frac{a}{s}\right) := \frac{sDa - aDs}{s^2} \qquad \left(\frac{a}{s} \in S^{-1}A\right).$$

Let S_0 be an arbitrary subset of A containing no divisors of zero; we write $\langle S_0 \rangle$ for the set of denominators generated by S_0. If $S_0 = \{s_1, \dots, s_n\}$, we also write $\langle s_1, \dots, s_n \rangle$ instead of $\langle S_0 \rangle$. Certainly, if $z \in A$ is such that Dz is not a divisor of zero, then Dz becomes invertible in $\langle Dz \rangle^{-1}A$.

For our main theorem, we require sets of denominators with a particular property:

DEFINITION 1.1: Let A be a commutative algebra, and let $S \subset A$ be a set of denominators. Then S is called *ample* if, for each $s_1, \dots, s_n \in S$, there is $t \in S$ such that t is divided by every element of $\langle s_1, \dots, s_n \rangle$.

It is not hard to conclude from the Mittag-Leffler theorem that, in a commutative, topologically simple Banach algebra, the set of non-zero elements is an ample set of denominators. A very similar argument, however, shows that much smaller ample sets of denominators exist in abundance:

PROPOSITION 1.2. *Let A be a commutative, topologically simple Banach algebra, and let $S_0 \subset A$ be a countable set of non-zero elements. Then there is an ample, countable set of denominators $S \subset A$ containing S_0.*

PROOF: Let $S_0 = \{s_n : n \in \mathbb{N}\}$. We shall define inductively an increasing sequence $(S_n)_{n=1}^{\infty}$ of countable sets of denominators in A such that

(i) $s_n \in S_n$ for $n \in \mathbb{N}$, and

(ii) for each $n \in \mathbb{N}$, there is $t \in S_{n+1}$ such that t is divided by every element of S_n.

By letting $S := \bigcup_{n=1}^{\infty} S_n$, we obtain a set with the desired properties.

To start the induction, let $S_1 := \langle s_1 \rangle$. Let $n \in \mathbb{N}$, and suppose that $S_1, \ldots, S_n$ have already been defined. Let $S_n = \{t_n : n \in \mathbb{N}\}$. Since A is topologically simple, we have $(t_n A)^- = A$ for all $n \in \mathbb{N}$. From Bourbaki's Mittag-Leffler theorem (see, for example, [7]), it follows that $\bigcap_{n=1}^{\infty} t_1 \cdots t_n A$ is also dense in A and thus, in particular, non-zero. Let t be an arbitrary, non-zero element of $\bigcap_{n=1}^{\infty} t_1 \cdots t_n A$. Then, by definition, t is divided by every element of S_n. Finally, let $S_{n+1} := \big\langle S_n \cup \{s_{n+1}, t\} \big\rangle$. ∎

2. A POWER SERIES STRUCTURE THEOREM

Let A be a commutative, topologically simple Banach algebra, let $D : A \to A$ be a derivation, let $z \in A$ be such that $Dz \neq 0$, and let $S \subset A$ be an ample set of denominators. Wish to prove an analogue of [**15**, Proposition 2.24], that is, a power series structure theorem for $S^{-1}A / \bigcap_{n=1}^{\infty} z^n S^{-1} A$. We follow [**15**] closely; whenever an argument is only a minor variation of one from [**15**], we just give a brief reference.

We begin with a purely algebraic lemma:

LEMMA 2.1. *Let A be a commutative algebra with identity, let $\Delta : A \to A$ be a derivation, and let $z \in A$ be such that $\Delta z = 1$. Define, for $n \in \mathbb{N}$,*

$$\theta_n(a) := \sum_{k=0}^{n} (-1)^k \frac{\Delta^k a}{k!} z^k \qquad (a \in A).$$

Then:

(i) $\theta_n(z) = 0 \qquad (n \in \mathbb{N})$;

(ii) $\Delta\big(\theta_n(a)\big) \in z^n A \qquad (a \in A,\ n \in \mathbb{N})$;

(iii) $\theta_n(ab) - \theta_n(a)\theta_n(b) \in z^{n+1} A \qquad (a, b \in A,\ n \in \mathbb{N})$;

(iv) $\theta_n\big(\theta_m(a)\big) - \theta_m(a) \in z^{m+1} A \qquad (a \in A,\ n, m \in \mathbb{N},\ m \geqslant n)$.

PROOF: The proof of [**15**, Lemma 2.15] is purely algebraic and thus carries over verbatim. ∎

Let A be a commutative Banach algebra, and let $z \in A$. Then z is said to have *finite closed descent* if $(z^n A)^- = (z^{n+1} A)^-$ for some $n \in \mathbb{N}$ [**2**].

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Lemma 2.2. *Let A be a commutative Banach algebra, let $z \in A$ have finite closed descent, let $S \subset A$ be an ample set of denominators, and let $s_1, \ldots, s_N \in S$. Then for any choice of sequences $(a_k)_{k=0}^\infty$ in A and $(\nu_{k,1})_{k=0}^\infty, \ldots, (\nu_{k,N})_{k=0}^\infty$ in $\mathbb{N}$, there is an element $y \in S^{-1}A$ and a sequence $(b_n)_{n=0}^\infty$ in $S^{-1}A$ such that*

$$\left(y - \sum_{k=0}^{n} \frac{a_k}{s_1^{\nu_{k,1}} \cdots s_N^{\nu_{k,N}}} z^k\right) - z^{n+1} b_n \in \bigcap_{k=1}^{\infty} z^k S^{-1}A \qquad (n \in \mathbb{N}_0).$$

Proof: Since S is ample, there is $t \in S$ such that t is divided (in A) by every element of $\langle s_1, \ldots, s_N \rangle$; in particular, for each $k \in \mathbb{N}$, there is $t_k \in A$ such that

$$t = t_k s_1^{\nu_{k,1}} \cdots s_N^{\nu_{k,N}}.$$

Define $\widetilde{a}_k := t_k a_k$ for $k \in \mathbb{N}$. Then the same Mittag-Leffler argument as in the proof of [**15**, Proposition 2.18] (which, in turn, goes back to [**1**]) yields $\widetilde{y} \in A$ and a sequence $(\widetilde{b}_n)_{n=0}^\infty$ in A such that

$$\left(\widetilde{y} - \sum_{k=0}^{n} \widetilde{a}_k z^k\right) - z^{n+1} \widetilde{b}_n \in \bigcap_{k=1}^{\infty} z^k A \qquad (n \in \mathbb{N}_0).$$

Division by t (in $S^{-1}A$) yields the claim with $y := \widetilde{y}/t$ and $b_n := \widetilde{b}_n/t$ for $n \in \mathbb{N}$. $\square$

Let A be any commutative algebra, and let $z \in A$ be such that $\bigcap_{n=1}^{\infty} z^n A = \{0\}$. The z-adic topology on A is the metric topology defined through

$$d(a, b) := \inf\{2^{-n} : n \in \mathbb{N}_0 \text{ and } a - b \in z^n A\} \qquad (a, b \in A),$$

where $z^0 A := A$. If $\bigcap_{n=1}^{\infty} z^n A \neq \{0\}$, we consider $A / \bigcap_{n=1}^{\infty} z^n A$. We write $\overline{a}$ for the coset of $a \in A$ in $A / \bigcap_{n=1}^{\infty} z^n A$. We then have a $\overline{z}$-adic topology on $A / \bigcap_{n=1}^{\infty} z^n A$.

In terms of the $\overline{z}$-adic topology, we obtain from Lemma 2.2 (compare [**15**, Proposition 2.18]):

Corollary 2.3. *Let A, z, S and $s_1, \ldots, s_N$ as well as y and $b_1, b_2, \ldots$ be as in Lemma 2.2. Then the infinite series $\sum_{k=0}^{\infty} \left(a_k/(s_1^{\nu_{k,1}} \cdots s_N^{\nu_{k,N}})\right)^{-} \overline{z}^k$ converges to $\overline{y}$ and, for each $n \in \mathbb{N}_0$, the infinite series $\sum_{k=n+1}^{\infty} \left(a_k/(s_1^{\nu_{k,1}} \cdots s_N^{\nu_{k,N}})\right)^{-} \overline{z}^{k-n-1}$ converges to $\overline{b}_n$ in the $\overline{z}$-adic topology on $S^{-1}A / \bigcap_{k=1}^{\infty} z^k S^{-1}A$.*

Corollary 2.4. *Let A be a commutative Banach algebra, let $z \in A$ have finite closed descent, let $D\colon A \to A$ be a derivation, and let $S \subset A$ be an ample set of*

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denominators such that Dz is invertible in $S^{-1}A$. Then, for each $\overline{x} \in S^{-1}A/\bigcap_{n=1}^{\infty} z^n S^{-1}A$, the infinite series

$$\theta(\overline{x}) := \sum_{k=0}^{\infty} (-1)^k \frac{\Delta^k \overline{x}}{k!} \overline{z}^k$$

converges in $S^{-1}A/\bigcap_{n=1}^{\infty} z^k S^{-1}A$, where Δ is the derviation on $S^{-1}A/\bigcap_{n=1}^{\infty} z^k S^{-1}A$ induced by $(Dz)^{-1}D$.

PROOF: Slightly abusing notation, we also use Δ to denote the derivation $(Dz)^{-1}D$ on $S^{-1}A$. Let $a \in A$ and $s \in S$ be arbitrary, and choose $b \in A$ and $t \in S$ such that $(Dz)^{-1} = b/t$. We claim that, for each $k \in \mathbb{N}_0$, there are $c_k \in A$ and $\nu_k, \mu_k \in \mathbb{N}$ such that

$$\Delta^k \left(\frac{a}{s}\right) = \frac{c_k}{s^{\nu_k} t^{\mu_k}}.$$

Since

$$\Delta^0 \left(\frac{a}{s}\right) = \frac{a}{s} = \frac{at}{st},$$

the claim holds for $k = 0$. Now suppose that the claim has been proved for arbitrary $k \in \mathbb{N}$. Then we have

$$\Delta^{k+1} \left(\frac{a}{s}\right) = \Delta \left(\frac{c_k}{s^{\nu_k} t^{\mu_k}}\right) = \frac{b}{r} D \left(\frac{c_k}{s^{\nu_k} t^{\mu_k}}\right) = \frac{b\big(s^{\nu_k} t^{\mu_k} Dc_k + c_k D(s^{\nu_k} t^{\mu_k})\big)}{s^{2\nu_k} t^{2\mu_k+1}},$$

which establishes the claim for $k + 1$. The assertion of Corollary 2.4 then follows from Corollary 2.3. ∎

Invoking Lemma 2.1, we obtain as in the proof of [**15**, Lemma 2.20]:

LEMMA 2.5. *Let A, z, D, S, θ and Δ be as in Corollary 2.4. Then:*

(i) $\theta(\overline{z}) = \overline{0}$;

(ii) $\Delta \circ \theta = 0$;

(iii) *θ is a homomorphism.*

(iv) *θ is a projection onto a unital subalgebra $\mathcal{A}_\theta$ of $S^{-1}A/\bigcap_{n=1}^{\infty} z^n S^{-1}A$. Moreover, no non-zero element of $\mathcal{A}_\theta$ is divided by $\overline{z}$ in $S^{-1}A/\bigcap_{n=1}^{\infty} z^n S^{-1}A$.*

As an analogue of [**15**, Lemma 2.21], we obtain:

PROPOSITION 2.6. *Let A, z, D, S, Δ, θ and $\mathcal{A}_\theta$ be as in Lemma 2.5, and suppose that z is not a divisor of zero in A. Then, for each element $\overline{a} \in S^{-1}A/\bigcap_{n=1}^{\infty} z^n S^{-1}A$, there is a unique sequence $(\overline{a}_k)_{k=0}^{\infty}$ in $\mathcal{A}_\theta$ such that $\overline{a} = \sum_{k=0}^{\infty} \overline{a}_k \overline{z}^k$. Moroever, $\overline{z}^m$ with $m \in \mathbb{N}$ divides $\overline{a}$ in $S^{-1}A/\bigcap_{n=1}^{\infty} z^n S^{-1}A$ if and only if $\overline{a}_0 = \cdots = \overline{a}_{m-1} = \overline{0}$.*

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PROOF: We begin with a proof of the uniqueness of $\overline{a}_0, \overline{a}_1, \ldots$. Most of the corresponding arguments in the proof of [**15**, Lemma 2.21] carry over verbatim. All we need is a substitute for [**15**, Lemma 2.8(ii)]. To this end, let $x \in \bigcap_{n=1}^{\infty} z^n S^{-1}A$, and let $y \in S^{-1}A$ be such that $x = zy$. Furthermore, for any $\nu \in \mathbb{N}$, there is $\widetilde{y} \in S^{-1}A$ such that $x = z^{\nu+1}\widetilde{y}$. Since z is not a divisor of zero in A, it cannot be a divisor of zero in $S^{-1}A$ either, that is, $y = z^{\nu}\widetilde{y}$. Since $\nu \in \mathbb{N}$ was arbitrary, this means that $y \in \bigcap_{n=1}^{\infty} z^n S^{-1}A$.

The existence of $\overline{a}_0, \overline{a}_1, \ldots$ is proved exactly as in the proof of [**15**, Lemma 2.21].

Finally, let $\overline{a} = \sum_{k=0}^{\infty} \overline{a}_k \overline{z}^k \in S^{-1}A / \bigcap_{n=1}^{\infty} z^n S^{-1}A$ with $\overline{a}_0, \overline{a}_1, \ldots \in \mathcal{A}_\theta$, and let $m \in \mathbb{N}$. As in the proof of [**15**, Lemma 2.21], we see that $\overline{a}_0 = \cdots = \overline{a}_{m-1} = \overline{0}$ if $\overline{z}^m$ divides $\overline{a}$. Conversely, suppose that $\overline{a}_0 = \cdots = \overline{a}_{m-1} = \overline{0}$. In the proof of the existence of $\overline{a}_0, \overline{a}_1, \ldots,$ it is actually shown that there are $b_0, b_1, \ldots$ such that

$$\overline{a} - \sum_{k=0}^{n} \overline{a}_k \overline{z}^k = \overline{z}^{n+1}\overline{b}_n \qquad (n \in \mathbb{N}).$$

For $n = m - 1$, this means $\overline{a} = \overline{z}^m \overline{b}_{m-1}$, that is, $\overline{z}^m$ divides $\overline{a}$. ∎

As a consequence we obtain (compare [**15**, Lemma 2.22], whose proof takes over):

COROLLARY 2.7. *Under the hypotheses of Proposition 2.6, we have the following:*

(i) $\mathcal{A}_\theta = \ker \Delta$.

Moreover, if $\overline{a} = \sum_{k=0}^{\infty} \overline{a}_k \overline{z}^k \in S^{-1}A / \bigcap_{n=1}^{\infty} z^n S^{-1}A$ *with* $\overline{a}_0, \overline{a}_1, \ldots \in \mathcal{A}_\theta$, *we have*

(ii) $\Delta(\overline{a}) = \sum_{k=1}^{\infty} k \overline{a}_k \overline{z}^{k-1}$;

(iii) $\overline{a}_k = \theta(\Delta^k(\overline{a}))/k!$ *for* $k \in \mathbb{N}_0$.

For any commutative algebra A, let $A[[Z]]$ denote the algebra of formal power series with coefficients in A. The *formal derivative* is the derivation

$$A[[Z]] \to A[[Z]], \quad \sum_{k=0}^{\infty} a_k Z^k \mapsto \sum_{k=1}^{\infty} k a_k Z^{k-1}.$$

We now have all the ingredients ready for the proof of the following structure result:

THEOREM 2.8. *Under the hypotheses of Proposition 2.6, the map*

$$(*) \qquad S^{-1}A / \bigcap_{n=1}^{\infty} z^n S^{-1}A \to \mathcal{A}_\theta[[Z]], \quad \overline{a} \mapsto \sum_{k=0}^{\infty} \frac{\theta(\Delta^k \overline{a})}{k!} Z^k$$

is an algebra monomorphism, whose range contains the polynomials with coefficients in $\mathcal{A}_\theta$. *Morover,* Δ *acts on the range of* $(*)$ *as the restriction of the formal derivative.*

PROOF: As noted in the proof of the corresponding assertion in [**15**] (Proposition 2.24), most of Theorem 2.8 has already been established (Lemma 2.5, Proposition 2.6, and Corollary 2.7).

What remains to be shown is the multiplicativity of $(*)$. Here, the corresponding argument from [**15**] requires some modification. Let $\overline{a}, \overline{b} \in S^{-1}A/\bigcap_{n=1}^{\infty} z^n S^{-1}A$ be such that $\overline{a} = \sum_{k=1}^{n} \overline{a}_k \overline{z}^k$ and $\overline{b} = \sum_{k=1}^{n} \overline{b}_k \overline{z}^k$ with $\overline{a}_0, \overline{b}_0, \ldots, \overline{a}_n, \overline{b}_n \in \mathcal{A}_\theta$. Then, as in the proof of [**15**, Proposition 2.24], we see that $(*)$ maps $\overline{a}\overline{b}$ to the product of the images of $\overline{a}$ and $\overline{b}$ under $(*)$. Since $(*)$ is continuous with respect to the $\overline{z}$-adic topology on $S^{-1}A/\bigcap_{n=1}^{\infty} z^n S^{-1}A$ and the Z-adic topology on $\mathcal{A}_\theta[[Z]]$, it follows that $(*)$ is indeed multiplicative. $\square$

Note that, unlike in [**15**, Proposition 2.24], we cannot conclude that $(*)$ is onto: This is due to the restriction on the denominators we had to impose in Lemma 2.2.

In a commutative, topologically simple Banach algebra, every non-zero element is not a divisor of zero and has finite closed descent. Hence, we have:

COROLLARY 2.9. *Let A be a commutative, topologically simple Banach algebra, let $S \subset A$ be an ample set of denominators, let $D\colon A \to A$ be a derivation, and let $z \in A$ be such that Dz is invertible in $S^{-1}A$. Then, with Δ denoting the derivation on $S^{-1}A/\bigcap_{n=1}^{\infty} z^n S^{-1}A$ induced by $(Dz)^{-1}D$ and with*

$$\theta\colon S^{-1}A/\bigcap_{n=1}^{\infty} z^n S^{-1}A \to S^{-1}A/\bigcap_{n=1}^{\infty} z^n S^{-1}A, \quad \overline{x} \mapsto \sum_{k=0}^{\infty} (-1)^k \frac{\Delta^k \overline{x}}{k!} \overline{z}^k,$$

the map $(*)$ is an algebra monomorphism, whose range contains the polynomials with coefficients in $\mathcal{A}_\theta$. Morover, Δ acts on the range of $(*)$ as the restriction of the formal derivative.

REMARKS 1. Let A be any commutative, topologically simple Banach algebra, let $D\colon A \to A$ be a derivation, and let $z \in A$ be such that $Dz \neq 0$. By Proposition 1.2, the element Dz is then contained in a countable, ample set $S \subset A$ of denominators. Then certainly Dz is invertible in $S^{-1}A$, so that Corollary 2.9 applies. However, we do not know that z is *not* invertible in $S^{-1}A$. If z is invertible, then both $S^{-1}A/\bigcap_{n=1}^{\infty} z^n S^{-1}A$ and $\mathcal{A}_\theta$ are just the zero algebra. In this situation, the conclusion of Corollary 2.9 — albeit true — is not very interesting.

2. The most interesting consequence of [**15**, Proposition 2.24] is certainly [**15**, Theorem 2.25], which eventually leads to a contradiction and thus to a proof of the Singer–Wermer conjecture. It is not hard to see that, in our setting, we have an analogue of [**15**, Proposition 2.24]. In [**15**], Thomas establishes a contradiction to [**15**, Theorem 2.25] through the construction of *recalcitrant systems* [**15**, Definition 3.3]. We have been unable

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so far to construct (analogues of) recalcitrant systems in localisations of commutative, topologically simple Banach algebras.

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