A DIFFUSION PROBLEM WITH SPHERICAL SYMMETRY: TEMPERATURE DISTRIBUTION FOR DIFFERENT INITIAL CONDITIONS

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1. Introduction

The present paper extends some earlier work [1] on heat flow in a composite system, in which a central, high-temperature region loses heat to a surrounding medium. An example of the type of situation in mind is the intrusion of igneous rock into a mass of cooler sedimentary material. As an idealization, spherical symmetry is assumed and the outer region is taken to be infinite in extent. In the earlier work, the central region and the surrounding medium were each taken to be at a constant temperature initially. The present paper gives solutions for a number of alternative situations. The temperature in the outer region is still taken to be constant initially but the temperature in the central region is represented by functions of type r^n or $(1/r)\sin kr$, where r is the distance from the centre of the system. As the system of equations for the temperature is linear, and any continuous function can be approximated arbitrarily closely by a polynomial or a Fourier series, the solutions given here can be superposed to give a solution for any continuous initial temperature distribution in the central region.

2. Notation and basic equations

We take the central region as a spherical core of radius a and use K_1, k_1, T_1 for the thermal conductivity, diffusivity and temperature in the core. Similarly, K_2, k_2 and T_2 are the corresponding quantities in the outer region. We treat K_1, K_2, k_1, k_2 as constants and T_1, T_2 as functions of r and t, with T_1 and T_2 specified at time $t = 0$. In particular, we take T_2 to be zero at $t = 0$ in each of the solutions developed below, thus measuring the temperature relative to the initial temperature in the outer region. The basic equations are:

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(1) $$\frac{\partial T_1}{\partial t} = k_1\left(\frac{\partial^2 T_1}{\partial r^2} + \frac{2}{r}\frac{\partial T_1}{\partial r}\right) \qquad 0 < r < a;$$

(2) $$\frac{\partial T_2}{\partial t} = k_2\left(\frac{\partial^2 T_2}{\partial r^2} + \frac{2}{r}\frac{\partial T_2}{\partial r}\right) \qquad r > a;$$

(3) $$T_1 = T_2, \quad K_1\left(\frac{\partial T_1}{\partial r}\right) = K_2\left(\frac{\partial T_2}{\partial r}\right) \qquad \text{at} \quad r = a.$$

Equation (3) expresses the requirement that the temperature and the heat flux should be continuous at the interface between the two regions. Additional conditions which must be satisfied are that T_1 is finite as $r \to 0$, that $T_2 \to 0$ as $r \to \infty$ and that T_1 and T_2 both tend to zero as $t \to \infty$.

For the initial conditions, we take $T_2(r, 0) = 0$ and consider two alternative forms for $T_1(r, 0)$, viz.

(4) $$T_1(r, 0) = T_0(r/a)^n, \qquad n = 0, 1, 2, \cdots.$$

(5) $$T_1(r, 0) = T_0\{\sin(\pi rh/a)\}/(\pi rh/a), \qquad 0 < h.$$

For convenience in describing the heat flow, we shall take T_0 as a positive constant; it serves as a scale factor for the temperature. The problem previously considered had $T_1(r, 0) = T_0$ and thus corresponds to $n = 0$ in (4) and to $h \to 0$ in (5).

As in [1], we write $\bar{T}_1(r, p)$ for the Laplace transform of $T_1(r, t)$ with respect to time. Thus

(6) $$\bar{T}_1(r, p) = \int_0^\infty T_1(r, t) \exp(-pt)dt$$

and in the same way $\bar{T}_2(r, p)$ denotes the Laplace transform of $T_2(r, t)$. The method of solution is to find $\bar{T}_1$ and $\bar{T}_2$ and to use contour integration to invert these transforms. This gives T_1 and T_2 as real, infinite integrals. The general form of these integrals can be deduced by looking for separable solutions of equations (1), (2) and (3); this is discussed in the next section since it throws some light on the structure of the integrals.

3. Separable form of solution

Using the condition that $T_2 \to 0$ as $t \to \infty$, the appropriate form for a separable solution of equation (2) is

(7) $$T_2 = (1/r)(a_1 \sin nr + a_2 \cos nr) \exp(-k_2 n^2 t)$$

where n can be taken as real and positive. The appropriate form for T_1 is similar. If we match the two solutions at $r = a$, using equation (3), and keep T_1 finite as $r \to 0$, we get

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(8) $$T_1 = AF_1(r, t, u), \quad T_2 = AF_2(r, t, u),$$

where A is arbitrary, u is real and positive, and

(9) $$F_1(r, t, u) = (Q/r)\{\sin (ur/a)\} \exp \{-k_1 u^2 t/a^2\},$$

(10) $$\begin{aligned} F_2(r, t, u) = {} & [(u \cos u + L \sin u) \sin \{u(r-a)/\sigma a\} \\ & + Qu \sin u \cos \{u(r-a)/\sigma a\}] \\ & \times (1/ru) \exp (-k_1 u^2 t/a^2), \end{aligned}$$

with the constants σ, L, Q defined by

(11) $$\sigma = \sqrt{(k_2/k_1)}, \quad L = (K_2-K_1)/K_1, \quad Q = K_2/K_1\sigma.$$

Equations (8) to (11) define the separable solution corresponding to u. The boundary condition that $T_2 \to 0$ as $r \to \infty$ imposes no additional restrictions on the choice of u, so we have a continuous spectrum of eigenvalues and can expect to obtain solutions of the form

(12) $$T_1(r, t) = \int_0^\infty A(u) F_1(r, t, u) du \qquad 0 < r \leqq a,$$

(13) $$T_2(r, t) = \int_0^\infty A(u) F_2(r, t, u) du \qquad r \geqq a.$$

The *amplitude factor* $A(u)$ is determined by the initial conditions; essentially we have to find a function $A(u)$ which satisfies the equations

(14) $$\begin{cases} T_1(r, 0) = \int_0^\infty A(u) F_1(r, 0, u) du & 0 < r \leqq a, \\ T_2(r, 0) = \int_0^\infty A(u) F_2(r, 0, u) du & r \geqq a, \end{cases}$$

with $T_1(r, 0)$ and $T_2(r, 0)$ specified.

When the Laplace transform method is used, the integrals obtained for $T_1(r, t)$ and $T_2(r, t)$ are of the form given by equations (12) and (13) and it is enough to obtain one of the two integrals. This determines $A(u)$ and the remaining integral can be written down immediately. In practice, it is usually more convenient to find $\bar{T}_2(r, p)$ and invert it to get $T_2(r, t)$ than to work from $\bar{T}_1(r, p)$, so in the later working the explicit form for $\bar{T}_1(r, p)$ and $T_1(r, t)$ is usually omitted.

4. Laplace transforms of T_1 and T_2

Taking the Laplace transform of equation (1),

(15) $$p\bar{T}_1 - T_1(r, 0) = (k_1/r)(\partial^2/\partial r^2)(r\bar{T}_1),$$

or if we put $p = k_1 q_1^2$ and $\bar{U}_1 = r\bar{T}_1$, the equation for $\bar{U}_1$ is

(16) $$\partial^2 \bar{U}_1/\partial r^2 = q_1^2 \bar{U}_1 - (r/k_1) T_1(r, 0).$$

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Hence

$$rT̄_1 = \bar{U}_1 = B_1 \sinh q_1 r + B_2 \cosh q_1 r + \bar{U}_0(r, p), \tag{17}$$

where B_1 and B_2 are functions of p and $\bar{U}_0$ is a Particular Integral of equation (16). As $r \to 0$, T_1 remains finite and hence $U_1 = rT_1$ must be zero at $r = 0$. This implies that $\bar{U}_1 = 0$ at $r = 0$ and so

$$B_2 = -\bar{U}_0(0, p). \tag{18}$$

In the same way, if $\bar{U}_2 = r\bar{T}_2$ and $p = k_2 q_2^2$, the equation for $\bar{U}_2$ is

$$\partial^2 \bar{U}_2 / \partial r^2 = q_2^2 \bar{U}_2 \tag{19}$$

and the appropriate solution is

$$r\bar{T}_2 = \bar{U}_2 = C(p) \exp \{-q_2(r-a)\}, \tag{20}$$

taking the real part of q_2 as positive. From equation (3), the conditions at $r = a$ are that

$$\bar{U}_1 = \bar{U}_2, \; K_1(\partial \bar{U}_1/\partial r) - K_2(\partial \bar{U}_2/\partial r) = (K_1 - K_2)(\bar{U}_2/a) \tag{21}$$

and this leads to the relations

$$\begin{aligned} B_1 G(p) = {} & \bar{U}_0(0, p)[(K_2 q_2 + K) \cosh q_1 a + K_1 q_1 \sinh q_1 a] \\ & - \bar{U}_0(a, p)\{K_2 q_2 + K\} - K_1 (\partial \bar{U}_0/\partial r)_{r=a}, \end{aligned} \tag{22}$$

$$\begin{aligned} CG(p) = {} & K_1 [q_1 \{\bar{U}_0(a, p) \cosh q_1 a - \bar{U}_0(0, p)\} \\ & - (\sinh q_1 a)(\partial \bar{U}_0/\partial r)_{r=a}], \end{aligned} \tag{23}$$

where $K = (K_2 - K_1)/a$ and

$$G(p) = K_1 q_1 \cosh q_1 a + (K_2 q_2 + K) \sinh q_1 a. \tag{24}$$

Specifying $T_1(r, 0)$ in equation (16) allows $\bar{U}_0$ to be written down and B_1, B_2 and C can then be found from equations (18), (22) and (23). If we use equation (4), i.e. $T_1(r, 0) = T_0(r/a)^n$, then $\bar{U}_0$ can be taken as a polynomial in r, namely

$$\bar{U}_0^{(n)} = \frac{T_0}{pa^n}\left[r^{n+1} + \frac{(n+1)nr^{n-1}}{q_1^2} + \frac{(n+1)(n)(n-1)(n-2)r^{n-3}}{q_1^4} + \cdots\right]. \tag{25}$$

If n is even, the series ends with a term in r and $\bar{U}_0(0, p)$ is zero; if n is odd, the series ends with a term $(n+1)!/q_1^{n+1}$ and the expressions for B_1, B_2 and C are slightly more complicated. Details of the solution for C and $\bar{T}_2$ are given in section 5.

If we use equation (5) to specify $T_1(r, 0)$, a suitable form for $\bar{U}_0$ is

$$\bar{U}_0^{(h)} = (aT_0/h\pi)\{\sin(\pi rh/a)\}/(p+p_0), \tag{26}$$

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where $p_0 = k_1\pi^2h^2/a^2$. In this case $B_2 = 0$ and the equations for B_1 and C become

$$(27)\quad B_1G(p) = -\{T_0/(p+p_0)\}[\{(K_2q_2+K)(\sin \pi h)/(\pi h/a)\}+K_1 \cos \pi h],$$

$$(28)\quad \begin{aligned} CG(p) = \{K_1T_0/(p+p_0)\}[(q_1a \cosh q_1a)\{(\sin \pi h)/\pi h\} \\ -\sinh q_1a \cos \pi h]. \end{aligned}$$

From these, explicit forms for $\bar{T}_1(r, p)$ and $\bar{T}_2(r, p)$ can be written down but we postpone discussion of these until section 6.

5. Solution when $T_1(r, 0) = T_0(r/a)^n$

For the geological problem mentioned in section 1 the only cases likely to be of practical interest are $n = 0, 1, 2$ and it is easy to find B_1, B_2, and C for these special cases. However, it is not too difficult to consider the general case (guided by the pattern that emerges when n is small). From equation (25), we have

$$(29)\quad \bar{U}_0^{(n+2)} = \frac{T_0r^{n+3}}{pa^{n+2}} + \frac{(n+3)(n+2)}{a^2q_1^2}\bar{U}_0^{(n)}$$

for $n = 0, 1, 2, \cdots$, and if we use equation (23) to form

$$G(p)[C^{(n+2)}-\{(n+3)(n+2)/a^2q_1^2\}C^{(n)}]$$

then the right-hand side simplifies, using equation (29), and we get

$$(30)\quad \begin{aligned} G(p)[C^{(n+2)}-\{(n+3)(n+2)/a^2q_1^2\}C^{(n)}] \\ = (K_1T_0/p)\{q_1a \cosh q_1a-(n+3) \sinh q_1a\}. \end{aligned}$$

If we change the notation slightly and write

$$(31)\quad C^{(n)}G(p) = (K_1T_0/p)C_n(p),$$

then the corresponding recurrence relation is

$$(32)\quad C_{n+2} = \{q_1a \cosh q_1a-(n+3) \sinh q_1a\}+\{(n+3)(n+2)/a^2q_1^2\}C_n,$$

for $n = 0, 1, 2, \cdots$, with

$$(33)\quad C_0(p) = q_1a \cosh q_1a-\sinh q_1a,$$

$$(34)\quad C_1(p) = (q_1a \cosh q_1a-2 \sinh q_1a)+(2/q_1a)(\cosh q_1a-1).$$

From equations (20) and (31),

$$(35)\quad \bar{T}_2(r, p) = \{(K_1T_0/rp)C_n(p)/G(p)\} \exp \{-q_2(r-a)\}$$

and the standard inversion integral [3] gives

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$$(36) \qquad T_2(r,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{K_1 T_0 C_n(p)}{rpG(p)} \exp\{pt - q_2(r-a)\}dp,$$

where the integration is along the line $\mathscr{R}(p) = \gamma > 0$ in the complex p-plane and the integrand is analytic for $\mathscr{R}(p) \geqq \gamma$. To evaluate this integral we follow Carslaw and Jaeger [2] and use the closed contour shown in Figure 1, where AC and FB are arcs of a large circle Γ_1 (centre O, radius R) and DE is a small circle Γ_2 (centre O, radius ε). A and B lie on the line $\mathscr{R}(p) = \gamma$. Because $\bar{T}_2(r, p)$ has a branch point at the origin, we make a cut along the negative real axis, complete the contour by lines CD and FE on either side of this cut and take $-\pi < \arg p < \pi$. This ensures that $\mathscr{R}(q_1)$ and $\mathscr{R}(q_2)$ are positive, an essential requirement.

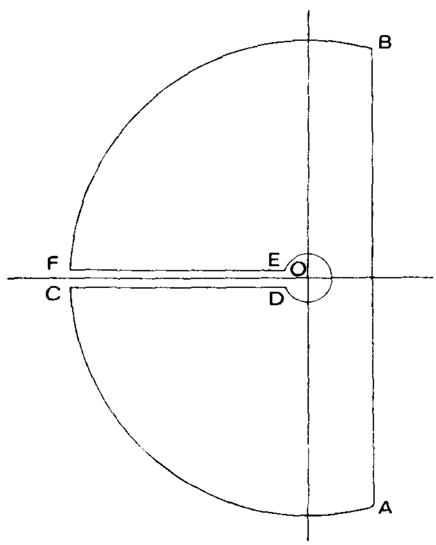

FIGURE 1

Carslaw and Jaeger prove that $G(p)$ has no zeros within or on this closed contour and as $C_n(p)$ can have a pole only at the origin it follows that the integrand in equation (36) has no poles within or on the contour. Hence integration from A to B along the straight line AB can be replaced by integration along the path $ACDEFB$ and in the limit, as $R \to \infty$ and $\varepsilon \to 0$, this gives $T_2(r, t)$. Following Carslaw and Jaeger [2], it may be shown that the integral around Γ_1 tends to zero as $R \to \infty$.

It can also be shown that the integral round Γ_2 tends to zero as $\varepsilon \to 0$. If we write $\alpha = q_1 a$, then it is easy to verify that (for $\alpha \neq 0$)

$$(37) \qquad C_0(p) = \alpha \cosh \alpha - \sinh \alpha = \int_0^\alpha u \sinh u\, du,$$

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(38) $$C_1(p) = (1/\alpha) \int_0^\alpha u^2 \sinh u\, du,$$

and for general n

(39) $$C_n(p) = \alpha^{-n} \int_0^\alpha u^{n+1} \sinh u\, du.$$

(Integration by parts shows that this form for $C_n(p)$ satisfies the recurrence relation (32)). Writing $\sinh u$ as a series now gives

(40) $$C_n(p) = \sum_{m=0}^{\infty} \left\{ \frac{\alpha^{2m+3}}{(2m+1)!\,(n+2m+3)} \right\}$$

and hence for $|p|$ small

(41) $$C_n(p) = (q_1 a)^3/(n+3) + O(q_1^5).$$

Also, for $|p|$ small,

(42) $$G(p) = K_2 q_1 + O(q_1^2)$$

and the remaining factor in the integrand is of order unity. The integral round Γ_2 is therefore of order q_1^2 and tends to zero as $\varepsilon \to 0$.

The integrals along CD and EF can be combined, following Carslaw and Jaeger, to give

(43) $$T_2(r, t) = 2\mathscr{R}(J),$$

where

(44) $$J = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \left\{ \frac{1}{2\pi i} \int_{EF} \bar{T}_2(r, p) \exp(pt) dp \right\}.$$

Along EF, we can write $p = k_1(u/a)^2 \exp(i\pi)$, with u positive. Then $q_1 = i(u/a)$, $q_2 = i(u/\sigma a)$ and from equation (40) $C_n(p)$ has its real part zero. If we write $C_n(p) = -i\mathscr{C}_n(u)$, then equations (33), (34) and (32) give

(45) $$\mathscr{C}_0(u) = \sin u - u \cos u,$$

(46) $$\mathscr{C}_1(u) = (2 \sin u - u \cos u) - (2/u)(1 - \cos u),$$

(47) $$\mathscr{C}_{n+2}(u) = \{(n+3) \sin u - u \cos u\} - \{(n+3)(n+2)/u^2\} \mathscr{C}_n(u),$$

for $n = 0, 1, 2, \cdots$. Also, on EF,

(48) $$\mathscr{R}\left[\frac{\exp\{pt - q_2(r-a)\}}{rG(p)}\right] = \frac{-auF_2(r, t, u)}{K_1 D^2},$$

where $F_2(r, t, u)$ is the function defined in equation (10) and

(49) $$D^2 = (u \cos u + L \sin u)^2 + (Qu \sin u)^2.$$

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This leads to

$$(50)\qquad T_2(r,t) = (2aT_0/\pi)\int_0^\infty \{\mathscr{C}_n(u)/D^2\}F_2(r,t,u)du.$$

Comparing equation (50) with equation (13) gives

$$(51)\qquad A(u) = (2aT_0/\pi)\{\mathscr{C}_n(u)/D^2\},$$

and $T_1(r, t)$ can now be written down from equation (12).

In analogy with equations (39) and (40), we can write

$$(52)\qquad \begin{aligned}\mathscr{C}_n(u) &= u^{-n}\int_0^u v^{n+1}\sin v\,dv\\ &= \sum_{m=0}^{\infty}\left\{\frac{(-1)^m u^{2m+3}}{(2m+1)!\,(n+2m+3)}\right\}.\end{aligned}$$

The series form is useful in obtaining information about the heat flow and temperature for large values of t (cf. Section 7).

6. Solution when $T_1(r, 0) = T_0\{\sin(\pi rh/a)\}/(\pi rh/a)$

From equations (20) and (28), we have in this case

$$(53)\qquad \bar{T}_2(r,p) = [\{K_1T_0/r(p+p_0)\}E(h,p)\exp\{-q_2(r-a)\}]/G(p)$$

where

$$(54)\qquad E(h,p) = (q_1a\cosh q_1a)\{(\sin\pi h)/\pi h\} - \sinh q_1a\cos\pi h.$$

We can write down the inversion integral for $T_2(r, t)$ as in section 5 and a similar contour in the p-plane can be used. At first sight, it looks as if the contour should be indented at $p = -p_0$ to allow for a pole at this point but closer scrutiny reveals that this is not necessary. The factor $E(h, p)$ is also zero when $p = -p_0$ and it can be proved that as $p \to -p_0$, either from below the cut in the p-plane or from above it, the ratio $E(h, p)/(p+p_0)$ approaches a finite limit. Hence we can use the same contour as before and integrate along $ACDEFB$ (Figure 1). For the large circle, the contribution to the integral again tends to zero as $R \to \infty$. (The argument follows the same lines as in the previous case; for R sufficiently large, $|p+p_0| > \frac{1}{2}|p|$ and hence it does not matter that there is a factor $p+p_0$ instead of a factor p in the denominator.)

On the small circle Γ_2,

$$(55)\qquad E(h,p) = q_1a\{-\cos\pi h + (\sin\pi h)/\pi h\} + O(q_1^3)$$

and $\{E(h, p)/G(p)\}$ approaches a finite limit as $|q_1| \to 0$, from equations (42) and (55). The differential dp gives a factor ε and the remaining factors are of order unity; hence the integral round Γ_2 tends to zero as $\varepsilon \to 0$.

This leaves only the contribution from integration along CD and EF and these can be combined, as in section 5, to give

$$T_2(r, t) = 2\mathscr{R}(J_1) \tag{56}$$

where J_1 is the limit, as $\varepsilon \to 0$ and $R \to \infty$, of the contribution from EF. Along EF, $q_1 = im = iu/a$ and hence

$$E(h, p) = i(u \cos u \sin \pi h - \pi h \sin u \cos \pi h)/\pi h = i\mathscr{E}(h, u), \tag{57}$$

$$p+p_0 = -k_1 m^2 + k_1(\pi h/a)^2 = (k_1/a^2)(\pi^2 h^2 - u^2), \tag{58}$$

$$(dp)/(p+p_0) = -2k_1 m\, dm/(p+p_0) = -2u\, du/(\pi^2 h^2 - u^2). \tag{59}$$

Using the above equations and equations (48) and (49) gives

$$T_2(r, t) = \frac{2aT_0}{\pi}\int_0^\infty u^2 \frac{\mathscr{E}(h, u)F_2(r, t, u)du}{(\pi^2 h^2 - u^2)D^2}. \tag{60}$$

In this case, the amplitude factor $A(u)$ of equation (13) is

$$A(u) = (2aT_0/\pi)\{u^2\mathscr{E}(h, u)\}/\{(\pi^2 h^2 - u^2)D^2\} \tag{61}$$

and this allows $T_1(r, t)$ to be written down.

We may note, as a check, that as h tends to zero $\mathscr{E}(h, u) \to -\mathscr{C}_0(u)$ and $A(u) \to (2aT_0/\pi)\{\mathscr{C}_0(u)/D^2\}$, which agrees with the form of $A(u)$ obtained for $n = 0$ in section 5 and with the results obtained in [1] for this special case.

7. Temperature and heat flow at interface

The temperature at the interface between the core and the outer region can be obtained by putting $r = a$ in either $T_1(r, t)$ or $T_2(r, t)$. Similarly, the heat flux from the core can be obtained from

$$\begin{aligned} H'(t) &= \text{heat flux across surface } r = a \\ &= -4\pi K_2 a^2 (\partial T_2/\partial r)_{r=a}. \end{aligned} \tag{62}$$

Integrating $H'(t)$ from 0 to t gives the heat loss from the core after time t and this will be denoted by $H(t)$. As $t \to \infty$, $T_1(r, t) \to 0$ and hence $H(t)$ must approach a limiting value, namely the total amount of heat energy initially available in the core. If we use H_0 for the latter quantity, then

$$H_0 = \int_0^a 4\pi r^2 \rho s T_1(r, 0)dr, \tag{63}$$

where ρ is the density of the core material and s is its specific heat. Since $k_1 = K_1/\rho s$, we can replace ρs in equation (63) by K_1/k_1.

If we apply these ideas to the case where $T_1(r, 0) = T_0(r/a)^n$, equations (10) and (50) give

$$(64) \qquad T_2(a, t) = (2QT_0/\pi) \int_0^\infty \{\mathscr{C}_n(u)/D^2\} \sin u \;\; X(t, u) du,$$

$$(65) \qquad H'(t) = (8QT_0aK_1) \int_0^\infty \{\mathscr{C}_n/D^2\}(\sin u - u \cos u) X(t, u) du,$$

$$(66) \qquad H(t) = (8QT_0a^3K_1/k_1) \int_0^\infty \{\mathscr{C}_n/D^2\}\{(\sin u - u \cos u)(1-X)/u^2\} du,$$

where $X(t, u)$ denotes the exponential factor $\exp(-k_1tu^2/a^2)$. For $u > 0$, $X \to 0$ as $t \to \infty$ and hence

$$(67) \qquad \begin{aligned} H_0 &= (8QT_0a^3K_1/k_1) \int_0^\infty \{\mathscr{C}_n/D^2\}\{(\sin u - u \cos u)/u^2\} du \\ &= 4\pi a^3K_1T_0/k_1(n+3). \end{aligned}$$

The second line in this equation comes from equation (63). From this, we obtain the incidental result that

$$(68) \qquad \int_0^\infty \{\mathscr{C}_0(u)\mathscr{C}_n(u)\}/(u^2D^2) du = \tfrac{1}{2}\pi/Q(n+3)$$

and the more important result that

$$(69) \qquad f = \{2Q(n+3)/\pi\} \int_0^\infty \{\mathscr{C}_0(u)\mathscr{C}_n(u)X(t, u)/u^2D^2\} du,$$

where $f = \{H_0 - H(t)\}/H_0$ is the fraction of the total heat flux from the core that has still to occur at time t.

For the case where $T_1(r, 0) = T_0\{\sin(\pi rh/a)\}/(\pi rh/a)$, equation (63) gives

$$(70) \qquad H_0 = (4\pi a^3K_1T_0/k_1)(\sin \pi h - \pi h \cos \pi h)/(\pi h)^3.$$

The expressions for $T_2(a, t)$, $H'(t)$ and $H(t)$ have the same structure as before; if we replace $\mathscr{C}_n(u)$ by $u^2\mathscr{E}(h, u)/(\pi^2h^2 - u^2)$ in equations (64), (65) and (66) we get the appropriate forms. This can be seen by comparing equations (50) and (60). The expression for f is more complicated in this case but can be written as

$$(71) \qquad f = \frac{2Q(\pi h)^3}{\pi\mathscr{C}_0(\pi h)} \int_0^\infty \frac{\mathscr{C}_0(u)\mathscr{E}(h, u)X(t, u)}{(\pi^2h^2 - u^2)D^2} du.$$

For large values of t, the main contribution to the integrals in equations (69) and (71) comes from the neighbourhood of $u = 0$, because of the exponential factor $X(t, u)$. For u small, the leading term in $\mathscr{C}_n(u)$ is $u^3/(n+3)$, from equation (52), and this leads to

$$(72) \qquad f = (a^3/6\sqrt{\pi})(K_1/K_2k_1\sqrt{k_2})t^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}),$$

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independent of n. Similarly, for u small,

$$\{\mathscr{E}(h, u)\}/(\pi^2 h^2 - u^2) = \{u\mathscr{C}_0(\pi h)\}/(\pi h)^3 + O(u^3), \tag{73}$$

and the corresponding expression for f is again given by equation (72). Thus the leading term in f is independent of $T_1(r, 0)$ for the forms of $T_1(r, 0)$ that have been examined here. Presumably this would also hold for any linear combination of these initial conditions.

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