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A NOTE ON HARDY-ORLICZ SPACES

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ABSTRACT. W. Deeb, R. Khalil and M. Marzuq have studied some properties of $H(\phi)$, the Hardy-Orlicz spaces. They introduced the functions class N_p $(0 < p \leqq 1)$ and discussed some properties of N_p. In the present short note we prove that $N_p = N^+$ for $0 < p \leqq 1$. We also give a condition of $H(\phi) = H(\psi)$.

Introduction. Let us first recall some definitions. We call a real-valued function ϕ defined on $[0, \infty)$ a modulus function if ϕ is an increasing continuous subadditive function and satisfies the condition that $\phi(x) = 0$ iff $x = 0$. Let D denote the unit disc in the complex plane and $H(D)$ the class of analytic functions in D. For a given modulus function ϕ, the Hardy-Orlicz space $H(\phi)$ is defined as

$$H(\phi) = \left\{ f \in H(D) : \sup_{0 \leqq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \phi\left(|f(re^{i\theta})|\right) d\theta < \infty \right\}.$$

Let

$$H^+(D) = \left\{ f \in H(D) : \lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta}) \quad \text{a.e. on } \partial D \right\}$$

and

$$H(\phi)^+ = \left\{ f \in H^+(D) \cap H(\phi) : \sup_{0 \leqq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \phi\left(|f(re^{i\theta})|\right) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \phi\left(|f(e^{i\theta})|\right) d\theta \right\}.$$

See [1] and [2] for more details about $H(\phi)$ and $H(\phi)^+$.

For $0 < p \leqq 1$, $\phi_p(x) = \log(1 + x^p)$ is a modulus function. Let N_p denote the space $H(\phi_p)^+$. (See [1]). Since $\phi_p(e^t) = \log(1 + e^{pt})$ is an increasing convex function of t, the function $\phi_p\left(|f(z)|\right) = \phi_p(e^{\log|f(z)|})$ is subharmonic provided that $f(z)$ is an analytic function in D. Thus, from Theorem 1.6 in [3],

$$M\phi_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \phi_p\left(|f(re^{i\theta})|\right) d\theta$$

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is an increasing function of r, so we can rewrite N_p as following

$$N_p = \left\{ f \in H^+(D) \cap H(\phi_p) : \lim_{r \to 1} \int_0^{2\pi} \log\left(1 + |f(re^{i\theta})|^p\right) d\theta = \int_0^{2\pi} \log\left(1 + |f(e^{i\theta})|^p\right) d\theta \right\}.$$

Let N denote the Nevanlinna class as usual and N^+ its subclass

$$N^+ = \left\{ f \in N : \lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{it})| dt = \int_0^{2\pi} \log^+ |f(e^{it})| dt \right\}.$$

In [1] W. Deeb, R. Khalil and M. Marzuq studied some properties of N_p. In this short note we prove $N_p = N^+$ for all $0 < p \leqq 1$ and discuss the conditions for $H(\phi) = H(\psi)$.

We can now prove our main result.

Proposition 1. *For all* $0 < p \leqq 1$, $N_p = N^+$.

Proof. Fix p, $0 < p \leqq 1$. Suppose $f(z)$ is an analytic function in D. Let $h(z) = \log\left(1 + |f(z)|^p\right) - p \log^+ |f(z)|$ for $z \in D$. It is not hard to see that $0 \leqq h(z) \leqq \log 2$.

Now suppose $f \in N^+$, then the radial limit of $f(re^{it})$, and hence of $h(re^{it})$, exists almost everywhere. We have

$$\begin{aligned} h(e^{it}) &= \lim_{r \to 1} h(re^{it}) \\ &= \log\left(1 + |f(e^{it})|^p\right) - p \log^+ |f(e^{it})| \quad \text{a.e.} \end{aligned}$$

and

$$\lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{it})| dt = \int_0^{2\pi} \log^+ |f(e^{it})| dt < \infty.$$

By dominated convergence we then have

$$\begin{aligned} &\lim_{r \to 1} \int_0^{2\pi} \log\left(1 + |f(re^{it})|^p\right) dt \\ &\quad = \lim_{r \to 1} \int_0^{2\pi} \left(h(re^{it}) + p \log^+ |f(re^{it})|\right) dt \\ &\quad = \lim_{r \to 1} \int_0^{2\pi} h(re^{it}) dt + p \lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{it})| dt \\ &\quad = \int_0^{2\pi} h(e^{it}) dt + p \int_0^{2\pi} \log^+ |f(e^{it})| dt \\ &\quad = \int_0^{2\pi} \log\left(1 + |f(e^{it})|^p\right) dt, \end{aligned}$$

and from the above

$$\begin{aligned}\int_0^{2\pi} \log\left(1+|f(e^{it})|^p\right)dt \\ &= \int_0^{2\pi} h(e^{it})dt + p\int_0^{2\pi}\log^+|f(e^{it})|dt \\ &\leqq 4\pi + p\int_0^{2\pi}\log^+|f(e^{it})|dt \\ &< \infty.\end{aligned}$$

This shows $f \in N_p$.

Conversely, let $f \in N_p$. By similar reasoning we can prove $f \in N^+$. Hence $N_p = N^+$ for $0 < p \leqq 1$. □

We now want to know when it follows that $H(\phi)^+ = H(\psi)^+$. Although necessary and sufficient conditions on ϕ and ψ are not known, we have the following observation.

Suppose ϕ is a modulus function such that $\phi_c(t) = \phi(e^t)$ is a convex function of t. Since $\phi(x)$ is increasing, so is $\phi_c(t)$. If f is an analytic function then $\log|f(z)|$ is subharmonic, and we have that $\phi\left(|f(z)|\right) = \phi_c\left(\log|f(z)|\right)$ is a subharmonic function. Some examples of such modulus ϕ are: x^p $(0 < p \leqq 1)$, $\log(1+x^p)$ $(0 < p \leqq 1)$, and $x/(\log)_n(e_n + x)$, where $e_1 = e$, $e_n = e^{e_{n-1}}$, and

$$(\log)_n(e_n + x) = \overbrace{\log\ldots\log}^{n\text{th}}(e_n + x), \quad n = 1, 2, 3, \ldots$$

Lemma 2. *Let $f \in N^+$, then*

$$\phi\left(|f(re^{i\theta})|\right) \leqq \frac{1}{2\pi}\int_0^{2\pi} p(r, \theta - t)\phi\left(|f(e^{it})|\right)dt.$$

Proof. From Theorem 5.4 in [4, p. 71], we have

$$\log|f(re^{i\theta})| \leqq \frac{1}{2\pi}\int_0^{2\pi} p(r, \theta - t)\log|f(e^{it})|dt.$$

Observing that $\phi_c(t)$ is increasing, we obtain by Jensen's inequality

$$\begin{aligned}\phi\left(|f(re^{i\theta})|\right) &= \phi_c\left(\log|f(re^{i\theta})|\right) \\ &\leqq \phi_c\left(\frac{1}{2\pi}\int_0^{2\pi} p(r, \theta - t)\log|f(e^{it})|dt\right) \\ &\leqq \frac{1}{2\pi}\int_0^{2\pi} p(r, \theta - t)\phi_c\left(\log|f(e^{it})|\right)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} p(r, \theta - t)\phi\left(|f(e^{it})|\right)dt.\end{aligned}$$

□

PROPOSITION 3. *Let $f \in H(\phi) \cap N^+$, then*

$$\lim_{r\to 1} \int_0^{2\pi} \phi\big(|f(re^{i\theta})|\big)d\theta = \int_0^{2\pi} \phi\big(|f(e^{i\theta})|\big)d\theta.$$

PROOF. By Lemma 2,

$$\phi\big(|f(re^{i\theta})|\big) \leqq \frac{1}{2\pi} \int_0^{2\pi} p(r, \theta - t)\phi\big(|f(e^{it})|\big)dt,$$

hence

$$\begin{aligned}\int_0^{2\pi} &\phi\big(|f(re^{i\theta})|\big)d\theta \\ &\leqq \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} p(r, \theta - t)d\theta\phi\big(|f(e^{it})|\big)dt \\ &= \int_0^{2\pi} \phi\big(|f(e^{it})|\big)dt.\end{aligned}$$

On the other hand, by the Fatou's lemma,

$$\int_0^{2\pi} \phi\big(|f(e^{i\theta})|\big)d\theta \leqq \varliminf_{r\to 1} \int_0^{2\pi} \phi\big(|f(re^{i\theta})|\big)d\theta.$$

Thus we have

$$\lim_{r\to 1} \int_0^{2\pi} \phi\big(|f(re^{i\theta})|\big)d\theta = \int_0^{2\pi} \phi\big(|f(e^{i\theta})|\big)d\theta. \qquad \square$$

REMARK 4. From Proposition 3, we know that $H(\phi) \cap N^+ \subset H(\phi)^+$. We believe that $H(\phi) \cap N^+ = H(\phi)^+$, so that $H(\phi)^+ = H(\psi)^+$ if and only if $H(\phi) = H(\psi)$.

We have the following

PROPOSITION 5. *Let ϕ and ψ be modulus functions. If*

$$\varlimsup_{x\to\infty} \frac{\phi(x)}{\psi(x)} < \infty,$$

then $H(\psi) \subset H(\phi)$.

PROOF. Suppose

$$\varlimsup_{x\to\infty} \frac{\phi(x)}{\psi(x)} < \infty,$$

then there are $x_0 > 0$ and $0 < M < \infty$ such that $\phi(x) < M\psi(x)$ for $x > x_0$. Let $f \in H(\psi)$, then

$$\begin{aligned}\frac{1}{2\pi} &\int_0^{2\pi} \phi\big(|f(re^{it})|\big)dt \\ &= \frac{1}{2\pi} \int_{\{\theta:|f|\leqq x_0\}} \phi\big(|f(re^{it})|\big)dt + \frac{1}{2\pi} \int_{\{\theta:|f|>x_0\}} \phi\big(|f(e^{it})|\big)dt \\ &\leqq \phi(x_0) + \frac{1}{2\pi} \int_{\{\theta:|f|>x_0\}} M\psi\big(|f(re^{it})|\big)dt \\ &\leqq \phi(x_0) + M\frac{1}{2\pi} \int_0^{2\pi} \psi\big(|f(re^{it})|\big)dt < M_0.\end{aligned}$$

where M_0 is a finite positive constant independent of r. Therefore $f \in H(\phi)$. This shows that $H(\psi) \subset H(\phi)$. □

REMARK 6. From Proposition 5, we know that $H(\phi) = H(\psi)$ if

$$0 < \varliminf_{x\to\infty} \frac{\phi(x)}{\psi(x)} \leqq \varlimsup_{x\to\infty} \frac{\phi(x)}{\psi(x)} < \infty.$$

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