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Versions of Schwarz's Lemma for Condenser Capacity and Inner Radius

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Abstract. We prove variants of Schwarz's lemma involving monotonicity properties of condenser capacity and inner radius. Also, we examine when a similar monotonicity property holds for the hyperbolic metric.

1 Introduction

Let f be a holomorphic function on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $0 < r < 1$, let $r\mathbb{D} = \{z \in \mathbb{C} : |z| < r\}$ and

$$\operatorname{Rad}\big(f(r\mathbb{D})\big) = \sup_{|z|<r} |f(z) - f(0)|.$$

According to the classical Schwarz lemma, the function

$$\Phi(r) = \frac{\operatorname{Rad}\big(f(r\mathbb{D})\big)}{r}, \quad 0 < r < 1,$$

is increasing. Schwarz's lemma was expressed in this form in [5], where R. B. Burckel, D. E. Marshall, D. Minda, P. Poggi-Corradini, and T. J. Ransford considered the size of the image set $f(r\mathbb{D})$, relative to several other geometric quantities, compared to the size of $r\mathbb{D}$; in particular they proved that the function

$$\Phi_T(r) = \frac{T\big(f(r\mathbb{D})\big)}{T(r\mathbb{D})}, \quad 0 < r < 1, \tag{1.1}$$

is increasing, where T may be area, diameter or logarithmic capacity. D. Betsakos [3] proved similar monotonicity properties of quasiregular mappings on the unit ball of $\mathbb{R}^n$. R. Laugesen and C. Morpurgo [13] and also T. Carroll and J. Ratzkin [6], under the additional assumption that f is univalent, proved that the function Φ_T in (1.1) is increasing when $T\big(f(r\mathbb{D})\big)$ is the first eigenvalue of the Laplacian with Dirichlet boundary data. Related results have recently appeared in [4, 9, 16].

Earlier appearances of this idea occurred in [2, 12, 14]. In [12], G. Julia proved that for $0 < p < \infty$, the function

$$\Phi_p(r) = \frac{\int_0^{2\pi} |f(re^{it})|^p \, dt}{r^p}, \quad 0 < r < 1,$$

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is increasing. E. F. Beckenbach [2] showed that if f is holomorphic in $\mathbb{D}$ and for all $r \in (0,1)$ and all $\theta \in [-\pi, \pi]$,

$$\int_0^r |f'(\rho e^{i\theta})|\, d\rho \leq 1,$$

then, in fact, for all $r \in (0,1)$ and for all $\theta \in [-\pi, \pi]$,

$$\int_0^r |f'(\rho e^{i\theta})|\, d\rho \leq r.$$

G. Pólya and G. Szegö [14, Problem 309] proved that the function

$$\Phi_L(r) = \frac{L\big(f(|z| = r)\big)}{2\pi r}, \quad 0 < r < 1,$$

(where $L\big(f(|z| = r)\big)$ is the length of the curve $f(|z| = r)$) is increasing.

In the present paper, we prove an analogous monotonicity property for the capacity of a condenser. A *condenser* in the complex plane $\mathbb{C}$ is a pair (D, K) where D is a Greenian open subset of $\mathbb{C}$ and K is a compact subset of D. Let h be the solution of the generalized Dirichlet problem on $D \setminus K$ with boundary values 0 on ∂D and 1 on ∂K. The function h is the *equilibrium potential* of the condenser (D, K). The *capacity* of (D, K) is

$$\mathrm{cap}(D, K) = \int_{D \setminus K} |\nabla h|^2.$$

The *equilibrium energy* of (D, K) is the extended real number

$$I(D, K) = \frac{2\pi}{\mathrm{cap}(D, K)}.$$

We set

$$\mathrm{C}_2(D, K) = e^{-I(D,K)}.$$

It is easy to compute the equilibrium energy of an annulus:

$$I(s\mathbb{D}, \overline{r\mathbb{D}}) = \log \frac{s}{r}, \quad r < s. \tag{1.2}$$

It follows that

$$\mathrm{C}_2(\mathbb{D}, \overline{r\mathbb{D}}) = r. \tag{1.3}$$

We refer to [10] and [8] for more information about condenser capacities.

Theorem 1 *Let $f\colon \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that $f(\mathbb{D})$ is a Greenian domain. Then the function*

$$\Phi_C(r) = \frac{\mathrm{C}_2\big(f(\mathbb{D}), f(\overline{r\mathbb{D}})\big)}{r}, \quad r \in (0,1),$$

is increasing. If Φ_C is not strictly increasing, there exists $d_0 \in (0,1]$ such that Φ_C is constant on $(0, d_0)$, Φ_C is strictly increasing on $(d_0, 1)$, and f is univalent on $d_0\mathbb{D}$.

In fact, Theorem 1 will follow from a more general result about condenser capacity. This result is Theorem 2.2 and will be proved in Section 2.

Let $D \subset \mathbb{C}$ be a Greenian domain with a Green function $G_D(x, y)$ and $z_0 \in D$. The limit

$$\gamma = \lim_{z \to z_0} \Big[G(z, z_0) - \log \frac{1}{z - z_0} \Big]$$

exists. The *inner radius* $R(D, z_0)$ of D at z_0 is [10, p. 123]

$$R(D, z_0) = e^{\gamma}.$$

A simple property of the inner radius is that if D is simply connected and f maps $\mathbb{D}$ conformally onto D with $f(0) = z_0$, then $R(D, z_0) = |f'(0)|$. It follows that $R(r\mathbb{D}, 0) = r$. In Section 3 we will prove the following monotonicity property for the inner radius $R\big(f(r\mathbb{D}), f(0)\big)$.

Theorem 2 *Let $f \colon \mathbb{D} \to \mathbb{C}$ be a holomorphic function. Then the function*

$$\Phi_R(r) = \frac{R\big(f(r\mathbb{D}), f(0)\big)}{r}, \quad 0 < r < 1,$$

is increasing. Moreover, if Φ_R is not strictly increasing, there exists an $s_0 \in (0, 1]$ such that Φ_R is constant on $(0, s_0)$ and strictly increasing on $(s_0, 1)$, and f is univalent on $s_0\mathbb{D}$.

We next recall the definition of the hyperbolic metric (see [1, p. 41], [11, p. 682]. A planar domain D is called *hyperbolic* provided that $\mathbb{C} \setminus D$ contains at least two points. Let Δ be a disk; a holomorphic function $f \colon \Delta \to D$ is a *universal covering map of D* if every point in D has an open neighborhood V such that $f^{-1}(V)$ is a disjoint union of open sets U_i and the restriction of f to U_i is a conformal map of U_i onto V. Let D be a hyperbolic domain and let $z \in D$. By the uniformization theorem, there exists a universal covering map $f \colon \mathbb{D} \to D$ with $f(0) = z$. The *density of the hyperbolic metric* for D at the point z is defined by

$$\lambda_D(z) = \frac{1}{|f'(0)|}.$$

Clearly, if D is simply connected, then

$$\lambda_D(z) = \frac{1}{R(D, z)}. \tag{1.4}$$

In view of Theorem 2, one may ask whether an analogous monotonicity property holds for the hyperbolic metric. We show that the answer is negative.

Theorem 3 *Let D be a hyperbolic domain in $\mathbb{C}$, let $w \in D$, and let $f \colon \mathbb{D} \to D$ be a universal covering map of D with $f(0) = w$. Let*

$$\Phi_H(r) = \frac{1}{r\lambda_{f(r\mathbb{D})}(w)}, \quad 0 < r < 1.$$

The function Φ_H is increasing if and only if D is simply connected (in which case f is univalent and Φ_H is constant).

This theorem will be proved in Section 4.

2 Schwarz-Type Lemma for Condenser Capacity

We start by stating some known results that we will need later. It is well known that holomorphic functions decrease the capacity of a condenser, that is

$$\text{(2.1)} \qquad \Cap\big(f(D), f(K)\big) \leq \Cap(D, K).$$

Moreover, equality holds only when f is univalent; see [15].

A condenser (D, K) will be called *regular* if the open set $D \setminus K$ is regular for the Dirichlet problem and every connected component of $D \setminus K$ has boundary points on both sets ∂D and K. Let h be the equilibrium potential of the regular condenser (D, K). The sets

$$\{z \in D \setminus K : h(z) = r\}, \quad 0 < r < 1,$$

are called *equipotential curves* of (D, K). We recall Grötzsch's lemma:

Theorem 2.1 ([8, p. 9]) *Let (D, K) be a condenser. Let G be an open set such that $K \subset G$ and $\overline{G} \subset D$. Then*

$$\text{(2.2)} \qquad I(D, K) \geq I(D, \overline{G}) + I(G, K).$$

If the condensers (D, K), $(D, \overline{G})$ and (G, K) are regular, then equality holds in (2.2) if and only if ∂G is an equipotential curve of (D, K).

Let (D, K) be a regular condenser and let h be its equilibrium potential. We extend h on D by setting $h = 1$ on K. Then h is a continuous superharmonic function on D. For every $r \in (0, 1)$, we consider the open set

$$D_r = \{x \in D : h(x) > 1 - r\}.$$

Then for $0 < r < s < 1$,

$$K \subset D_r \subset \overline{D_r} \subset D_s \subset \overline{D_s} \subset D.$$

Therefore the condensers $(D, \overline{D_r})$ and $(D_s, \overline{D_r})$ are well defined and regular. We have the following monotonicity theorem for condenser capacity.

Theorem 2.2 *Let (D, K) be a regular condenser and let D_r, $0 < r < 1$, be as above. Also, let f be a holomorphic function which is non-constant on every connected component of D and let G be a Greenian open set such that $f(D) \subset G$ and $(G, f(K))$ is a regular condenser. Then the functions*

$$\Phi_K(r) = \frac{1}{\Cap\big(f(D_r), f(K)\big)} - \frac{1}{\Cap(D_r, K)}, \quad r \in (0, 1),$$

and

$$\Phi_D(r) = \frac{1}{\Cap(D, \overline{D_r})} - \frac{1}{\Cap\big(G, f(\overline{D_r})\big)}, \quad r \in (0, 1),$$

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are increasing. If Φ_K *is not strictly increasing, there exists* $k_0 \in (0,1]$ *such that* $\Phi_K = 0$ *on* $(0,k_0)$, Φ_K *is strictly increasing on* $(k_0,1)$, *and* f *is univalent on* D_{k_0}. *If* Φ_D *is not strictly increasing, there exists* $d_0 \in (0,1]$ *such that* Φ_D *is constant on* $(0,d_0)$, Φ_D *is strictly increasing on* $(d_0,1)$, *and* f *is univalent on* D_{d_0}. *Also,* Φ_D *is constant on* $(0,1)$ (i.e., $d_0 = 1$) *if and only if* f *is univalent on* D *and* $\partial f(D_r)$, $r \in (0,1)$, *is an equipotential curve of the condenser* $\big(G, f(K)\big)$.

Proof Let $0 < r < s < 1$. We will consider the functions Φ_K and Φ_D separately.

First we consider the function Φ_K. By inequality (2.1) and Grötzsch's lemma,

$$\begin{aligned} I(D_s,K) - I(D_r,K) &= I(D_s,\overline{D_r}) \\ &\le I\big(f(D_s), f(\overline{D_r})\big) \\ &\le I\big(f(D_s), f(K)\big) - I\big(f(D_r), f(K)\big). \end{aligned} \tag{2.3}$$

Therefore

$$\frac{1}{\Cap\big(f(D_r), f(K)\big)} - \frac{1}{\Cap(D_r,K)} \le \frac{1}{\Cap\big(f(D_s), f(K)\big)} - \frac{1}{\Cap(D_s,K)},$$

which means that

$$\Phi_K(r) \le \Phi_K(s). \tag{2.4}$$

Suppose that equality holds in (2.4). Then the inequality (2.3) must be an equality. By [15], f must be univalent in D_s and therefore $\Phi_K = 0$ on $(0,s)$. Let

$$k_0 = \sup\{s \in (0,1) : \text{ there exists } 0 < r < s \text{ such that } \Phi_K(r) = \Phi_K(s)\} > 0.$$

Then $\Phi_K = 0$ on $(0,k_0)$, Φ_K is strictly increasing on $(k_0,1)$, and f is univalent on D_{k_0}.

Now we consider the function Φ_D. By inequality (2.1) and Grötzsch's lemma,

$$I(D,\overline{D_r}) - I(D,\overline{D_s}) = I(D_s,\overline{D_r})$$

$$\le I\big(f(D_s), f(\overline{D_r})\big) \tag{2.5}$$

$$\le I\big(G, f(\overline{D_r})\big) - I\big(G, f(\overline{D_s})\big). \tag{2.6}$$

Therefore

$$\frac{1}{\Cap(D,\overline{D_r})} - \frac{1}{\Cap\big(G, f(\overline{D_r})\big)} \le \frac{1}{\Cap(D,\overline{D_s})} - \frac{1}{\Cap\big(G, f(\overline{D_s})\big)},$$

which means

$$\Phi_D(r) \le \Phi_D(s). \tag{2.7}$$

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Suppose that equality holds in (2.7). Then the inequalities (2.5) and (2.6) must be equalities. By [15] and the equality in (2.5), f must be univalent in D_s. The equilibrium potential of the condenser (D_s, K) is the function

$$h_s(x) = \frac{h(x) - (1-s)}{s}, \quad x \in D_s \setminus K.$$

Therefore, the equilibrium potential of the condenser $\big(f(D_s), f(K)\big)$ is the function

$$u_s(x) = h_s\big(f^{-1}(x)\big) = \frac{h\big(f^{-1}(x)\big) - (1-s)}{s}, \quad x \in f(D_s) \setminus f(K)$$

and $f(\partial D_r)$ is an equipotential curve of $\big(f(D_s), f(K)\big)$. By the equality case in Grötzsch's lemma and the equality in (2.6), $\partial f(D_s)$ must be an equipotential curve of the condenser $\big(G, f(\overline{D_r})\big)$. Let u_r be the equilibrium potential of $\big(G, f(\overline{D_r})\big)$ and let c_s be the constant value of u_r on $\partial f(D_s)$. Since

$$u_s(x) = \frac{h\big(f^{-1}(x)\big) - (1-s)}{s} = \frac{s-r}{s}, \quad x \in \partial f(D_r),$$

by the maximum principle we obtain that

$$\frac{s}{(s-r)} u_s(x) = \frac{u_r(x) - c_s}{1 - c_s}, \quad x \in f(D_s) \setminus f(\overline{D_r}),$$

or

$$u_r(x) = \frac{s(1-c_s)}{(s-r)} u_s(x) + c_s, \quad x \in f(D_s) \setminus f(\overline{D_r}).$$

Let

$$a = \frac{s(1-c_s)}{(s-r)} + c_s > 0.$$

Then the function

$$u(x) = \begin{cases} \frac{u_r(x)}{a}, & x \in G \setminus f(\overline{D_s}), \\ \frac{1}{a}\Big(\frac{s(1-c_s)}{(s-r)} u_s(x) + c_s\Big), & x \in f(\overline{D_s}) \setminus f(K) \end{cases}$$

is the equilibrium potential of the condenser $\big(G, f(K)\big)$. Therefore $\partial f(D_t)$ is an equipotential curve of the condenser $\big(G, f(K)\big)$ for every $t \in (0, s]$. Also, by Grötzsch's lemma and the fact that f is univalent on D_s,

$$\begin{aligned} I(D, \overline{D_t}) - I(D, \overline{D_s}) &= I(D_s, \overline{D_t}) \\ &= I\big(f(D_s), f(\overline{D_t})\big) \\ &= I\big(f(D), f(\overline{D_t})\big) - I\big(f(D), f(\overline{D_s})\big) \end{aligned}$$

and therefore

$$\Phi_D(t) = \Phi_D(s),$$

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for every $t \in (0, s)$. Let

$$d_0 = \sup\{s \in (0,1) : \text{ there exists } 0 < r < s \text{ such that } \Phi_D(r) = \Phi_D(s)\} > 0.$$

Then Φ_D is constant on $(0, d_0)$, Φ_D is strictly increasing on $(d_0, 1)$, and f is univalent on D_{d_0}.

Finally, if $d_0 = 1$ then f is univalent on D and $\partial f(D_t)$, $t \in (0, 1)$, is an equipotential curve of the condenser $\big(G, f(K)\big)$. The converse is obvious. ∎

We proceed to prove Theorem 1.

Proof of Theorem 1 Recall first the definition

$$\mathrm{C}_2(D,K) = e^{-I(D,K)}$$

which is equivalent to

$$\frac{1}{\Cap(D,K)} = \log\frac{1}{\mathrm{C}_2(D,K)}.$$

Let $0 < \epsilon < r < s < 1$ and consider the condenser $(\mathbb{D}, \overline{\epsilon\mathbb{D}})$. Then $\partial(r\mathbb{D})$ and $\partial(s\mathbb{D})$ are equipotential curves of $(\mathbb{D}, \overline{\epsilon\mathbb{D}})$ and by (1.3) and Theorem 2.2,

$$\begin{aligned}\frac{\mathrm{C}_2\big(f(\mathbb{D}), f(\overline{r\mathbb{D}})\big)}{r} &= \frac{\mathrm{C}_2\big(f(\mathbb{D}), f(\overline{r\mathbb{D}})\big)}{\mathrm{C}_2(\mathbb{D}, \overline{r\mathbb{D}})}\\ &\le \frac{\mathrm{C}_2\big(f(\mathbb{D}), f(\overline{s\mathbb{D}})\big)}{\mathrm{C}_2(\mathbb{D}, \overline{s\mathbb{D}})}\\ &= \frac{\mathrm{C}_2\big(f(\mathbb{D}), f(\overline{s\mathbb{D}})\big)}{s}.\end{aligned}$$

The equality statement follows from the corresponding equality statement of Theorem 2.2. ∎

3 Schwarz-Type Lemma for the Inner Radius

We will use the following representation for the inner radius (see [10, p. 127])

$$\begin{aligned}\log R(D, z_0) &= \lim_{\epsilon\to 0}\Big[\frac{2\pi}{\Cap\big(D, \overline{B(z_0,\epsilon)}\big)} - \log\frac{1}{\epsilon}\Big]\\ &= \lim_{\epsilon\to 0}\Big[I\big(D, \overline{B(z_0,\epsilon)}\big) - \log\frac{1}{\epsilon}\Big].\end{aligned}$$

If f is univalent on $\mathbb{D}$, then [10, p. 124] $R\big(f(\mathbb{D}), f(0)\big) = |f'(0)|$. If $r \in (0, 1)$, the function

$$g(z) = f(rz), \quad z \in \mathbb{D},$$

is univalent and

$$R\big(f(r\mathbb{D}), f(0)\big) = R\big(g(\mathbb{D}), g(0)\big) = |g'(0)| = r|f'(0)|.$$

Therefore

$$\frac{R\big(f(r\mathbb{D}), f(0)\big)}{r} = |f'(0)|, \tag{3.1}$$

for all $r \in (0,1)$.

We proceed to prove Theorem 2.

Proof of Theorem 2 Let $0 < r < s < 1$. By (1.2), (2.1) and (2.2), we obtain that for every ϵ with $0 < \epsilon < r$,

$$\begin{aligned}\log\frac{s}{r} &= I(s\mathbb{D}, \overline{r\mathbb{D}})\\ &\le I\big(f(s\mathbb{D}), f(\overline{r\mathbb{D}})\big)\\ &\le I\big(f(s\mathbb{D}), \overline{f(0)+\epsilon\mathbb{D}}\big) - I\big(f(r\mathbb{D}), \overline{f(0)+\epsilon\mathbb{D}}\big)\\ &= I\big(f(s\mathbb{D}), \overline{f(0)+\epsilon\mathbb{D}}\big) - \log\frac{1}{\epsilon} - \Big[I\big(f(r\mathbb{D}), \overline{f(0)+\epsilon\mathbb{D}}\big) - \log\frac{1}{\epsilon}\Big].\end{aligned} \tag{3.2}$$

Therefore

$$\begin{aligned}\log\frac{s}{r} &\le \lim_{\epsilon\to 0} I\big(f(s\mathbb{D}), \overline{f(0)+\epsilon\mathbb{D}}\big) - \log\frac{1}{\epsilon} - \Big[I\big(f(r\mathbb{D})\overline{f(0)+\epsilon\mathbb{D}}\big) - \log\frac{1}{\epsilon}\Big]\\ &= \log R\big(f(s\mathbb{D}), f(0)\big) - \log R\big(f(r\mathbb{D}), f(0)\big)\end{aligned}$$

and

$$\Phi_R(r) = \frac{R\big(f(r\mathbb{D}), f(0)\big)}{r} \le \frac{R\big(f(s\mathbb{D}), f(0)\big)}{s} = \Phi_R(s).$$

Suppose that $\Phi_R(r) = \Phi_R(s)$ for some $0 < r < s < 1$. Then we must have equality in (3.2). By [15], f is univalent on $s\mathbb{D}$ and by (3.1) Φ_R is constant and equal to $|f'(0)|$ on $(0, s)$. Let

$$s_0 = \sup\{s \in (0,1) : \text{ there exists } 0 < r < s \text{ such that } \Phi_R(r) = \Phi_R(s)\} > 0.$$

Then Φ_R is constant on $(0, s_0)$ and strictly increasing on $(s_0, 1)$ and f is univalent on $s_0\mathbb{D}$. ∎

4 On the Hyperbolic Metric

For the proof of Theorem 3, we will need two fundamental theorems for the hyperbolic metric and the universal covering maps. The first is the *principle of the hyperbolic metric.*

Theorem 4.1 ([11, p. 682] or [1, p. 43]) *Let Δ be a disk, D be a hyperbolic domain, and $f\colon \Delta \to D$ be a holomorphic function. Then for all $z \in \Delta$,*

$$\lambda_D\big(f(z)\big)\,|f'(z)| \le \lambda_\Delta(z). \tag{4.1}$$

If there exists a point in Δ such that equality holds in (4.1), then f is a universal covering map.

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Let D be a hyperbolic domain and let $f\colon \mathbb{D} \to D$ be a universal covering map of D. Also, let γ be a curve in D. A curve $\tilde{\gamma}$ in $\mathbb{D}$ is called a *lifting* of γ if $f \circ \tilde{\gamma} = \gamma$. We need the following theorem.

Theorem 4.2 ([7, p. 246]) *Let $D \subset \mathbb{C}$ be a hyperbolic domain with $w_0 \in D$ and let $f\colon \mathbb{D} \to D$ be a universal covering map of D. If γ is a curve in D with initial point w_0 and $f(z_0) = w_0$, then there is a unique lifting $\tilde{\gamma}$ of γ with initial point z_0.*

We proceed to prove Theorem 3.

Proof of Theorem 3 If D is simply connected, then f is a Riemann map and by (3.1) and (1.4), Φ_H is constant.

Conversely, assume that D is not simply connected. Then there exists a point a in the complement of D and a simple closed curve $\gamma\colon [0,1] \to D$ such that $\gamma(0) = \gamma(1) = w$ and the winding number of γ around a is 1. Let $\tilde{\gamma}$ be the lifting of γ with $\tilde{\gamma}(0) = 0$. Since $\tilde{\gamma}([0,1])$ is a compact subset of $\mathbb{D}$, there exists $r_o \in (0,1)$ such that $\tilde{\gamma}([0,1]) \subset r_o\mathbb{D}$.

For $n \in \mathbb{N}$, let $\gamma_n\colon [0,1] \to D$ be the curve obtained by tracing the curve γ n times and let $\tilde{\gamma}_n$ be the lifting of γ_n with $\tilde{\gamma}_n(0) = 0$. Let $0 = t_1 < t_2 < \cdots < t_n < 1$ be the points in $[0,1]$ with $\gamma_n(t_i) = w$, $i = 1,2,\dots,n$. Since $f \circ \tilde{\gamma}_n = \gamma_n$, we have $f\big(\tilde{\gamma}_n(t_i)\big) = w$, *i.e.*, the points $\tilde{\gamma}_n(t_i) \in \mathbb{D}$, $i = 1,2,\dots,n$, are zeros of the function $f - w$. We claim that

$$\tilde{\gamma}_n(t_i) \neq \tilde{\gamma}_n(t_j), \quad i \neq j.$$

Suppose that $\tilde{\gamma}_n(t_i) = \tilde{\gamma}_n(t_j)$, for some $i < j$. Consider the closed curve

$$\tilde{\delta}(t) = \tilde{\gamma}_n(t), \quad t \in [t_i, t_j]$$

which lies in $\mathbb{D}$ and note that the number of the zeros of the function $f - a$ in the interior of $\tilde{\delta}$ is $N_{f-a} = 0$. On the other hand, the winding number of $\delta = f \circ \tilde{\delta}$ around the point a is $\operatorname{Ind}_\delta(a) = j - i > 0$. By the argument principle,

$$0 = N_{f-a} = \frac{1}{2\pi i}\int_{\tilde{\delta}} \frac{f'}{f-a} = \operatorname{Ind}_\delta(a) = j - i > 0.$$

This contradiction proves the claim above.

Let N be the number of zeros of $f - w$ in $\overline{r_o\mathbb{D}}$. If $n > N$, then our claim above implies that $\tilde{\gamma}_n([0,1]) \not\subseteq r_o\mathbb{D}$. This shows that the restriction of f on $r_o\mathbb{D}$ is *not* a universal covering map for the domain $f(r_o\mathbb{D})$. The principle of the hyperbolic metric gives

$$\lambda_{f(r_o\mathbb{D})}(w)|f'(0)| < \lambda_{r_o\mathbb{D}}(0).$$

Since $|f'(0)| = \frac{1}{\lambda_{f(\mathbb{D})}(w)}$ and $\lambda_{r_o\mathbb{D}}(0) = \frac{1}{r_o}$, we have

$$\Phi_H(r_o) > \frac{1}{\lambda_{f(\mathbb{D})}(w)}. \tag{4.2}$$

If Φ_H were increasing, then for every $s > r_o$

$$\Phi_H(r_o) \leq \Phi_H(s) = \frac{1}{s\lambda_{f(s\mathbb{D})}(w)} \leq \frac{1}{s\lambda_{f(\mathbb{D})}(w)}$$

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and therefore
$$\Phi_H(r_o) \leq \lim_{s\to 1} \frac{1}{s\lambda_{f(\mathbb{D})}(w)} = \frac{1}{\lambda_{f(\mathbb{D})}(w)},$$
contradicting (4.2). ■

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