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THE 7-REGULAR AND 13-REGULAR PARTITION FUNCTIONS MODULO 3

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Abstract

Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. In this paper we establish a formula for $b_{13}(3n+1)$ modulo 3 and use this to find exact criteria for the 3-divisibility of $b_{13}(3n+1)$ and $b_{13}(3n)$. We also give analogous criteria for $b_7(3n)$ and $b_7(3n+2)$.

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1. Introduction

A *partition* of n is a nonincreasing sequence of positive integers whose sum is n. As usual we denote the number of partitions of n by $p(n)$. Ramanujan proved that the congruences

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7$$

and

$$p(11n+6) \equiv 0 \pmod{11}$$

hold for all nonnegative integers n, and Ahlgren and Ono demonstrated that for any positive integer m coprime to 6, there exist infinitely many congruences of the form $p(An+B) \equiv 0 \pmod m$ [1, 3, 12].

For $\ell > 1$, a partition is called *ℓ-regular* if none of its parts is divisible by ℓ; we denote the number of ℓ-regular partitions of n by $b_\ell(n)$. The generating function for the ℓ-regular partition function satisfies the identity

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \prod_{n=1}^{\infty}\Big(\frac{1-q^{\ell n}}{1-q^n}\Big). \tag{1.1}$$

Many results on the arithmetic of $b_\ell(n)$ modulo m have been proven for various values of ℓ and m (see, for example, [2, 5–8, 10, 14]). In [11] Lovejoy and the second author gave a formula for $b_3(n)$ modulo 9, and recently Webb [16] showed that $b_{13}(n)$ satisfies the following infinite family of congruences modulo 3.

THEOREM 1.1 [16]. *For all* $\alpha, n \geq 0$,

$$b_{13}\Big(3^{\alpha+2}n + \frac{5 \cdot 3^{\alpha+1} - 1}{2}\Big) \equiv 0 \pmod 3.$$

In this paper we establish a formula for $b_{13}(3n+1)$ modulo 3 in terms of the prime factorisation of $2n+1$ (see Theorem 3.2) by relating the appropriate generating function to a weight one Hecke eigenform arising from binary quadratic forms. This formula yields the following criteria for the 3-divisibility of $b_{13}(3n+1)$. (Here, for p prime, $\mathrm{ord}_p(m)$ denotes the largest integer t such that $p^t \mid m$.)

THEOREM 1.2. *Let n be a nonnegative integer. Then* $b_{13}(3n+1) \equiv 0 \pmod 3$ *if and only if there is a prime p such that one of the following holds:*

(1) $p \equiv 2 \pmod 3$ *and* $\mathrm{ord}_p(2n+1)$ *is odd;*
(2) $p \equiv 1 \pmod 3$, $(p/13) = -1$ *and* $\mathrm{ord}_p(2n+1)$ *is odd;*
(3) $p \equiv 1 \pmod 3$, $(p/13) = 1$ *and* $\mathrm{ord}_p(2n+1) \equiv 2 \pmod 3$.

Theorem 1.2 implies the following families of congruences. In addition, Theorem 1.1 follows from case (1) of Theorem 1.2.

THEOREM 1.3. *Let* $p \notin \{2, 3, 13\}$ *be prime and* $0 \leq \beta \leq p-1$ *with* $\beta \neq \frac{1}{2}(p-1)$.

(1) *Suppose that* $p \equiv 2 \pmod 3$, *or that* $p \equiv 1 \pmod 3$ *and* $(p/13) = -1$. *Then for all* $\alpha, n \geq 0$,

$$b_{13}\Big(3p^{2\alpha+2}n + \frac{(6\beta+3)p^{2\alpha+1} - 1}{2}\Big) \equiv 0 \pmod 3.$$

(2) *Suppose that* $p \equiv 1 \pmod 3$ *and* $(p/13) = 1$. *Then for all* $\alpha, n \geq 0$,

$$b_{13}\Big(3p^{3\alpha+3}n + \frac{(6\beta+3)p^{3\alpha+2} - 1}{2}\Big) \equiv 0 \pmod 3.$$

(3) *Suppose that* $\gamma \geq 0$, $\gamma \equiv 1 \pmod 3$ *and* $((2\gamma+1)/13) = -1$. *Then for all* $\alpha, n \geq 0$,

$$b_{13}\Big(3 \cdot 13^{\alpha+1}n + \frac{(2\gamma+1) \cdot 13^{\alpha} - 1}{2}\Big) \equiv 0 \pmod 3.$$

Webb arrived at Theorem 1.1 by proving the modularity of the values of $b_{13}(3n+1)$ modulo 3. Here we show that the modularity of $b_{13}(3n)$ modulo 3 can be established in a similar way (see Theorem 4.1). We use this to demonstrate a connection between these values and those of $b_{13}(3n+1)$, which yields an analogue of Theorem 1.2 for $b_{13}(3n)$ (see Theorem 4.3). Lastly, we show that similar phenomena hold for $b_7(n)$ (see Theorem 5.1).

In Section 2 we give the necessary background on modular forms. We prove our results for $b_{13}(n)$ in Sections 3 and 4, and those for $b_7(n)$ in Section 5.

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2. Modular forms

Given a Dirichlet character χ modulo N and an integer k, denote by $M_k(\Gamma_0(N), \chi)$ the complex vector space of holomorphic modular forms on $\Gamma_0(N)$ of weight k and character χ. We will often identify a modular form $f(z) \in M_k(\Gamma_0(N), \chi)$ with its Fourier expansion at infinity:

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n \in \mathbb{C}[\![q]\!] \quad (q := e^{2\pi i z}).$$

One can verify the congruence of a pair of modular forms modulo a prime p via a theorem of Sturm. Given $f(z) \in \mathbb{Z}[\![q]\!]$, we define $\mathrm{ord}_p(f(z)) := \min\{n \geq 0 : p \nmid a(n)\}$ provided this set is nonempty and write $\mathrm{ord}_p(f(z)) = \infty$ otherwise. If $g(z) \in \mathbb{Z}[\![q]\!]$ and $\mathrm{ord}_p(f(z) - g(z)) = \infty$, we write $f(z) \equiv g(z) \pmod{p}$.

Theorem 2.1 [15]. *Suppose that $f(z), g(z) \in M_k(\Gamma_0(N), \chi) \cap \mathbb{Z}[\![q]\!]$ and*

$$\mathrm{ord}_p(f(z) - g(z)) > \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma_0(N)],$$

where $[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \cdot \prod_{d \text{ prime}, d \mid N}(1 + d^{-1})$. Then $f(z) \equiv g(z) \pmod{p}$.

For a prime p the operator U_p is defined by

$$f(z) \mid U_p := \sum_{n=0}^{\infty} a(pn)q^n,$$

while the Hecke operator $T_{p,k,\chi}$ of index p, weight k and character χ acts via

$$f(z) \mid T_{p,k,\chi} := \sum_{n=0}^{\infty} (a(pn) + \chi(p)p^{k-1}a(n/p))q^n.$$

Recall that $T_{p,k,\chi}$ preserves $M_k(\Gamma_0(N), \chi)$ and that the same holds for U_p when $p \mid N$. We will often abbreviate $T_{p,k,\chi}$ by $T_{p,k}$ or T_p.

We require Dedekind's eta function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.1}$$

Results of Gordon, Hughes, Newman and Ligozat (see [13, Theorems 1.64 and 1.65]) giving conditions under which an eta-quotient is a modular form will be used without comment.

We will also employ twists of modular forms by Dirichlet characters.

Proposition 2.2 [4]. *Let $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$, where χ has conductor L, and let ψ be a Dirichlet character modulo M. Then*

$$f(z) \otimes \psi := \sum_{n=0}^{\infty} \psi(n)a(n)q^n \in M_k(\Gamma_0(\tilde{N}), \chi\psi^2),$$

where $\tilde{N} = \mathrm{lcm}(N, LM, M^2)$.

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3. Proof of results for $b_{13}(3n+1)$

Denote by χ_d the character $\chi_d(\bullet) = (d/\bullet)$, and by $\chi_{0,n}$ the principal character modulo n. In [16] Webb proved the existence of a modular form $H(z)$ in the space $M_{12}(\Gamma_0(156), \chi_{13}) \cap q^3\mathbb{Z}[\![q^6]\!]$ such that

$$H(z) \equiv \sum_{n=0}^{\infty} b_{13}(3n+1)q^{6n+3} \pmod 3.$$

Define the form $\mathcal{H}_{13,1}(z) \in M_{12}(\Gamma_0(156), \chi_{13})$ by $\mathcal{H}_{13,1}(z) := H(z) \mid U_3$. Then

$$\mathcal{H}_{13,1}(z) \equiv \sum_{n=0}^{\infty} b_{13}(3n+1)q^{2n+1} \pmod 3. \tag{3.1}$$

Proposition 3.1. *Write $\mathcal{H}_{13,1}(z) := \sum_{n=1}^{\infty} c(n)q^n$. Then for all odd primes p,*

$$\mathcal{H}_{13,1}(z) \mid T_p \equiv c(p) \cdot \mathcal{H}_{13,1}(z) \pmod 3.$$

Proof. Suppose that $Q(x,y) = ax^2 + bxy + cy^2 \in \mathbb{Z}[x,y]$ is a positive definite binary quadratic form of discriminant $D < 0$. For $n \geq 0$ let

$$r(Q,n) := \#\{(x,y) \in \mathbb{Z}^2 : Q(x,y) = n\},$$

and denote by $\theta_{a,b,c}(z)$ the function defined by

$$\theta_{a,b,c}(z) := \sum_{n=0}^{\infty} r(Q,n)q^n.$$

It is well known that $\theta_{a,b,c}(z) \in M_1(\Gamma_0(|D|), \chi_D)$. The set of reduced primitive positive definite binary quadratic forms of discriminant -156 is

$$\{x^2 + 39y^2, 3x^2 + 13y^2, 5x^2 \pm 2xy + 8y^2\}.$$

As this group is cyclic, by [9, Theorem 12],

$$\tfrac{1}{2}[\theta_{1,0,39}(z) - \theta_{3,0,13}(z) + i\theta_{5,2,8}(z) - i\theta_{5,-2,8}(z)] = \tfrac{1}{2}[\theta_{1,0,39}(z) - \theta_{3,0,13}(z)]$$

is a normalised eigenform for the Hecke operator $T_{p,1}$ for every odd prime p.

Define

$$E(z) := \frac{\eta^3(z)}{\eta(3z)} \in M_1(\Gamma_0(9), \chi_{-3}).$$

Then

$$\tfrac{1}{2}[(\theta_{1,0,39}(z) - \theta_{3,0,13}(z)) \otimes \chi_{0,2}] \cdot E(z)^{11} \in M_{12}(\Gamma_0(468), \chi_{13}) \cap \mathbb{Z}[\![q]\!]$$

and, on comparing their q-expansions out to their q^{1008} terms, Sturm's theorem yields that

$$\tfrac{1}{2}[(\theta_{1,0,39}(z) - \theta_{3,0,13}(z)) \otimes \chi_{0,2}] \cdot E(z)^{11} \equiv \mathcal{H}_{13,1}(z) \pmod 3. \tag{3.2}$$

Recall that T_p and $\chi_{0,2}$ commute for odd primes p. Then, since $E(z) \equiv 1 \pmod 3$ and $f(z) \mid T_{p,12,\chi_{13}} \equiv f(z) \mid T_{p,1,\chi_{-156}} \pmod 3$ for all odd primes p and all $f(z) \in \mathbb{Z}[\![q]\!]$, our result follows. □

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We now prove an exact formula for $b_{13}(3n+1)$ modulo 3.

THEOREM 3.2. *Let n be a nonnegative integer and write*

$$2n+1=\prod_{i=1}^{m} p_i^{e_i},$$

where $p_1, p_2, \ldots, p_m$ *are distinct primes. For each* $1 \le i \le m$*, let* $\alpha_i = (p_i/3)(p_i/13)$*, define* β_i *to be* $(-1)^{\lfloor e_i/2 \rfloor}$ *if* $(p_i/3) = (p_i/13) = -1$ *and* 1 *otherwise, set*

$$\gamma_i = \begin{cases} e_i + 1 & \text{if} \left(\frac{p_i}{3}\right) = \left(\frac{p_i}{13}\right) = 1, \\ 2 - (-1)^{\alpha_i e_i} & \text{otherwise} \end{cases}$$

and define δ_i *to be* $(-1)^{e_i}$ *if* p_i *is represented by* $3x^2 + 13y^2$ *and* 1 *otherwise. Then*

$$b_{13}(3n+1) \equiv \prod_{i=1}^{m} \beta_i \gamma_i \delta_i \pmod 3.$$

PROOF. Note first that by (3.2),

$$c(2n+1) \equiv r(3x^2+13y^2, 2n+1) - r(x^2+39y^2, 2n+1) \pmod 3 \tag{3.3}$$

for all $n \ge 0$. By classical results on quadratic forms, an odd prime p is represented by a binary quadratic form of discriminant -156 if and only if $p \mid -156$ or $(-156/p) = 1$, and an odd prime p with $1 = (-156/p) = (p/3)(p/13)$ has four representations by reduced forms of discriminant -156. As the forms $x^2 + 39y^2$ and $3x^2 + 13y^2$ (respectively $5x^2 \pm 2xy + 8y^2$) represent no integer congruent to 2 (respectively 1) modulo 3, it follows that a prime $p \notin \{2, 3, 13\}$ is represented by $x^2 + 39y^2$ or $3x^2 + 13y^2$ if and only if $p \equiv 1 \pmod 3$ and $(p/13) = 1$. Further, since $3 = 3 \cdot (\pm 1)^2 + 13 \cdot 0^2$ and $13 = 3 \cdot 0^2 + 13 \cdot (\pm 1)^2$, we conclude by (3.3) that for an odd prime p,

$$c(p) \equiv \begin{cases} 2 \pmod 3 & \text{if } p \in \{3, 13\}, \\ 2 \pmod 3 & \text{if } p \text{ is represented by } x^2+39y^2, \\ 1 \pmod 3 & \text{if } p \notin \{3,13\} \text{ is represented by } 3x^2+13y^2 \text{ and} \\ 0 \pmod 3 & \text{if } p \equiv 2 \pmod 3, \text{ or } p \equiv 1 \pmod 3 \text{ and } (p/13) = -1. \end{cases} \tag{3.4}$$

Moreover, Proposition 3.1 implies that

$$c(mn) \equiv c(m)c(n) \pmod 3 \quad \text{if } (m,n) = 1 \tag{3.5}$$

and

$$c(p^{k+1}) \equiv c(p)c(p^k) - \chi_{13}(p) \cdot p \cdot c(p^{k-1}) \pmod 3 \tag{3.6}$$

for all $k \ge 1$ and all odd primes p. Recalling (3.1), our result now follows inductively from (3.4), (3.5) and (3.6). □

PROOF OF THEOREM 1.2. Theorem 1.2 follows directly from Theorem 3.2. □

PROOF OF THEOREM 1.3. We prove only part (3), as the other parts can be proven in a similar fashion. Since 3 is a quadratic residue modulo 13, the conditions $\gamma \equiv 1 \pmod 3$ and $((2\gamma+1)/13) = -1$ imply that

$$\mathrm{ord}_p\Big(13^\alpha\Big(26n + \frac{2\gamma+1}{3}\Big)\Big)$$

is odd for some odd prime p with $(p/13) = -1$. Our result now follows from cases (1) and (2) of Theorem 1.2. □

4. Modularity of $b_{13}(3n)$ modulo 3

THEOREM 4.1. *There exists a modular form* $\mathcal{H}_{13,0}(z) \in M_{20}(\Gamma_0(468), \chi_{13}) \cap \mathbb{Z}[\![q]\!]$ *such that*

$$\mathcal{H}_{13,0}(z) \equiv \sum_{n=0}^{\infty} b_{13}(3n) q^{6n+1} \pmod 3. \tag{4.1}$$

PROOF. For $m \in \mathbb{Z}$, define

$$f_m(z) := \eta^{61-2m}(13z)\eta^{2m-1}(z).$$

One can check that $f_m(z) \in M_{30}(\Gamma_0(13), \chi_{13})$ for $-2 \le m \le 33$, and also that

$$A(z) := \eta^{19}(13z)\eta^{17}(z) = q^{11} \cdot \prod_{n=1}^{\infty}(1-q^{13n})^{19}(1-q^n)^{17} \in M_{18}(\Gamma_0(13), \chi_{13}).$$

Let $F(z) := f_0(z) \mid U_3$. Then $F(z) \in M_{30}(\Gamma_0(39), \chi_{13})$ and, since (1.1) and (2.1) give

$$f_0(z) = \Big(\sum_{n=0}^{\infty} b_{13}(n) q^{n+33}\Big) \cdot \prod_{n=1}^{\infty}(1-q^{13n})^{60},$$

we see that

$$F(z) \equiv \Big(\sum_{n=0}^{\infty} b_{13}(3n) q^{n+11}\Big) \cdot \prod_{n=1}^{\infty}(1-q^{13n})^{20} \pmod 3. \tag{4.2}$$

Upon checking that the two sides agree modulo 3 out to their q^{420} terms, Sturm's theorem yields

$$F(z) \equiv A(z) \cdot E(z)^{12} + \sum_{m=-2}^{33} \epsilon_m f_m(z) \pmod 3, \tag{4.3}$$

where $\epsilon_m = 1$ for $m \in \{1, 3, 6, 10, 13, 14, 15, 16, 20\}$, $\epsilon_m = 2$ for $m \in \{-1, 0, 12, 17, 19, 21\}$ and $\epsilon_m = 0$ otherwise. Note that by (4.2),

$$\frac{F(6z)}{q^{65} \cdot \prod_{n=1}^{\infty}(1-q^{78n})^{20}} \equiv \sum_{n=0}^{\infty} b_{13}(3n) q^{6n+1} \pmod 3. \tag{4.4}$$

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Next, letting

$$g_m(z) := \eta^{41-2m}(78z)\eta^{2m-1}(6z),$$

we find that $g_m(z) \in M_{20}(\Gamma_0(468), \chi_{13})$ for $-1 \le m \le 22$ and

$$g_m(z) = q^{133-6m} \cdot \prod_{n=1}^{\infty}(1-q^{78n})^{41-2m}(1-q^{6n})^{2m-1} = \frac{f_m(6z)}{q^{65} \cdot \prod_{n=1}^{\infty}(1-q^{78n})^{20}}. \quad (4.5)$$

Moreover, note that

$$\frac{A(6z)}{q^{65} \cdot \prod_{n=1}^{\infty}(1-q^{78n})^{20}} = q \cdot \prod_{n=1}^{\infty} \frac{(1-q^{6n})^{17}}{(1-q^{78n})} = \frac{\eta^{17}(6z)}{\eta(78z)} \in M_8(\Gamma_0(468), \chi_{13}). \quad (4.6)$$

Thus,

$$\mathcal{H}_{13,0}(z) := \frac{\eta^{17}(6z)}{\eta(78z)} \cdot E(z)^{12} + \sum_{m=-1}^{22} \epsilon_m g_m(z)$$

lies in $M_{20}(\Gamma_0(468), \chi_{13})$, and our result follows from (4.3)–(4.6). □

Corollary 4.2. *For every $n \ge 0$,*

$$b_{13}(3n) \equiv b_{13}(9n+1) \pmod 3.$$

Proof. Note that $\mathcal{H}_{13,1}(z) \otimes \chi_{0,3} \in M_{12}(\Gamma_0(468), \chi_{13})$. Then

$$(\mathcal{H}_{13,1}(z) \otimes \chi_{0,3}) \cdot E(z)^8 \in M_{20}(\Gamma_0(468), \chi_{13}) \cap \mathbb{Z}[\![q]\!]$$

and one can check that this form and $\mathcal{H}_{13,0}(z)$ are congruent modulo 3 out to their q^{1680} terms. Then Sturm's theorem, (3.1) and (4.1) yield

$$\sum_{n=0}^{\infty} b_{13}(3n)q^{6n+1} \equiv \sum_{\substack{n=0 \\ n \not\equiv 1 \ (\mathrm{mod}\ 3)}}^{\infty} b_{13}(3n+1)q^{2n+1} \pmod 3,$$

and our result immediately follows. □

Combining Corollary 4.2 and Theorem 1.2 gives the following criteria for the 3-divisibility of $b_{13}(3n)$.

Theorem 4.3. *Let n be a nonnegative integer. Then $b_{13}(3n) \equiv 0 \pmod 3$ if and only if there is a prime p such that one of the following holds:*

(1) $p \equiv 2 \pmod 3$ *and* $\mathrm{ord}_p(6n+1)$ *is odd;*
(2) $p \equiv 1 \pmod 3$, $(p/13) = -1$ *and* $\mathrm{ord}_p(6n+1)$ *is odd;*
(3) $p \equiv 1 \pmod 3$, $(p/13) = 1$ *and* $\mathrm{ord}_p(6n+1) \equiv 2 \pmod 3$.

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5. 3-divisibility results for $b_7(n)$

In this section we establish results on the 3-divisibility of $b_7(n)$ analogous to those we have proven for $b_{13}(n)$. For brevity we will not state the analogues of Proposition 3.1 and Theorem 3.2.

THEOREM 5.1. *Let n be a nonnegative integer. Then* $b_7(3n) \equiv 0 \pmod 3$ *if and only if there is a prime p such that one of the following holds:*

(1) $p \equiv 2 \pmod 3$ *and* $\mathrm{ord}_p(12n+1)$ *is odd;*
(2) $p \equiv 1 \pmod 3$, $(7/p) = -1$ *and* $\mathrm{ord}_p(12n+1)$ *is odd;*
(3) $p \equiv 1 \pmod 3$, $(7/p) = 1$ *and* $\mathrm{ord}_p(12n+1) \equiv 2 \pmod 3$.

Moreover, $b_7(3n+2) \equiv 0 \pmod 3$ *if and only if there is a prime p such that one of the following holds:*

(4) $p \equiv 2 \pmod 3$ *and* $\mathrm{ord}_p(4n+3)$ *is odd;*
(5) $p \equiv 1 \pmod 3$, $(7/p) = -1$ *and* $\mathrm{ord}_p(4n+3)$ *is odd;*
(6) $p \equiv 1 \pmod 3$, $(7/p) = 1$ *and* $\mathrm{ord}_p(4n+3) \equiv 2 \pmod 3$.

PROOF. As in our proof of Theorem 4.1, one can show that the modular forms

$$\mathcal{H}_{7,0}(z) := \sum_{m=0}^{6} \mu_m \eta^{21-4m}(84z)\eta^{4m-1}(12z) \in M_{10}(\Gamma_0(1008), \chi_7) \cap \mathbb{Z}[\![q]\!]$$

and

$$\mathcal{H}_{7,2}(z) := \sum_{m=-1}^{10} \lambda_m \eta^{37-4m}(84z)\eta^{4m-1}(12z) \in M_{18}(\Gamma_0(336), \chi_7) \cap \mathbb{Z}[\![q]\!]$$

satisfy

$$\mathcal{H}_{7,0}(z) \equiv \sum_{n=0}^{\infty} b_7(3n) q^{12n+1} \pmod 3$$

and

$$\mathcal{H}_{7,2}(z) \equiv \sum_{n=0}^{\infty} b_7(3n+2) q^{12n+9} \pmod 3,$$

where

$$(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (1, 2, 2, 2, 2, 2, 1)$$

and

$$(\lambda_{-1}, \lambda_0, \ldots, \lambda_{10}) = (1, 2, 1, 1, 2, 1, 2, 1, 2, 2, 1, 2).$$

The group

$$\{x^2 + 84y^2, 3x^2 + 28y^2, 4x^2 + 21y^2, 7x^2 + 12y^2, 5x^2 \pm 2xy + 17y^2, 8x^2 \pm 4xy + 11y^2\}$$

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of reduced primitive positive definite binary quadratic forms of discriminant −336 is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4$. As above, for every odd prime p,

$$\tfrac{1}{2}[\theta_{1,0,84}(z) + \theta_{3,0,28}(z) - \theta_{4,0,21}(z) - \theta_{7,0,12}(z)]$$

is a normalised eigenform for the Hecke operator $T_{p,1}$. Note that for any odd prime p, $(-336/p) = (p/3)(7/p)$. Then, since the forms $x^2 + 84y^2$, $3x^2 + 28y^2$, $4x^2 + 21y^2$ and $7x^2 + 12y^2$ (respectively $5x^2 \pm 2xy + 17y^2$ and $8x^2 \pm 4xy + 11y^2$) represent no integer congruent to 2 (respectively 1) modulo 3, it follows that a prime $p \notin \{2, 3, 7\}$ is represented by one of the first four forms if and only if $p \equiv 1 \pmod 3$ and $(7/p) = 1$.

Finally, on verifying that the forms

$$\tfrac{1}{2}[\theta_{1,0,84}(z) + \theta_{3,0,28}(z) - \theta_{4,0,21}(z) - \theta_{7,0,12}(z)] \cdot E(z)^{17}$$

and

$$\mathcal{H}_{7,0}(z)E(z)^8 - \mathcal{H}_{7,2}(z) - (\mathcal{H}_{7,2}(z) \mid U_3)$$

lying in $M_{18}(\Gamma_0(1008), \chi_7)$ are congruent modulo 3, by similar arguments as in our proofs of Proposition 3.1 and Theorem 3.2, our result follows. □

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