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# SEMISIMPLICITY OF HECKE AND (WALLED) BRAUER ALGEBRAS

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**Abstract**

We show how to use Jantzen's sum formula for Weyl modules to prove semisimplicity criteria for endomorphism algebras of $\mathbf{U}_q$-tilting modules (for any field $\mathbb{K}$ and any parameter $q \in \mathbb{K} - \{0, -1\}$). As an application, we recover the semisimplicity criteria for the Hecke algebras of types **A** and **B**, the walled Brauer algebras and the Brauer algebras from our more general approach.

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## 1. Introduction

Fix a reductive Lie algebra $\mathfrak{g}$, a field $\mathbb{K}$ and any $q \in \mathbb{K}^*$, where $\mathbb{K}^* = \mathbb{K} - \{0, -1\}$ if $\mathrm{char}(\mathbb{K}) > 2$ and $\mathbb{K}^* = \mathbb{K} - \{0\}$ otherwise. Let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ be the $q$-deformed enveloping algebra of $\mathfrak{g}$ over $\mathbb{K}$ and let $T$ be a $\mathbf{U}_q$-tilting module.

In this paper we give a semisimplicity criterion for the algebra $\mathrm{End}_{\mathbf{U}_q}(T)$, which relies only on the combinatorics of the root and weight data associated to $\mathfrak{g}$. The crucial observation we use here is that $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple if and only if all Weyl factors of $T$ are simple $\mathbf{U}_q$-modules—a property which can be checked using (versions of) Jantzen's sum formula.

We apply our methods to four explicit examples: the Hecke algebras of types **A** and **B**, the walled Brauer algebras and the Brauer algebras. For all of these we obtain full semisimplicity criteria by using the corresponding combinatorics of roots

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and weights. In all of these cases the semisimplicity criteria were obtained before, but using specific properties of the algebras in question; see Remarks 5.3, 6.10 and 7.14. However, our approach has the advantage that it provides a quite general method to deduce semisimplicity criteria. The necessary calculations to prove these are always the same (*mutatis mutandis*, depending on the associated root and weight data). Hence, our approach unites the known semisimplicity criteria of these algebras in our more general framework.

**1.1. The setup.** The category $\mathbf{U}_q$-**Mod** of finite-dimensional representations of $\mathbf{U}_q$ (of type 1) provides an interesting example of a tensor category. The structure of $\mathbf{U}_q$-**Mod** heavily depends on the field $\mathbb{K}$ and on $q \in \mathbb{K}^*$. If $\mathrm{char}(\mathbb{K}) = 0$ and $q = 1$, then we are in the *classical case*, where $\mathbf{U}_q$-**Mod** behaves like the category $\mathfrak{g}$-**Mod** of complex, finite-dimensional representations of $\mathfrak{g}$ and, hence, is in particular semisimple. But $\mathbf{U}_q$-**Mod** is nonsemisimple in case $\mathrm{char}(\mathbb{K}) > 0$ and $q = 1$, or in case $\mathrm{char}(\mathbb{K}) \geq 0$ and $q \in \mathbb{K}^*, q \neq 1$ is a root of unity.

In this paper we consider an arbitrary field $\mathbb{K}$ and arbitrary $q \in \mathbb{K}^*$ and study particular pieces of the category $\mathbf{U}_q$-**Mod** in more detail. To be more specific, we show how *Jantzen's sum formula* can be used to deduce the semisimplicity of modules in $\mathbf{U}_q$-**Mod**.

As an application, we provide semisimplicity criteria for well-known algebras $\mathcal{A}$ arising in invariant theory, namely for *Hecke algebras* $\mathcal{H}_d^{\mathbf{A}}(q)$ and $\mathcal{H}_d^{\mathbf{B}}(q)$ *of types* **A** *and* **B** (see Theorem 5.1), for the *walled Brauer algebra* $\mathcal{B}_{r,s}(\delta)$ (see Theorem 6.1) and for the *Brauer algebra* $\mathcal{B}_d(\delta)$ (see Theorem 7.1). These examples are however just the tip of an iceberg: our approach should work to provide semisimplicity criteria for a big class of algebras (see also Remark 1.1). But in this paper we restrict to these examples, and we obtain explicit necessary and sufficient conditions for the semisimplicity of these algebras $\mathcal{A}$ (over any field $\mathbb{K}$ and any $q \in \mathbb{K}^*$). For instance, when $\mathcal{A} = \mathcal{H}_d^{\mathbf{A}}(q)$ is the Hecke algebra of the symmetric group $S_d$ in $d$ letters, we get the following result.

Theorem (Semisimplicity criterion for the Hecke algebra of type **A**). *$\mathcal{H}_d^{\mathbf{A}}(q)$ is semisimple if and only if one of the following conditions holds:*

(1) $\mathrm{char}(\mathbb{K}) > d$ *and* $q = 1$;
(2) $\mathrm{char}(\mathbb{K}) = 0$ *and* $q = 1$;
(3) $q \in \mathbb{K}^*, q \neq 1$ *is a root of unity with* $\mathrm{ord}(q^2) > d$;
(4) $q \in \mathbb{K}^*, q \neq 1$ *is a nonroot of unity.*

The Hecke algebra of type **A** and its semisimplicity criterion stated above is a particular nice example of our general approach, since the corresponding combinatorics is very easy in this case.

To explain our methods in more detail, we consider the full, additive tensor subcategory $\mathcal{T}$ of $\mathbf{U}_q$-**Mod** given by all $\mathbf{U}_q$*-tilting modules* (a notion that we recall in Section 2). For any $\mathbf{U}_q$-tilting module $T \in \mathcal{T}$, we have, as observed in [5, Theorems 4.11 and 5.13], that

$$\mathrm{End}_{\mathbf{U}_q}(T) \text{ is semisimple if and only if } T \text{ is a semisimple } \mathbf{U}_q\text{-module.} \tag{1.1}$$

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Moreover, $T$ is a semisimple $\mathbf{U}_q$-module if and only if all Weyl modules $\Delta_q(\lambda)$ appearing in the Weyl filtration of $T$ are simple $\mathbf{U}_q$-modules; see Lemma 2.4. The important step here is now to use (a version of) Jantzen's sum formula for the Weyl modules $\Delta_q(\lambda)$, see Theorem 2.9, to translate the semisimplicity problem into a purely algorithmic problem in terms of roots, weights and the combinatorics of the (affine) Weyl group $W$:

$$\text{a Weyl module } \Delta_q(\lambda) \text{ is simple if and only if} \\ \text{Jantzen's sum formula of } \Delta_q(\lambda) \text{ vanishes.} \tag{1.2}$$

To state some explicit consequences, let us restrict ourselves to Lie algebras $\mathfrak{g}$ of type $\mathbf{A}_{m-1}, \mathbf{B}_m, \mathbf{C}_m$ or $\mathbf{D}_m$. We then have the quantum analogue $V \in \mathcal{T}$ of the *vector representation* of $\mathfrak{g}$ and its dual $V^* \in \mathcal{T}$ (which are isomorphic in types $\mathbf{B}_m, \mathbf{C}_m$ and $\mathbf{D}_m$).

Let $n = \dim(V)$ and take the $\mathbf{U}_q$-module $T_n^{r,s} = V^{\otimes r} \otimes (V^*)^{\otimes s}$. Since $V \in \mathcal{T}$ and, hence, $T_n^{r,s}$ is a tensor product of $\mathbf{U}_q$-tilting modules (except if $\mathrm{char}(\mathbb{K}) = 2$ in type $\mathbf{B}_m$), it is itself a $\mathbf{U}_q$-tilting module; see Proposition 2.3. Thus, (1.1) and (1.2) apply.

By (generalized versions of) *Schur–Weyl duality*, see Section 3, the above-mentioned algebras $\mathcal{A}$ arise, for suitable choices of $\mathfrak{g}, n, r, s$, as endomorphism algebras of the form $\mathrm{End}_{\mathbf{U}_q}(T_n^{r,s})$. Hence, our method implies directly explicit semisimplicity criteria as long as $\mathcal{A} \cong \mathrm{End}_{\mathbf{U}_q}(T_n^{r,s})$.

It remains to deal with the cases where the algebras $\mathcal{A}$ do not appear as such endomorphism algebras. This could happen because of the following reasons.

- The natural map from $\mathcal{A}$ to $\mathrm{End}_{\mathbf{U}_q}(T_n^{r,s})$ is not injective (this happens in case $r + s$ is large compared to $n$) or not surjective (this happens in case $\mathfrak{g} = \mathfrak{so}_{2m}$).
- The algebra $\mathcal{A}$ does not appear as an algebra of the form $\mathrm{End}_{\mathbf{U}_q}(T_n^{r,s})$ at all (this happens for $\mathcal{B}_d(\delta)$ in case $\mathrm{char}(\mathbb{K}) = 0$ and $\delta \in \mathbb{Z}_{<0}$ is odd).

We note that, by quantum Schur–Weyl duality as in Theorems 3.4 and 3.6, Hecke algebras of type $\mathbf{A}$ or $\mathbf{B}$ can always be obtained as endomorphism algebras of some $\mathbf{U}_q$-tilting module.

To deal with these cases for the (walled) Brauer algebras, we first observe that passing to a field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = p > 2$ has several advantages (our approach is in fact easier in positive characteristic). First of all, the (walled) Brauer algebra for parameter $\delta$ equals the (walled) Brauer algebra for parameter $\delta \pm ap$ (for any $a \in \mathbb{Z}$), which allows us to pass from even values of $\delta$ to odd values of $\delta$. Second, since under the corresponding Schur–Weyl duality $n$ depends on $\delta$, we can avoid that $r + s$ is large compared to $n$ by adding $p$ to $n$ often enough. Using both observations we can always achieve $\mathcal{A} \cong \mathrm{End}_{\mathbf{U}_q}(T_n^{r,s})$. However, adding $p$ makes Jantzen's sum formula more involved. We therefore prefer to argue differently: in some 'boundary cases' we can determine the kernel of the action of $\mathcal{A}$ on $\mathrm{End}_{\mathbf{U}_q}(T_n^{r,s})$ explicitly, see Section 4, and deduce in this way the (non)semisimplicity of $\mathcal{A}$ from the (non)semisimplicity of $\mathrm{End}_{\mathbf{U}_q}(T_n^{r,s})$.

Finally, it remains to treat the case $\mathrm{char}(\mathbb{K}) = 0$. We observe that the algebra $\mathcal{A}$ in question is semisimple if and only if it is semisimple over fields of large enough characteristic. One way to pass at least to the complex numbers is to use the theory of *ultraproducts*, see, for example, [51, Ch. 2], and realize the complex numbers as an *ultralimit* of fields of positive characteristics. Since the semisimplicity can be described by an integral polynomial expression (namely a determinant), the algebra $\mathcal{A}$ is semisimple over the complex numbers if and only if it is semisimple over fields of large enough characteristics. Instead of the (way more powerful) theory of ultraproducts, we use the probably more common tool of *trace forms* to pass from positive characteristic to characteristic zero; see Appendix A. Note that both arguments rely on the fact that our algebras $\mathcal{A}$ in question can be defined over $\mathbb{Z}$.

REMARK 1.1. Our methods to deduce semisimplicity criteria work more generally and not just for the category $\mathbf{U}_q$-**Mod**. Our arguments in [5] (which are the basis of the criterion from (1.1)) do depend on the existence of weight spaces such that [5, Lemma 4.5] makes sense, and the semisimplicity criterion itself relies on the existence of Jantzen's sum formula, which also involves weight computations and is not available in general. But as long as these notions are available, our method works. As an explicit generalization, we could for instance work with category $\boldsymbol{O}$, its tilting theory (see, for example, [26, Ch. 11]) and the corresponding Jantzen's sum formulas (see, for example, [26, Ch. 5, Section 3]). For brevity, we stay with $\mathbf{U}_q$-**Mod** in this paper.

### 1.2. Outline of the paper.

The paper is organized as follows.

- In Section 2 we recall some facts about $\mathbf{U}_q$-tilting modules. Moreover, we recall the two main ingredients for our proofs of semisimplicity.
  - The semisimplicity criterion of endomorphism algebras of $\mathbf{U}_q$-tilting modules.
  - Jantzen's sum formula, which provides a method to check whether a given Weyl module $\Delta_q(\lambda)$ is a simple $\mathbf{U}_q$-module.
- In Section 3 we list, for the convenience of the reader, some Schur–Weyl-like dualities, which we need in a rather complete form. In Section 4 we additionally describe in some 'boundary cases' the kernels of the homomorphisms appearing in the Schur–Weyl-like dualities. We need the explicit description in some of these cases for our proof, but the explicit descriptions are interesting in their own right.
- In Sections 5–7 we give the semisimplicity criteria for the Hecke algebras of types **A** and **B**, the walled Brauer algebras and the Brauer algebras.
- In Appendix A we describe in detail some tools to compare semisimplicity in characteristic $p$ and in characteristic zero. Moreover, in Appendix B we recall the root and weight data in types $\mathbf{A}_{m-1}, \mathbf{B}_m, \mathbf{C}_m$ and $\mathbf{D}_m$ that we use in this paper.

CONVENTIONS 1.2. Throughout, we denote by $\mathbb{K}$ an arbitrary field, by $q$ any element in $\mathbb{K}^*$ and by $p \in \mathbb{Z}_{>0}$ a prime number (usually $p = \mathrm{char}(\mathbb{K})$). We call the case of

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$\mathrm{char}(\mathbb{K}) = 0$ and $q = 1$ the *classical case*. We exclude the quasi-classical case $q = -1$ for technical reasons in case $\mathrm{char}(\mathbb{K}) > 2$ (the notion quasi-classical was coined in [36, Section 33.2], where Lusztig also proved that, if $\mathrm{char}(\mathbb{K}) = 0$, then the $q = -1$ case is equivalent to the $q = 1$ case).

Let $\mathrm{ord}(q^2) = \ell$ with $\ell \in \mathbb{Z}_{\geq 0}$ be the order of $q^2$, that is, the smallest integer $\ell \in \mathbb{Z}_{\geq 0}$ such that $q^{2\ell} = 1$ (or $\ell = 0$ if no such number exists). In case $q \neq 1$ and $\ell \neq 0$, we say that $q$ is a *root of unity*. If $\ell = 0$, then we call $q$ a *nonroot of unity*.

By an algebra $\mathcal{A}$ we always mean a unital, associative algebra over $\mathbb{Z}$ or $\mathbb{K}$. All modules are finite-dimensional, left $\mathcal{A}$-modules throughout the paper. As usual in the case $\mathcal{A} = \mathbf{U}_q$, we consider only $\mathbf{U}_q$-modules of *type* 1 (see [29, Ch. 5, Section 2]).

## 2. $\mathbf{U}_q$-tilting modules and semisimplicity

We start by briefly recalling some notions from the theory of $\mathbf{U}_q(\mathfrak{g})$-tilting modules. The reader unfamiliar with these is referred to [1, 4, 5, 30] or [50] (and the references therein).

Here we denote by $\mathbf{U}_q(\mathfrak{g})$ the *quantized enveloping algebra* specialized at $q \in \mathbb{K}^*$ for a reductive Lie algebra $\mathfrak{g}$ with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$ attached to a choice of *positive roots* $\Phi^+ \subset \Phi$ inside all roots $\Phi$. Let $\Pi \subset \Phi^+$ be the set of *simple roots*, $X$ the *integral weight lattice* and $X^+$ the set of *dominant integral weights*.

For the main calculations in the paper it is enough to restrict ourselves to the classical Lie algebras $\mathfrak{g} = \mathfrak{gl}_m$, $\mathfrak{g} = \mathfrak{so}_{2m+1}$, $\mathfrak{g} = \mathfrak{sp}_{2m}$ or $\mathfrak{g} = \mathfrak{so}_{2m}$ for some fixed $m \in \mathbb{Z}_{>0}$. We usually let $n$ denote the dimension of the corresponding (quantized) vector representation $V = \Delta_q(\omega_1)$ (that is $n = m$ for $\mathfrak{gl}_m$, $n = 2m + 1$ for $\mathfrak{g} = \mathfrak{so}_{2m+1}$ and $n = 2m$ for $\mathfrak{g} = \mathfrak{sp}_{2m}$ (respectively $\mathfrak{g} = \mathfrak{so}_{2m}$)). For convenience, we have listed in Appendix B the necessary explicit root and weight data for our purpose in the Dynkin types $\mathbf{A}_{m-1}, \mathbf{B}_m, \mathbf{C}_m$ and $\mathbf{D}_m$ (together with some standard notations that we use throughout). We study the category $\mathbf{U}_q$-**Mod** of finite-dimensional representations of $\mathbf{U}_q$ (of type 1) in what follows.

Remark 2.1. In the 'generic' cases (for example, $q = \pm 1, \mathrm{char}(\mathbb{K}) = 0$), $\mathbf{U}_q$-**Mod** is semisimple and behaves combinatorially as $\mathfrak{g}$-**Mod** for the corresponding Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. (For nonroots of unity or $q = \pm 1, \mathrm{char}(\mathbb{K}) = 0$, see [3, Theorem 9.4] and [36, Section 33.2] for $q = -1$.)

The algebra $\mathbf{U}_q$ has a triangular decomposition $\mathbf{U}_q = \mathbf{U}_q^+ \mathbf{U}_q^0 \mathbf{U}_q^-$. This gives for each $\lambda \in X^+$ a *Weyl* $\mathbf{U}_q$*-module* $\Delta_q(\lambda)$ and a *dual Weyl* $\mathbf{U}_q$*-module* $\nabla_q(\lambda)$. The $\mathbf{U}_q$-module $\Delta_q(\lambda)$ has a unique simple head $L_q(\lambda)$ which is the unique simple socle of $\nabla_q(\lambda)$. Let $\mathrm{ch}(M)$ denote the *(formal) character of* $M \in \mathbf{U}_q$-**Mod**, that is,

$$\mathrm{ch}(M) = \sum_{\lambda \in X} (\dim(M_\lambda)) e^\lambda \in \mathbb{Z}[X],$$

where $M_\lambda = \{m \in M \mid um = \lambda(u)m, u \in \mathbf{U}_q^0\}$ is the $\lambda$-weight space of $M$ (here we regard $\lambda$ as a character of $\mathbf{U}_q^0$) and $\mathbb{Z}[X]$ is the group algebra of the additive group $X$.

The following proposition is crucial in the nonsemisimple cases.

Proposition 2.2. *The characters* $\mathrm{ch}(\Delta_q(\lambda))$ *and* $\mathrm{ch}(\nabla_q(\lambda))$ *are independent of* $\mathrm{char}(\mathbb{K})$ *and of* $q \in \mathbb{K}^*$. *In particular, they are given as in the classical case.*

Proof. The statement follows directly from the definitions and the $q$-version of Kempf's vanishing theorem, which can be found in [48, Theorem 5.5]. □

We say that $M \in \mathbf{U}_q$-**Mod** has a $\Delta_q$*-filtration* if there exist $i \in \mathbb{Z}_{\geq 0}$ and a descending sequence

$$M = M_0 \supset M_1 \supset \cdots \supset M_{i'} \supset \cdots \supset M_{i-1} \supset M_i = 0,$$

such that for all $i' = 0, \ldots, i-1$ we have $M_{i'} \in \mathbf{U}_q$-**Mod**, $M_{i'}/M_{i'+1} \cong \Delta_q(\lambda_{i'})$ with $\lambda_{i'} \in X^+$.

A $\nabla_q$*-filtration* is defined similarly, but using $\nabla_q(\lambda)$ instead of $\Delta_q(\lambda)$ and an ascending sequence of $\mathbf{U}_q$-submodules, that is,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{i'} \subset \cdots \subset M_{i-1} \subset M_i = M,$$

such that for all $i' = 0, \ldots, i-1$ we have $M_{i'+1}/M_{i'} \cong \nabla_q(\lambda_{i'})$ with $\lambda_{i'} \in X^+$.

A $\mathbf{U}_q$*-tilting module* is a $\mathbf{U}_q$-module $T \in \mathbf{U}_q$-**Mod** which has both a $\Delta_q$- and a $\nabla_q$-filtration. These filtrations are unique up to reordering of factors (this can be verified using standard arguments; see, for example, [14, Proposition A2.2] or [30, Section 4.16, Remark (4)]) and we henceforth call the appearing factors *Weyl* or *dual Weyl factors* of $T$, respectively.

The *category* $\mathcal{T}$ *of* $\mathbf{U}_q$*-tilting modules* is the full subcategory $\mathcal{T} \subset \mathbf{U}_q$-**Mod** with objects consisting of all $\mathbf{U}_q$-tilting modules. The category $\mathcal{T}$ is an additive Krull–Schmidt category, closed under direct sums, duality and finite tensor products. The latter is in general nontrivial to prove. Apart from type $\mathbf{B}_m$, the following proposition has an elementary proof.

Proposition 2.3. *Let* $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ *with* $\mathfrak{g}$ *of type* $\mathbf{A}_{m-1}$, $\mathbf{B}_m$, $\mathbf{C}_m$ *or* $\mathbf{D}_m$. *Then the vector representation* $V$ *of* $\mathbf{U}_q$ *is a* $\mathbf{U}_q$*-tilting module. Moreover,* $T_n^d = V^{\otimes d} \in \mathcal{T}$ *is a* $\mathbf{U}_q$*-tilting module for all* $d \in \mathbb{Z}_{\geq 0}$ *as well. The dimension* $\dim(\mathrm{End}_{\mathbf{U}_q}(T_n^d))$ *depends only on* $\mathfrak{g}$ *and* $d$.

(Here we need that $\mathrm{char}(\mathbb{K}) \neq 2$ in type $\mathbf{B}_m$ and we assume this throughout if we work in this type.)

Proof. For the types $\mathbf{A}_{m-1}$, $\mathbf{C}_m$ and $\mathbf{D}_m$, see [4, Proposition 3.10] for an elementary proof. In type $\mathbf{B}_m$ it was observed in [27, page 20] that $V$ is a $\mathbf{U}_q$-tilting module as long as $\mathrm{char}(\mathbb{K}) \neq 2$. By [43, Theorem 3.3], it follows that $V^{\otimes d} \in \mathcal{T}$. To see that $\dim(\mathrm{End}_{\mathbf{U}_q}(T_n^d))$ depends only on $\mathfrak{g}$ and $d$, first note that $\dim(\mathrm{End}_{\mathbf{U}_q}(T_n^d)) = \sum_{\lambda \in X^+}(T : \Delta_q(\lambda))^2$ (which can be derived from the Ext-vanishing; see, for example, [5, Theorem 3.1]). Now use the fact that $\mathrm{ch}(\Delta_q(\lambda))$ is as in the classical case, which implies the statement. □

Lemma 2.4. *A* $\mathbf{U}_q$*-tilting module* $T \in \mathcal{T}$ *is a semisimple* $\mathbf{U}_q$*-module if and only if all Weyl factors* $\Delta_q(\lambda)$ *of* $T$ *are simple* $\mathbf{U}_q$*-modules if and only if all dual Weyl factors* $\nabla_q(\lambda)$ *of* $T$ *are simple* $\mathbf{U}_q$*-modules.*

PROOF. The second equivalence is evident. If $T$ is semisimple, then clearly all Weyl factors $\Delta_q(\lambda)$ of $T$ are simple $\mathbf{U}_q$-modules. If all Weyl factors are simple, hence $\Delta_q(\lambda) \cong \nabla_q(\lambda)$, then the statement follows by using Ext-vanishing (see for example [5, Theorem 3.1]) and induction on the length of a $\Delta_q$-filtration of $T$. □

THEOREM 2.5 (Semisimplicity criterion for $\mathrm{End}_{\mathbf{U}_q}(T)$). *Let $T \in \mathcal{T}$ be a $\boldsymbol{U}_q$-tilting module. Then the algebra $\mathrm{End}_{\boldsymbol{U}_q}(T)$ is semisimple if and only if $T$ is a semisimple $\boldsymbol{U}_q$-module.*

PROOF. This is a consequence of [5, Theorems 4.11 and 5.13]. □

Thus, by Lemma 2.4 and Theorem 2.5, the question whether $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple is equivalent to the question whether all (dual) Weyl factors of $T$ are simple $\mathbf{U}_q$-modules.

COROLLARY 2.6. *The algebra $\mathrm{End}_{\boldsymbol{U}_q}(T)$ is semisimple if and only if all Weyl factors $\Delta_q(\lambda)$ of $T$ are simple $\boldsymbol{U}_q$-modules if and only if all dual Weyl factors $\nabla_q(\lambda)$ of $T$ are simple $\boldsymbol{U}_q$-modules.*

A method to check if a given Weyl module is a simple $\mathbf{U}_q$-module is provided by *Jantzen's sum formula*. In order to state it, we need some preparations. First, for any $a \in \mathbb{Z}_{\geq 0}$ and any $p$, we denote by $\nu_p(a)$ its *p-adic valuation*, that is, the largest nonnegative integer such that $p^{\nu_p(a)}$ divides $a$. Second, let $W$ be the *Weyl group* associated to $\mathfrak{g}$ (recall that $W$ is generated by the simple reflections $s_i = s_{\alpha_i}$ for each simple root $\alpha_i \in \Pi$) and let $l(w)$ denote the length of an element $w \in W$. The Weyl group $W$ acts on $X$ in two ways:

$$s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i \quad \text{for } \lambda \in X \quad \text{respectively} \quad s_i.\lambda = s_i(\lambda + \rho) - \rho \quad \text{for } \lambda \in X.$$

Here we use the notation $\rho$ as in Appendix B.

DEFINITION 2.7. Let $\lambda \in X^+, \mu \in X$ and assume that $w.\lambda = \mu$ for some $w \in W$. Then we set

$$\chi(\lambda) = \mathrm{ch}(\Delta_q(\lambda)) \quad \text{and} \quad \chi(\mu) = (-1)^{l(w)} \chi(\lambda). \tag{2.1}$$

In particular, $\chi(\lambda) = 0$ for all *dot-singular* $\mathbf{U}_q$-weights $\lambda \in X$ (a $\mathbf{U}_q$-weight $\lambda$ is dot-singular if there exists $\alpha \in \Phi$ such that $\langle \lambda + \rho, \alpha^\vee \rangle = 0$). On the other hand, any *dot-regular* (that is, non-dot-singular) $\mu \in X$ is of the form $w.\lambda$ for some unique $\lambda \in X^+$, which makes the assignments well defined.

CONVENTIONS 2.8. What we call dot-singular is often called singular in the literature. In contrast, we call a $\mathbf{U}_q$-weight $\lambda \in X^+$ *singular* if there exists $\alpha \in \Phi$ with $\langle \lambda, \alpha^\vee \rangle = 0$. Similarly for regular $\mathbf{U}_q$-weights. We recall for the root systems of classical type equivalent criteria for being (dot-)singular (which we use for calculations) in Appendix B.

Formulas (2.2)–(2.4) are called *Jantzen's sum formulas* because they originate in Jantzen's work [28]. We tend to write *JSF* to abbreviate 'Jantzen's sum formula'.

THEOREM 2.9 (Jantzen's sum formula). *Let $\lambda \in X^+$. Then $\Delta_q(\lambda)$ has a filtration*

$$\Delta_q(\lambda) = \Delta_q^0(\lambda) \supset \Delta_q^1(\lambda) \supset \Delta_q^2(\lambda) \supset \ldots,$$

*called Jantzen filtration, such that all $\Delta_q^{k'}(\lambda) \in \mathbf{U}_q$-**Mod**, $\Delta_q(\lambda)/\Delta_q^1(\lambda) \cong L_q(\lambda)$ and:*

- *if* $\mathrm{char}(\mathbb{K}) = 0$ *and* $q = 1$ *or* $q \in \mathbb{K}^*$ *is a nonroot of unity, then* $\Delta_q^1(\lambda) = 0$*;*
- *if* $\mathrm{char}(\mathbb{K}) = 0$ *and* $q \in \mathbb{K}^*$ *is a root of unity with* $\mathrm{ord}(q^2) = \ell$*, then*

$$\sum_{k'>0} \mathrm{ch}(\Delta_q^{k'}(\lambda)) = -\sum_{\alpha\in\Phi^+} \sum_{\substack{0<k\ell \\ <\langle\lambda+\rho,\alpha^\vee\rangle}} \chi(\lambda - k\ell\alpha); \tag{2.2}$$

- *if* $\mathrm{char}(\mathbb{K}) = p > 0$ *and* $q \in \mathbb{K}^*$ *is a root of unity with* $\mathrm{ord}(q^2) = \ell$*, then*

$$\sum_{k'>0} \mathrm{ch}(\Delta_q^{k'}(\lambda)) = -\sum_{\alpha\in\Phi^+} \sum_{\substack{0<k\ell \\ <\langle\lambda+\rho,\alpha^\vee\rangle}} p^{\nu_p(k)}\chi(\lambda - k\ell\alpha); \tag{2.3}$$

- *if* $\mathrm{char}(\mathbb{K}) = p > 0$ *and* $q = 1$*, then*

$$\sum_{k'>0} \mathrm{ch}(\Delta_1^{k'}(\lambda)) = -\sum_{\alpha\in\Phi^+} \sum_{\substack{0<kp \\ <\langle\lambda+\rho,\alpha^\vee\rangle}} \nu_p(kp)\chi(\lambda - kp\alpha). \tag{2.4}$$

*(The sums on the right-hand sides run over all $k \in \mathbb{Z}_{\geq 0}$ such that the indicated inequalities hold.) In particular, $\Delta_q(\lambda)$ is a simple $\mathbf{U}_q$-module if and only if the corresponding JSF is zero.*

PROOF. See [2, Theorem 6.3], [30, Proposition II.8.19] and [52, Theorem 5.1]. □

We first show in an example how Theorem 2.9 together with Corollary 2.6 can be used in practice to determine whether $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple.

EXAMPLE 2.10. Consider $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{gl}_5)$ over $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 5$. Let $n = m = 5$, $d = 3$ and $V = \Delta_1(\omega_1) \cong \Delta_1(\varepsilon_1)$ be the vector representation of $\mathbf{U}_1$ and set $T_n^d = V^{\otimes d}$. We want to check whether $\mathrm{End}_{\mathbf{U}_1}(T_n^d)$ is a semisimple algebra. Since $V \in \mathcal{T}$, so is $T_n^d$ by Proposition 2.3. By Corollary 2.6, it remains to check whether $T_n^d$ has only Weyl factors which are simple $\mathbf{U}_1$-modules. We see (using Proposition 2.2) that the Weyl factors of $T_n^d$ have highest weights

$$\lambda = 3\varepsilon_1 = (3,0,0,0,0), \quad \mu = 2\varepsilon_1 + \varepsilon_2 = (2,1,0,0,0),$$
$$\nu = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = (1,1,1,0,0).$$

In order to see whether these Weyl factors are simple $\mathbf{U}_1$-modules, we use *JSF* from (2.4). We have $\rho = (4,3,2,1,0)$ (see Appendix B) and thus (for all $\alpha \in \Phi^+$)

$$\langle\lambda+\rho,\alpha^\vee\rangle \leq \langle\lambda+\rho,(\varepsilon_1-\varepsilon_5)^\vee\rangle = 7, \quad \langle\mu+\rho,\alpha^\vee\rangle \leq \langle\lambda+\rho,(\varepsilon_1-\varepsilon_5)^\vee\rangle = 6,$$
$$\langle\nu+\rho,\alpha^\vee\rangle \leq \langle\lambda+\rho,(\varepsilon_1-\varepsilon_5)^\vee\rangle = 5 = \mathrm{char}(\mathbb{K}).$$

*JSF* for $\nu$ is zero: it totally collapses since the second sum on the right-hand side of (2.4) is empty. Hence, $\Delta_1(\nu)$ is a simple $\mathbf{U}_1$-module. For $\mu$ we see that the only possible contribution to *JSF* comes from the positive root $\alpha \in \Phi^+$ of the form $\alpha = \varepsilon_1 - \varepsilon_5$. But

$$\mu + \rho - 5(\varepsilon_1 - \varepsilon_5) = (\boxed{1}, 4, 2, \boxed{1}, 5).$$

Hence, $\mu + \rho - 5(\varepsilon_1 - \varepsilon_5)$ is a singular $\mathbf{U}_1$-weight. (We illustrate with boxes with black numbers the entries which make a $\mathbf{U}_q$-weight singular. Moreover, we illustrate with boxes with white numbers, the numbers relevant for the calculation in the regular cases.) Thus, $\chi(\mu - 5(\varepsilon_1 - \varepsilon_5)) = 0$ and so *JSF* is zero, which again implies that $\Delta_1(\mu)$ is a simple $\mathbf{U}_1$-module.

For $\lambda$ the only possible contributions can come from the positive roots $\alpha \in \Phi^+$ of the form $\alpha = \varepsilon_1 - \varepsilon_5$ or $\alpha = \varepsilon_1 - \varepsilon_4$. We calculate

$$\lambda + \rho - 5(\varepsilon_1 - \varepsilon_5) = (\boxed{2}, 3, \boxed{2}, 1, 5), \quad \lambda + \rho - 5(\varepsilon_1 - \varepsilon_4) = (\boxed{2}, 3, \boxed{2}, 6, 0).$$

Hence, $\Delta_1(\lambda)$ is again a simple $\mathbf{U}_1$-module, which shows that all Weyl factors of $T_n^d$ are simple $\mathbf{U}_1$-modules. Thus, $\mathrm{End}_{\mathbf{U}_1}(T_n^d)$ is a semisimple algebra under the assumption $\mathrm{char}(\mathbb{K}) = 5$.

The situation changes if $\mathrm{char}(\mathbb{K}) = 3$: in contrast to the above there can now possibly be contributions to *JSF* of $\Delta_1(\lambda)$ for the four positive roots $\alpha \in \Phi^+$ given by $\alpha = \varepsilon_1 - \varepsilon_5$ (for $k = 1, 2$), $\alpha = \varepsilon_1 - \varepsilon_4$, $\alpha = \varepsilon_1 - \varepsilon_3$ or $\alpha = \varepsilon_1 - \varepsilon_2$. We calculate

$$\begin{aligned} &\lambda + \rho - 3(\varepsilon_1 - \varepsilon_5) = (4, \boxed{3}, 2, 1, \boxed{3}), &&\lambda + \rho - 3(\varepsilon_1 - \varepsilon_3) = (\boxed{4}, 3, \boxed{5}, 1, 0),\\ &\lambda + \rho - 6(\varepsilon_1 - \varepsilon_5) = (\boxed{1}, 3, 2, \boxed{1}, 6), &&\lambda + \rho - 3(\varepsilon_1 - \varepsilon_2) = (\boxed{4}, \boxed{6}, 2, 1, 0),\\ &\lambda + \rho - 3(\varepsilon_1 - \varepsilon_4) = (\boxed{4}, 3, 2, \boxed{4}, 0). \end{aligned}$$

Thus, *JSF* of $\Delta_1(\lambda)$ is nonzero. Hence, $\Delta_1(\lambda)$ provides a Weyl factor of $T_n^d$ which is a nonsimple $\mathbf{U}_1$-module, showing that $\mathrm{End}_{\mathbf{U}_1}(T_n^d)$ is not a semisimple algebra any more.

Remark 2.11. We deduced in Example 2.10 that $\mathrm{End}_{\mathbf{U}_1}(T_n^d)$ is nonsemisimple from the appearance of nonzero summands in *JSF*. We did not pay attention to possible cancellations which could occur because of the sign in (2.1). This is however justified: in type $\mathbf{A}_{m-1}$ such cancellations cannot occur; see [2, Section 7.4]. Thus, in type $\mathbf{A}_{m-1}$ it suffices to give one nonzero summand to conclude that the corresponding *JSF* is nonzero.

We illustrate now nontrivial cancellations.

Example 2.12. Let $\mathrm{char}(\mathbb{K}) = 2$ and $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{sp}_6)$. Consider the $\mathbf{U}_1$-module $\Delta_1(\lambda)$ with $\lambda = \varepsilon_1 + \varepsilon_2 = (1, 1, 0)$. We claim that $\Delta_1(\lambda)$ is a simple $\mathbf{U}_1$-module. First note that $\rho = (3, 2, 1)$. As in Example 2.10, we see that the only positive roots $\alpha \in \Phi^+$ that could contribute to *JSF* from (2.4) are

$$2\varepsilon_1, \quad 2\varepsilon_2, \quad \varepsilon_1 - \varepsilon_3, \quad \varepsilon_1 + \varepsilon_2, \quad \varepsilon_1 + \varepsilon_3, \quad \varepsilon_2 + \varepsilon_3.$$

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We leave it to the reader to verify that $\alpha = 2\varepsilon_1$, $\alpha = 2\varepsilon_2$, $\alpha = \varepsilon_1 - \varepsilon_3$ and $\alpha = \varepsilon_2 + \varepsilon_3$ do not contribute to *JSF* of $\Delta_1(\lambda)$. For the others, $\langle \lambda + \rho, (\varepsilon_1 + \varepsilon_2)^\vee \rangle = 7$ and $\langle \lambda + \rho, (\varepsilon_1 + \varepsilon_3)^\vee \rangle = 5$. Thus, we have to deal with $k = 1, 2, 3$ or $k = 1, 2$ in *JSF* of $\Delta_1(\lambda)$:

$$\lambda + \rho - 2(\varepsilon_1 + \varepsilon_2) = (2, 1, 1), \qquad \lambda + \rho - 2(\varepsilon_1 + \varepsilon_3) = (2, 3, -1),$$
$$\lambda + \rho - 4(\varepsilon_1 + \varepsilon_2) = (0, -1, 1), \qquad \lambda + \rho - 4(\varepsilon_1 + \varepsilon_3) = (0, 3, -3),$$
$$\lambda + \rho - 6(\varepsilon_1 + \varepsilon_2) = (-2, -3, 1).$$

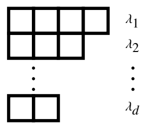
Permuting the two remaining regular $\mathbf{U}_1$-weights into dominant $\mathbf{U}_1$-weights gives different signs. Thus, the contributions cancel in *JSF* of $\Delta_1(\lambda)$ by Definition 2.7. Hence, *JSF* of $\Delta_1(\lambda)$ is zero (although not all summands are zero). Thus, $\Delta_1(\lambda)$ is a simple $\mathbf{U}_1$-module.

## 3. Several versions of Schur–Weyl dualities

In this section we recall a few known examples of Schur–Weyl-like dualities.

Conventions 3.1. Let $\Lambda^+(d) = \{\lambda \in \mathbb{Z}^d_{\geq 0} \mid \lambda_1 \geq \cdots \geq \lambda_d \geq 0, \sum_{i=1}^d \lambda_i = d\}$ denote the set of all *partitions of* some $d \in \mathbb{Z}_{>0}$. We identify these with *Young diagrams with d nodes*:

$$\lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda^+(d) \quad \leftrightsquigarrow$$




We always use the English notation for our Young diagrams, that is, starting with $\lambda_1$ nodes in the top row. Using the notation from Appendix B, we can associate to each Young diagram $\lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda^+(d)$ a $\mathbf{U}_q(\mathfrak{gl}_m)$-weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_d\varepsilon_d \in X^+$ and, hence, a Weyl module $\Delta_q(\lambda)$ for $\mathbf{U}_q(\mathfrak{gl}_m)$ (analogously for $\mathbf{U}_q(\mathfrak{g})$ with $\mathfrak{g}$ of types $\mathbf{B}_m$, $\mathbf{C}_m$ and $\mathbf{D}_m$). Here, by convention, $\Delta_q(\lambda) = 0$ if $\lambda_{m+1} > 0$.

Similarly, given a *pair* of Young diagrams $(\lambda, \mu) \in \Lambda^+(d_1) \times \Lambda^+(d_2)$, then we can associate to it a dominant $\mathbf{U}_q(\mathfrak{gl}_m \oplus \mathfrak{gl}_m)$-weight in the evident way and therefore a Weyl module for $\mathbf{U}_q(\mathfrak{gl}_m \oplus \mathfrak{gl}_m)$ that we denote by $\Delta_q(\lambda, \mu)$. Again, $\Delta_q(\lambda, \mu) = 0$ if $\lambda_{m+1} > 0$ or $\mu_{m+1} > 0$.

We can also associate to such a pair $(\lambda, \mu) \in \Lambda^+(r) \times \Lambda^+(s)$ a $\mathbf{U}_q(\mathfrak{gl}_{2m})$-weight $(\lambda, \overline{\mu})$ via

$$(\lambda, \overline{\mu}) = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r - \mu_s\varepsilon_{r+1} - \cdots - \mu_1\varepsilon_{r+s}$$

and, thus, a Weyl module $\Delta_q(\lambda, \overline{\mu})$ for $\mathbf{U}_q(\mathfrak{gl}_{2m})$. Again, $\Delta_q(\lambda, \overline{\mu}) = 0$ if $(\lambda, \overline{\mu})_{2m+1} > 0$.

### 3.1. Type A: the Hecke algebra of type A and (quantum) Schur–Weyl duality.

We fix $q \in \mathbb{K}^*$ in what follows. We denote by $S_d$ the symmetric group in $d \in \mathbb{Z}_{>0}$ letters.

Definition 3.2. Let $d \in \mathbb{Z}_{>0}$. The *Hecke algebra* $\mathcal{H}^{\mathbf{A}}_d(q)$ *of type* $\mathbf{A}$ is the $\mathbb{K}$-algebra generated by $\{H_{s_i} = H_i \mid s_i \in S_d\}$ for all transpositions $s_i = (i, i+1) \in S_d$ subject to the relations

$$H_i^2 = (q - q^{-1})H_i + 1 \quad \text{for } i = 1, \ldots, d-1,$$
$$H_iH_j = H_jH_i \quad \text{for } |i-j| > 1, \qquad H_iH_jH_i = H_jH_iH_j \quad \text{for } |i-j| = 1.$$

The *group algebra of the symmetric group* is $\mathbb{K}[S_d] = \mathcal{H}^{\mathbf{A}}_d(1)$.

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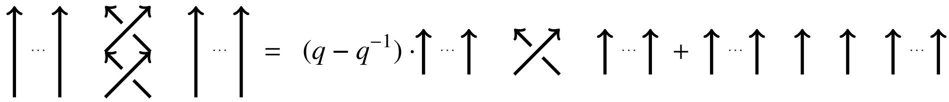
REMARK 3.3. One can think of the generators $H_i$ of $\mathcal{H}_d^{\mathbf{A}}(q)$ diagrammatically as crossings because there is a surjection from the group algebra $\mathbb{K}[B_d]$ of the braid group $B_d$ in $d$ strands to $\mathcal{H}_d^{\mathbf{A}}(q)$ given by sending the braid group generator $b_i$ (with strand $i$ crossing over strand $i+1$) to $H_i$. For example, the first relation from Definition 3.2 then reads as



(We read all diagrams in this paper from left to right and bottom to top.) Similarly, the algebra $\mathbb{K}[S_d]$ can then be thought of as the quotient of $\mathbb{K}[B_d]$ given by forgetting the information of over- and undercrossings and working with permutation diagrams.

Let $\mathrm{char}(\mathbb{K}) = 0$ and $q = 1$, or let $q \in \mathbb{K}^*$ be a nonroot of unity. Then there are simple $\mathcal{H}_d^{\mathbf{A}}(q)$-modules $D_q^\lambda$ for each $\lambda \in \Lambda^+(d)$; see, for example, [39, Ch. 3, Section 4]. These are lifts of the classical Specht modules of $S_d$ to its Hecke algebra $\mathcal{H}_d^{\mathbf{A}}(q)$.

Let $V = \Delta_q(\omega_1)$ denote the $(n = m)$-dimensional vector representation of $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{gl}_m)$. There is an action of $\mathcal{H}_d^{\mathbf{A}}(q)$ on $T_n^d = V^{\otimes d}$ by so-called *R-matrices*; see, for example, [17, (1.1)].

THEOREM 3.4 ((Quantum) Schur–Weyl duality, type $\mathbf{A}$).

(a) *The actions of $\mathbf{U}_q$ and $\mathcal{H}_d^{\mathbf{A}}(q)$ on $T_n^d$ commute.*

(b) *Let $\Phi_{q\mathrm{SW}}^{\mathbf{A}}$ be the algebra homomorphism induced by the action of $\mathcal{H}_d^{\mathbf{A}}(q)$ on $T_n^d$. Then*

$$\Phi_{q\mathrm{SW}}^{\mathbf{A}} : \mathcal{H}_d^{\mathbf{A}}(q) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_q}(T_n^d) \quad \text{and} \quad \Phi_{q\mathrm{SW}}^{\mathbf{A}} : \mathcal{H}_d^{\mathbf{A}}(q) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_q}(T_n^d) \quad \text{if } n \geq d.$$

(c) *Let $\mathrm{char}(\mathbb{K}) = 0$ and $q = 1$, or let $q \in \mathbb{K}^*$ be a nonroot of unity. Then there is a $(\mathbf{U}_q, \mathcal{H}_d^{\mathbf{A}}(q))$-bimodule decomposition*

$$T_n^d \cong \bigoplus_{\lambda \in \Lambda^+(d)} \Delta_q(\lambda) \otimes D_q^\lambda,$$

*with simple $\mathbf{U}_q$-modules $\Delta_q(\lambda) \cong L_q(\lambda)$.*

PROOF. Part (a) and surjectivity in (b) are proven in [17, Theorem 6.3]. Injectivity in (b) follows from Proposition 2.3. The statement (c) is the known $q$-analogue to the classical Schur–Weyl duality; see, for example, [19, Theorem 9.1.2] for the classical case (which holds almost word-by-word in the semisimple, quantized case as well). □

**3.2. Type $\mathbf{A} \oplus \mathbf{A}$: the Hecke algebra of type B and (quantum) Schur–Weyl duality.** We fix again $q \in \mathbb{K}^*$ in what follows. If $q = 1$, then we assume that $p \neq 2$.

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DEFINITION 3.5. Let $d \in \mathbb{Z}_{>0}$. The (one-parameter) *Hecke algebra* $\mathcal{H}_d^{\mathbf{B}}(q)$ *of type* $\mathbf{B}$ is the $\mathbb{K}$-algebra generated by $\{H_i \mid s_i \in S_d\} \cup \{H_0\}$ subject to the relations

$$H_i^2 = (q - q^{-1})H_i + 1 \quad \text{for } i = 0, \ldots, d-1, \quad H_iH_j = H_jH_i \quad \text{for } |i-j| > 1,$$
$$H_iH_jH_i = H_jH_iH_j \quad \text{for } |i-j| = 1,\ i, j \neq 0, \quad H_0H_1H_0H_1 = H_1H_0H_1H_0.$$

The *group algebra of the type* $\mathbf{B}_d$ *Weyl group* is $\mathbb{K}[S_d \ltimes (\mathbb{Z}/2\mathbb{Z})^d] = \mathcal{H}_d^{\mathbf{B}}(1)$.

Let $\mathrm{char}(\mathbb{K}) = 0$ and $q = 1$, or let $q \in \mathbb{K}^*$ be a nonroot of unity. For each pair of Young diagrams $(\lambda, \mu) \in \Lambda^+(d_1) \times \Lambda^+(d_2)$, define $D_q^{\lambda,\mu}$ via induction; see, for example, [40, Section 2.6] (which works in the semisimple, quantized case as well). That is,

$$D_q^{\lambda,\mu} = \mathcal{H}_d^{\mathbf{B}}(q) \otimes_{\mathcal{H}_{d_1}^{\mathbf{A}}(q) \otimes \mathcal{H}_{d_2}^{\mathbf{A}}(q)} (D_q^{\lambda} \otimes D_q^{\mu}).$$

Here $D_q^{\lambda}$ and $D_q^{\mu}$ are the quantum Specht modules of type $\mathbf{A}$.

Take $\mathfrak{g} = \mathfrak{gl}_m \oplus \mathfrak{gl}_m$ and set $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$. Let $V = \Delta_q(\omega_1)$ denote the $(n = 2m)$-dimensional vector representation of $\mathbf{U}_q(\mathfrak{gl}_{2m})$ restricted to $\mathbf{U}_q$. There is an action of $\mathcal{H}_d^{\mathbf{B}}(q)$ on $T_n^d = V^{\otimes d}$; see, for example, [23, Section 1].

THEOREM 3.6 ((Quantum) Schur–Weyl duality, type $\mathbf{A} \oplus \mathbf{A}$).

(a) *The actions of* $\mathbf{U}_q$ *and* $\mathcal{H}_d^{\mathbf{B}}(q)$ *on* $T_n^d$ *commute.*

(b) *Let* $\Phi_{q\mathrm{SW}}^{\mathbf{B}}$ *be the algebra homomorphism induced by the action of* $\mathcal{H}_d^{\mathbf{B}}(q)$ *on* $T_n^d$. *Then*

$$\Phi_{q\mathrm{SW}}^{\mathbf{B}} : \mathcal{H}_d^{\mathbf{B}}(q) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_q}(T_n^d) \quad \textit{and} \quad \Phi_{q\mathrm{SW}}^{\mathbf{B}} : \mathcal{H}_d^{\mathbf{B}}(q) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_q}(T_n^d), \ \textit{if } \tfrac{1}{2}n \geq d.$$

(c) *Let* $\mathrm{char}(\mathbb{K}) = 0$ *and* $q = 1$, *or let* $q \in \mathbb{K}^*$ *be a nonroot of unity. Then there is a* $(\mathbf{U}_q, \mathcal{H}_d^{\mathbf{B}}(q))$*-bimodule decomposition*

$$T_{m,m}^d \cong \bigoplus_{d_1+d_2=d} \bigoplus_{\substack{\lambda \in \Lambda^+(d_1) \\ \mu \in \Lambda^+(d_2)}} \Delta_q(\lambda, \mu) \otimes D_q^{\lambda,\mu},$$

*with simple* $\mathbf{U}_q$*-modules* $\Delta_q(\lambda, \mu) \cong L_q(\lambda, \mu)$.

PROOF. The statements (a) and (b), in the classical case, are proven in [40, Theorem 9] (see also [40, Remark 12] for the isomorphism criterion). The arguments given there go through for arbitrary $\mathbb{K}$ and $q \in \mathbb{K}^*$ as well. Statement (c) can be deduced from [40, Lemma 11], which again works in the semisimple, quantized case as well. See also [23, Theorem 4.3]. □

REMARK 3.7. The statements of Theorem 3.6 can be extended to the so-called *Ariki–Koike algebras* (the Hecke algebras for the complex reflection groups $G(m, 1, d)$); see for example [25] or [49]. Moreover, the approach taken in [40, Section 4] is set up such that it can be quantized as well. Hence, it should give a quantum Schur–Weyl duality for $G(m, p, d)$ as well.

**3.3. Mixed type A: the walled Brauer algebras and mixed Schur–Weyl duality.** The following algebra is called the *walled Brauer algebra* (or the *oriented Brauer algebra*), and was independently introduced in [32] and [53].

DEFINITION 3.8. Let $r, s \in \mathbb{Z}_{\geq 0}$ not both zero, $\delta \in \mathbb{K}$. The *walled Brauer algebra* $\mathcal{B}_{r,s}(\delta)$ is the $\mathbb{K}$-algebra generated by $\{\sigma_i \mid i = 1, \dots, r+s-1,\ i \neq r\} \cup \{u_r\}$ subject to the relations

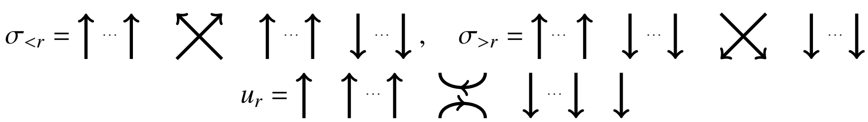
$$\sigma_i^2 = 1 \quad \text{for } i = 1, \dots, r+s-1,\ i \neq r,$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| > 1, \qquad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \quad \text{for } |i-j| = 1,$$
$$u_r^2 = \delta u_r, \quad u_r = u_r\sigma_{r-1}u_r = u_r\sigma_{r+1}u_r, \quad u_r\sigma_j = \sigma_j u_r \quad \text{for } |r-j| > 1,$$
$$u_r\sigma_{r-1}\sigma_{r+1}u_r\sigma_{r-1}\sigma_{r+1} = u_r\sigma_{r-1}\sigma_{r+1}u_r = \sigma_{r-1}\sigma_{r+1}u_r\sigma_{r-1}\sigma_{r+1}u_r.$$

Note that $\mathbb{K}[S_r]$ is a subalgebra of $\mathcal{B}_{r,s}(\delta)$ as well as a quotient of $\mathcal{B}_{r,s}(\delta)$ (given by killing the ideal generated by $u_r$). Similarly for $s$ instead of $r$.

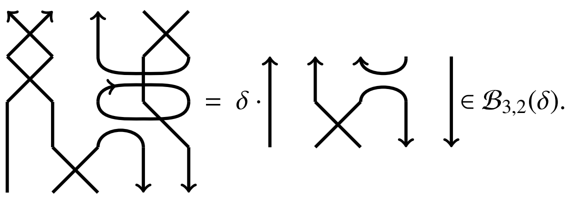
LEMMA 3.9. *Let* $0 < \mathrm{char}(\mathbb{K}) = p \leq \min\{r, s\}$. *Then* $\mathcal{B}_{r,s}(\delta)$ *is nonsemisimple.*

PROOF. Assume that $r \leq s$ and $0 < p \leq r$. Note that $\mathbb{K}[S_r]$ is a nonsemisimple quotient of $\mathcal{B}_{r,s}(\delta)$ by Maschke's theorem. Since quotients of semisimple algebras are semisimple, $\mathcal{B}_{r,s}(\delta)$ cannot be semisimple. Dually for $r \geq s$ and $0 < p \leq s$. Hence, the statement follows. □

REMARK 3.10. One can think of the generators of $\mathcal{B}_{r,s}(\delta)$ as being generators of Kauffman's *oriented* tangle algebra with $r$ left upwards and $s$ right downwards pointing arrows as follows:




In this setting, the relations from Definition 3.8 can be interpreted in the usual, topological sense of Kauffman's tangle algebra (each internal circle can be removed and gives a factor $\delta \in \mathbb{K}$). Here is an example of a typical element in $\mathcal{B}_{3,2}(\delta)$:




A *primitive (walled Brauer) diagram* is a single diagram (instead of a linear combination) of Kauffman's oriented tangle algebra without internal circles. These form a basis of $\mathcal{B}_{r,s}(\delta)$.

One could also define a *quantized* walled Brauer algebra $\mathcal{B}_{r,s}([\delta])$; see [12, Definition 2.2].

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Conventions 3.11. From now on, if $\mathrm{char}(\mathbb{K}) = p$, then we additionally assume that $\delta \in \mathbb{F}_p \subset \mathbb{K}$. Here $\mathbb{F}_p$ is the field with $p$ elements. Hence, there is a minimal $\delta_p \in \mathbb{Z}_{\geq 0}$ such that $\delta \equiv \delta_p \bmod p$. By convention, $\delta \in \mathbb{Z}$ and $\delta_0 = |\delta|$ if $\mathrm{char}(\mathbb{K}) = 0$.

We can associate to each pair of Young diagrams $(\lambda, \mu)$ with $\lambda \in \Lambda^+(r-i)$ and $\mu \in \Lambda^+(s-i)$ (for $i = 0, 1, \dots, \min\{r, s\}$) a $\mathcal{B}_{r,s}(\delta)$-module via induction; see [10, (2.9)] for the classical case and [11, Theorem 2.7] for the general case. That is,

$$D_1^{\lambda,\mu} = \mathcal{B}_{r,s}(\delta) \otimes_{\mathbb{K}(S_r)\otimes\mathbb{K}(S_s)} (D_1^{\lambda} \otimes D_1^{\mu}). \tag{3.1}$$

If $\mathbb{K} = \mathbb{C}$ and $r + s \leq \delta_0 + 1$, then these $\mathcal{B}_{r,s}(\delta)$-modules are exactly the simple $\mathcal{B}_{r,s}(\delta)$-modules; see [11, Theorem 2.7].

Let $V = \Delta_1(\omega_1)$ denote the $(n = m)$-dimensional vector representation of $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{gl}_m)$ and let $V^*$ be its dual. We set $T_n^{r,s} = V^{\otimes r} \otimes (V^*)^{\otimes s}$. By [12, Section 3], there is an action of $\mathcal{B}_{r,s}(\delta)$ on $T_n^{r,s}$ for $n = \delta_p$. This action is given by letting $\mathbb{K}(S_r)$ and $\mathbb{K}(S_s)$ act by permutations. In order to explain the action of $u_r$, denote by $\{1, \dots, n\}$ and $\{\overline{1}, \dots, \overline{n}\}$ a basis of $V$ and its dual basis of $V^*$, respectively. Let $v, w \in \{1, \dots, n\}$ with $v \neq w$. Then $u_r$ acts on the components $r$ and $r+1$ of a primitive tensor in $T_n^{r,s}$ by sending $v \otimes \overline{v}$ to $\sum_{i=1}^{n} i \otimes \overline{i}$ and $v \otimes \overline{w}$ to zero.

Recall the following 'mixed' version of Schur–Weyl duality.

Theorem 3.12 (Mixed Schur–Weyl duality, type **A**). *Let $n = \delta_p$.*

(a) *The actions of $\boldsymbol{U}_1$ and $\mathcal{B}_{r,s}(\delta)$ on $T_n^{r,s}$ commute.*

(b) *Let $\Phi_{\mathrm{wBr}}$ be the algebra homomorphism induced by the action of $\mathcal{B}_{r,s}(\delta)$ on $T_n^{r,s}$. Then*

$$\Phi_{\mathrm{wBr}} : \mathcal{B}_{r,s}(\delta) \twoheadrightarrow \mathrm{End}_{\boldsymbol{U}_1}(T_n^{r,s}) \quad \text{and} \quad \Phi_{\mathrm{wBr}} : \mathcal{B}_{r,s}(\delta) \xrightarrow{\cong} \mathrm{End}_{\boldsymbol{U}_1}(T_n^{r,s}), \text{ if } n \geq r + s.$$

(c) *Let $\mathbb{K} = \mathbb{C}$ and $r + s \leq \delta_0 + 1$. Moreover, set $Y = \{0, 1, \dots, \min\{r, s\}\}$. Then there is a $(\boldsymbol{U}_1, \mathcal{B}_{r,s}(\delta))$-bimodule decomposition*

$$T_n^{r,s} \cong \bigoplus_{i \in Y} \bigoplus_{\substack{\lambda \in \Lambda^+(r-i) \\ \mu \in \Lambda^+(s-i)}} \Delta_1(\lambda, \overline{\mu}) \otimes D_1^{\lambda,\overline{\mu}},$$

*with simple $\boldsymbol{U}_1$-modules $\Delta_1(\lambda, \overline{\mu}) \cong L_1(\lambda, \overline{\mu})$.*

Proof. Parts (a) and (b) are proven in [12, Theorem 7.1 and Corollary 7.2]. The statement (c) can be derived from [32, Theorem 1.1] together with (3.1). □

Remark 3.13.

(a) The assumption $r + s \leq \delta_0 + 1$ in (c) of Theorem 3.12 will turn out to be necessary to ensure that $\mathcal{B}_{r,s}(\delta)$ is semisimple.

(b) Note that there is also a quantized version of Theorem 3.12; see [12].

**3.4. Types B, C, D: the Brauer algebras and Schur–Weyl–Brauer duality.** The following algebra, called the *Brauer algebra*, goes back to work of Brauer [8].

DEFINITION 3.14. Let $d \in \mathbb{Z}_{>0}$, $\delta \in \mathbb{K}$. The *Brauer algebra* $\mathcal{B}_d(\delta)$ is the $\mathbb{K}$-algebra generated by $\{\sigma_i, u_i \mid i = 1, \dots, d-1\}$ subject to the relations

$$\sigma_i^2 = 1 \quad \text{for } i = 1, \dots, d-1,$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| > 1, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \quad \text{for } |i-j| = 1,$$
$$u_i^2 = \delta u_i \quad \text{for } i = 1, \dots, d-1, \quad u_iu_j = u_ju_i \quad \text{for } |i-j| > 1,$$
$$u_iu_{i+1}u_i = u_i \quad \text{for } i = 1, \dots, d-2, \quad u_iu_{i-1}u_i = u_i \quad \text{for } i = 2, \dots, d-1,$$
$$\sigma_iu_ju_i = \sigma_ju_i \quad \text{for } |i-j| = 1, \quad \sigma_iu_i = u_i = u_i\sigma_i \quad \text{for } i = 1, \dots, d-1.$$

Note that $\mathbb{K}[S_d]$ is a subalgebra of $\mathcal{B}_d(\delta)$ as well as a quotient of $\mathcal{B}_d(\delta)$ (given by killing the ideal generated by the $u_i$).

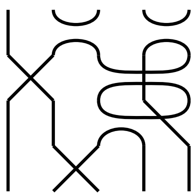
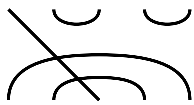
LEMMA 3.15. *Let* $\mathrm{char}(\mathbb{K}) = p \leq d$. *Then* $\mathcal{B}_d(\delta)$ *is nonsemisimple.*

PROOF. Analogously to Lemma 3.9. □

REMARK 3.16. One can think of the generators of $\mathcal{B}_d(\delta)$ as being generators of Kauffman's *unoriented* tangle algebra with $d$ strands as follows:

$\sigma_i =$ , $u_i =$ 

Removing circles gives a factor $\delta \in \mathbb{K}$ again. An example is

 $= \delta \cdot$  $\in \mathcal{B}_5(\delta)$.

A *primitive (Brauer) diagram* is a single diagram (instead of a linear combination) of Kauffman's unoriented tangle algebra without internal circles. These form a basis of $\mathcal{B}_d(\delta)$.

There is also again a *quantized* Brauer algebra $\mathcal{B}_d([\delta])$ (called *Birman–Murakami–Wenzl or BMW algebra*); see, for example, [41, Section 2].

For the Brauer algebras we use from now on the same conventions for the parameter $\delta_p$ as for the walled Brauer algebras; see Conventions 3.11. Recall that we can associate to each Young diagram $\lambda$ with $\lambda \in \Lambda^+(d-2i)$ (for $i = 0, 1, \dots, \lfloor \frac{1}{2}d \rfloor$) a $\mathcal{B}_d(\delta)$-module $D_1^\lambda$. If $\mathbb{K} = \mathbb{C}$ and $2d \leq \delta_0 + 1$, then these are simple $\mathcal{B}_d(\delta)$-modules; see, for example, [20, Section 4].

Let $V = \Delta_1(\omega_1)$ denote the $n$-dimensional vector representation of $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{g})$ for $\mathfrak{g}$ being $\mathfrak{so}_{2m+1}$, $\mathfrak{sp}_{2m}$ or $\mathfrak{so}_{2m}$ (here $n = 2m+1$ for $\mathfrak{g} = \mathfrak{so}_{2m+1}$ or $n = 2m$ for $\mathfrak{g} = \mathfrak{sp}_{2m}$ and for $\mathfrak{g} = \mathfrak{so}_{2m}$). By [18, Theorem 3.11], there is an action of $\mathcal{B}_d(\delta)$ on $T_n^d = V^{\otimes d}$ for $n = \delta_p$. The action is very similar to the one for $\mathcal{B}_{r,s}(\delta)$ recalled above. We point out that in type $\mathbf{C}_m$ the action of $u_i \in \mathcal{B}_d(\delta)$ has an additional sign coming from the part of odd parity from the super action studied in [18, Theorem 3.11].

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Theorem 3.17 (Schur–Weyl–Brauer duality, types $\mathbf{B}, \mathbf{C}, \mathbf{D}$). *Let $n = \delta_p$.*

(a) *The actions of $\boldsymbol{U}_1$ and $\mathcal{B}_d(\delta)$ on $T_n^d$ commute.*

(b) *Let $\Phi_{\mathrm{Br}}$ be the algebra homomorphism induced by the action of $\mathcal{B}_d(\delta)$ on $T_n^d$. Then*

$$\boldsymbol{B}_m : \Phi_{\mathrm{Br}} : \mathcal{B}_d(\delta) \twoheadrightarrow \mathrm{End}_{\boldsymbol{U}_1}(T_n^d) \quad \textit{and} \quad \Phi_{\mathrm{Br}} : \mathcal{B}_d(\delta) \xrightarrow{\cong} \mathrm{End}_{\boldsymbol{U}_1}(T_n^d) \quad \textit{if } n \geq d.$$
$$\boldsymbol{C}_m : \Phi_{\mathrm{Br}} : \mathcal{B}_d(\delta) \twoheadrightarrow \mathrm{End}_{\boldsymbol{U}_1}(T_n^d) \quad \textit{and} \quad \Phi_{\mathrm{Br}} : \mathcal{B}_d(\delta) \xrightarrow{\cong} \mathrm{End}_{\boldsymbol{U}_1}(T_n^d) \quad \textit{if } n \geq 2d,$$
$$\boldsymbol{D}_m : \Phi_{\mathrm{Br}} : \mathcal{B}_d(\delta) \hookrightarrow \mathrm{End}_{\boldsymbol{U}_1}(T_n^d) \quad \textit{if } n \geq d \quad \textit{and}$$
$$\Phi_{\mathrm{Br}} : \mathcal{B}_d(\delta) \xrightarrow{\cong} \mathrm{End}_{\boldsymbol{U}_1}(T_n^d) \quad \textit{if } n \geq 2d+1.$$

(c) *Let $\mathbb{K} = \mathbb{C}$, $d \leq \delta_0 + 1$ and $\boldsymbol{U}_1 = \boldsymbol{U}_1(\mathfrak{sp}_{2m})$ (thus, we have $2m = n = -\delta$). Moreover, set $Y = \{0, 1, \ldots, \lfloor \frac{1}{2} d \rfloor\}$. Then there is a $(\boldsymbol{U}_1, \mathcal{B}_d(\delta))$-bimodule decomposition*

$$T_n^d \cong \bigoplus_{i \in Y} \bigoplus_{\lambda \in \Lambda^+(d-2i)} \Delta_1(\lambda) \otimes D_1^\lambda,$$

*with simple $\boldsymbol{U}_1$-modules $\Delta_1(\lambda) \cong L_1(\lambda)$.*

Proof. The parts (a) and (b) are proven in [18, Theorem 5.5] (published version), but the criterion given there is not optimal in type $\mathbf{B}_m$. The above bound holds in type $\mathbf{B}_m$: note that $T_n^d \in \mathcal{T}$ (since we assume that $\mathrm{char}(\mathbb{K}) \neq 2$). Thus, by Proposition 2.3, $\dim(\mathrm{End}_{\mathbf{U}_1(\mathfrak{so}_{2m+1})}(T_n^d))$ is as in the classical case. Hence, the above bound follows from the classical bound (which can already be found implicitly in Brauer's work [8]). The statement (c) is given in [24, Theorem 1.1]. □

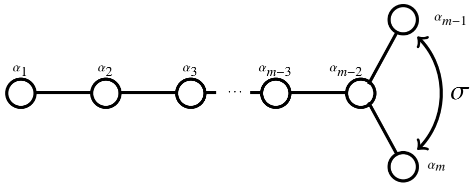
Remark 3.18.

(a) The assumption $d \leq \delta_0 + 1$ in (c) of Theorem 3.17 again turns out to be necessary to ensure that $\mathcal{B}_d(\delta)$ is semisimple.

(b) Note that there are also quantized versions of Theorem 3.17 (in some cases and for appropriate parameters); see, for example, [24, Theorem 1.3] or [35, (9.6)].

**3.5. A slightly stronger statement in type D.** Let $m \geq 1$. Then surjectivity of $\Phi_{\mathrm{Br}}$ fails in general for $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{so}_{2m})$. In the remaining part of this section we will determine the image and show that $\mathrm{im}(\Phi_{\mathrm{Br}}) \cong \mathrm{End}_{\tilde{\mathbf{U}}_1}(T_n^d)$; see Theorem 3.24. Here, as we explain below, $\tilde{\mathbf{U}}_1$ is obtained from a nontrivial symmetry of the Dynkin diagram of type $\mathbf{D}_m$ as in (3.2). The proof of Theorem 3.24 is slightly involved and the main part is a counting argument comparing multiplicities of $\tilde{\mathbf{U}}_1$-modules to multiplicities of $\mathbf{U}_1(\mathfrak{so}_{2m'})$-modules for some large enough $m'$; see Lemma 3.22. We point out that our approach is inspired partly by [34, Section 8].

Suppose that $\mathrm{char}(\mathbb{K}) \neq 2$. Denote by $\sigma : \mathbf{U}_1 \to \mathbf{U}_1$ the involution induced by a graph automorphism of the Dynkin diagram of type $\mathbf{D}_m$. For $m \geq 4$, the automorphism is given via




(3.2)

Moreover, if $m = 1$, then $\mathfrak{so}_2$ is one dimensional and $\sigma$ is trivial. If $m = 2$, then $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$ and $\sigma$ swaps the two components of the Dynkin diagram of type $\mathbf{A}_1 \times \mathbf{A}_1$. For $m = 3$, we have $\mathfrak{so}_6 \cong \mathfrak{sl}_4$ and $\sigma$ swaps the two extremal nodes of the Dynkin diagram of type $\mathbf{A}_3$.

Suppose that $M \in \mathbf{U}_1$-**Mod**. Denote by ${}^{\sigma}M$ the *$\sigma$-twist of $M$*, that is, ${}^{\sigma}M = M$ as $\mathbb{K}$-vector spaces with $\mathbf{U}_1$ action via $x\overline{m} = \overline{\sigma(x)m}$ for all $x \in \mathbf{U}_1, m \in M$. Here we have written $\overline{m}$ for an element $m \in M$ when considered as an element of ${}^{\sigma}M$. Set

$$\tilde{M} = M \oplus {}^{\sigma}M. \tag{3.3}$$

Denote by $\tilde{\mathbf{U}}_1$ the *skew group ring* $\mathbf{U}_1 \rtimes \mathbb{K}(\mathbb{Z}/2\mathbb{Z})$ and by $\tau$ the generator of $\mathbb{K}(\mathbb{Z}/2\mathbb{Z})$. The elements of $\tilde{\mathbf{U}}_1$ are of the form $x + y\tau$ for $x, y \in \mathbf{U}_1$ and multiplication in $\tilde{\mathbf{U}}_1$ is such that $\mathbf{U}_1$ and $\mathbb{K}(\mathbb{Z}/2\mathbb{Z})$ are subalgebras together with

$$\tau x = \sigma(x)\tau \quad \text{for all } x \in \mathbf{U}_1. \tag{3.4}$$

As a semidirect product of Hopf algebras, $\tilde{\mathbf{U}}_1$ is itself a Hopf algebra with Hopf subalgebras $\mathbf{U}_1$ and $\mathbb{K}(\mathbb{Z}/2\mathbb{Z})$. In particular, $\tau$ is *group-like* and acts on a tensor product as $\tau \otimes \tau$. Moreover, $\tilde{M}$ from above is a $\tilde{\mathbf{U}}_1$-module with $\tau$-action given via $\tau(m, \overline{n}) = (n, \overline{m})$ for all $m, n \in M$ (a computation shows that, under this convention, (3.4) is preserved).

Now suppose that $\mathrm{char}(\mathbb{K}) = 0$. Then the simple $\mathbf{U}_1$-modules are the Weyl modules $\Delta_1(\lambda)$ for $\lambda = \sum_{i=1}^{m} \lambda_i \varepsilon_i \in X^+$ with highest weight vector $v_\lambda$. As $\mathbf{U}_1$-modules,

$${}^{\sigma}\Delta_1(\lambda) \cong \Delta_1(\overline{\lambda}) \quad \text{for all } \lambda \in X^+,$$

where $\overline{\lambda} = \sum_{i=1}^{m-1} \lambda_i \varepsilon_i - \lambda_m \varepsilon_m$. In particular, ${}^{\sigma}\Delta_1(\lambda)$ has $\overline{v}_\lambda$ as highest weight vector. We use the abbreviation $\tilde{\Delta}_1(\lambda)$ for the $\tilde{\mathbf{U}}_1$-module given as in (3.3) (for $M = \Delta_1(\lambda)$).

In case $\lambda_m = 0$ (thus, $\lambda = \overline{\lambda}$), set $w_\lambda = (v_\lambda, \overline{v}_\lambda)$ and $w'_\lambda = (v_\lambda, -\overline{v}_\lambda)$. Then the $\tilde{\mathbf{U}}_1$-module $\tilde{\Delta}_1(\lambda)$ decomposes into $\pm 1$ eigenspaces of $\tau$ given by

$$\tilde{\Delta}_1^{+}(\lambda) = \mathbf{U}_1 w_\lambda, \quad \tilde{\Delta}_1^{-}(\lambda) = \mathbf{U}_1 w'_\lambda.$$

For example, the vector representation $V$ of $\mathbf{U}_1$ is a $\tilde{\mathbf{U}}_1$-module with trivial action of $\tau$ and $V \cong \tilde{\Delta}_1^{+}(\omega_1)$. These notations enable us to classify simple $\tilde{\mathbf{U}}_1$-modules in case $\mathrm{char}(\mathbb{K}) = 0$.

Proposition 3.19. *Let* $\mathrm{char}(\mathbb{K}) = 0$ *and* $\lambda \in X^+$. *Then* $\tilde{\mathbf{U}}_1$-***Mod*** *is semisimple, and:*

(a) *if* $\lambda_m \neq 0$, *then* $\tilde{\Delta}_1(\lambda) \cong \tilde{\Delta}_1(\overline{\lambda})$ *is a simple* $\tilde{\mathbf{U}}_1$*-module;*

(b) *if* $\lambda_m = 0$, *then* $\tilde{\Delta}_1^{\pm}(\lambda)$ *are simple* $\tilde{\mathbf{U}}_1$*-modules;*

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(c) *up to isomorphism, the set*

$$\{\tilde{\Delta}_1(\lambda) \mid \lambda \in X^+, \lambda_m > 0\} \cup \{\tilde{\Delta}_1^{\pm}(\lambda) \mid \lambda \in X^+, \lambda_m = 0\}$$

*is a complete list of nonisomorphic, simple $\tilde{\boldsymbol{U}}_1$-modules.*

Proof. By the above discussion and standard Clifford theory; see, for example, [42, Section 2] or [44, Appendix] (both references treat a more general case). □

Let still $\mathrm{char}(\mathbb{K}) = 0$ and recall the following decomposition of $\Delta_1(\lambda) \otimes V$ as a $\mathbf{U}_1$-module:

$$\Delta_1(\lambda) \otimes V \cong \bigoplus_{i=1}^{m} \Delta_1(\lambda \pm \varepsilon_i), \quad \text{where } \Delta_1(\lambda \pm \varepsilon_i) = \Delta_1(\lambda + \varepsilon_i) \oplus \Delta_1(\lambda - \varepsilon_i). \tag{3.5}$$

Here $\Delta_1(\mu) = 0$ (and, hence, $\tilde{\Delta}_1(\mu) = 0$) if $\mu \notin X^+$. This leads to the following lemma.

Lemma 3.20. *Let* $\mathrm{char}(\mathbb{K}) = 0$, $\lambda \in X^+$ *and* $\epsilon \in \{+, -\}$. *Then, as $\tilde{\boldsymbol{U}}_1$-modules:*

$$\text{if } \lambda_m > 1: \quad \tilde{\Delta}_1(\lambda) \otimes V \cong \bigoplus_{i=1}^{m} \tilde{\Delta}_1(\lambda \pm \varepsilon_i);$$

$$\text{if } \lambda_m = 1: \quad \tilde{\Delta}_1(\lambda) \otimes V \cong \bigoplus_{i=1}^{m-1} \tilde{\Delta}_1(\lambda \pm \varepsilon_i) \oplus \tilde{\Delta}_1(\lambda + \varepsilon_m) \oplus \tilde{\Delta}_1^{+}(\lambda - \varepsilon_m) \oplus \tilde{\Delta}_1^{-}(\lambda - \varepsilon_m);$$

$$\text{if } \lambda_m = 0: \quad \tilde{\Delta}_1^{\epsilon}(\lambda) \otimes V \cong \bigoplus_{i=1}^{m-1} \tilde{\Delta}_1^{\epsilon}(\lambda \pm \varepsilon_i) \oplus \tilde{\Delta}_1(\lambda + \varepsilon_m).$$

Proof. We have $\tilde{\Delta}_1(\lambda) \cong \tilde{\Delta}_1(\overline{\lambda})$ for $\lambda_m \neq 0$ and $\tilde{\Delta}_1(\lambda) \cong \Delta_1^{+}(\lambda) \oplus \Delta_1^{-}(\lambda)$ for $\lambda_m = 0$. Using Proposition 3.19 and (3.5), the statement follows. □

This leads to the following multiplicity formulas of $\mathbf{U}_1$-modules.

Proposition 3.21. *Let* $\mathrm{char}(\mathbb{K}) = 0$, $\lambda \in X^+$ *and* $\epsilon \in \{+, -\}$. *As usual, let* $T_n^d = V^{\otimes d}$. *Then:*

$$\text{if } \lambda_m > 1: \quad (T_n^d : \tilde{\Delta}_1(\lambda)) = \sum_{i=1}^{m} (T_n^{d-1} : \tilde{\Delta}_1(\lambda \pm \varepsilon_i));$$

$$\begin{aligned} \text{if } \lambda_m = 1: \quad (T_n^d : \tilde{\Delta}_1(\lambda)) = {} & \sum_{i=1}^{m-1} (T_n^{d-1} : \tilde{\Delta}_1(\lambda \pm \varepsilon_i)) + (T_n^{d-1} : \tilde{\Delta}_1(\lambda + \varepsilon_m)) \\ & + (T_n^{d-1} : \tilde{\Delta}_1^{+}(\lambda - \varepsilon_m)) + (T_n^{d-1} : \tilde{\Delta}_1^{-}(\lambda - \varepsilon_m)); \end{aligned}$$

$$\text{if } \lambda_m = 0: \quad (T_n^d : \tilde{\Delta}_1^{\epsilon}(\lambda)) = \sum_{i=1}^{m-1} (T_n^{d-1} : \tilde{\Delta}_1^{\epsilon}(\lambda \pm \varepsilon_i)) + (T_n^{d-1} : \tilde{\Delta}_1(\lambda + \varepsilon_m)),$$

*where any multiplicity is zero if the corresponding weight is not in $X^+$.*

Proof. For $d = 1$, see Lemma 3.20. The general statement for $d > 1$ follows recursively. □

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Let $m' \in \mathbb{Z}_{\geq 1}$ and $n' = 2m'$ be such that $1 \leq d \leq 2m < m'$. We consider $\mathbf{U}'_1 = \mathbf{U}_1(\mathfrak{so}_{2m'})$ and use notations as $V'$, $(T')^d_{n'}$, $X'$, $\lambda'$ etc to distinguish these from the data for $\mathbf{U}_1$. Moreover, let $\lambda' \in (X')^+$ with $|\lambda'| \leq d$. Then $\lambda'_{m'} = 0$ and $\lambda'_{m+1} \leq 1$. Given now a $\mathbf{U}'_1$-weight $\lambda' \in (X')^+$, define a $\mathbf{U}_1$-weight $\tilde{\lambda} = \sum_{i=1}^m (\lambda'_i - \lambda'_{2m-i+1})\varepsilon_i \in X^+$. Note that, if $\lambda'_{m+1} = 0$, then $\tilde{\lambda} = \lambda'$.

The main step now is to compare $\mathbf{U}'_1$-multiplicities to $\tilde{\mathbf{U}}_1$-multiplicities.

Lemma 3.22. *Let* $\mathrm{char}(\mathbb{K}) = 0$. *With the notation from above, we have the following.*

(a) *If* $\lambda'_{m+1} = 0$, *then*

$$((T')^d_{n'} : \Delta'_1(\lambda')) = \begin{cases} (T^d_n : \tilde{\Delta}_1(\tilde{\lambda})) & \text{if } \lambda'_m > 0, \\ (T^d_n : \tilde{\Delta}^+_1(\tilde{\lambda})) & \text{if } \lambda'_m = 0. \end{cases}$$

(b) *If* $\lambda'_{m+1} = 1$, *then*

$$((T')^d_{n'} : \Delta'_1(\lambda')) = (T^d_n : \tilde{\Delta}^-_1(\tilde{\lambda})).$$

Proof. We use induction on $d$. If $d = 1$, then $(T')^d_{n'} = V' \cong \Delta'_1(\omega'_1)$ and $T^d_n = V \cong \tilde{\Delta}^+_1(\omega_1)$. Since the $\mathbf{U}_1$-weight $\tilde{\lambda}$ associated to $\omega'_1$ is $\omega_1$, (a) and (b) follow.

Assume that $d > 1$. Then the $'$-version of (3.6) can be used to express the left-hand sides in (a) and (b) in terms of $\mathbf{U}'_1$-multiplicities in $(T')^{d-1}_{n'}$ and Proposition 3.21 expresses the right-hand sides in terms of $\tilde{\mathbf{U}}_1$-multiplicities in $T^{d-1}_n$. It is now a matter of bookkeeping to check that the induction hypothesis gives the stated equalities. We give details only for the first case of (a). So we have $\lambda'_{m+1} = 0$. Assume first that $\lambda'_m > 1$. Then the $'$-version of (3.6) gives

$$((T')^d_{n'} : \Delta'_1(\lambda')) = \sum_{i=1}^{m+1} ((T')^{d-1}_{n'} : \Delta'_1(\lambda' \pm \varepsilon'_i)). \tag{3.6}$$

Note that $\Delta'_1(\lambda' - \varepsilon'_{m+1}) = 0$ and $|\lambda' + \varepsilon'_{m+1}| \geq 2m + 1 > d - 1$. Thus, the $(m+1)$th summand in (3.6) is zero ($((T')^d_{n'} : \Delta'_1(\lambda')) = 0$ unless $|\lambda'| \leq d$). Now, by the induction hypothesis, $((T')^{d-1}_{n'} : \Delta'_1(\lambda' \pm \varepsilon'_i)) = (T^{d-1}_n : \tilde{\Delta}_1(\tilde{\lambda} \pm \varepsilon_i))$ for all $i = 1, \ldots, m$ (we write $\tilde{\lambda} \pm \varepsilon_i$ short for $\tilde{\mu}$ with $\mu = \lambda' \pm \varepsilon'_i$, if $i = 1, \ldots, m$, and $\mu = \lambda' \mp \varepsilon'_i$, if $i = m+1, \ldots, m'$). Then Proposition 3.21 gives the desired equality.

Now assume that $\lambda'_{m+1} = 0$ and $\lambda'_m = 1$. Arguing as before,

$$\begin{aligned} ((T')^d_{n'} : \Delta'_1(\lambda')) &= \sum_{i=1}^{m} ((T')^{d-1}_{n'} : \Delta'_1(\lambda' \pm \varepsilon'_i)) + ((T')^{d-1}_{n'} : \Delta'_1(\lambda' + \varepsilon'_{m+1})) \\ &= \sum_{i=1}^{m-1} (T^{d-1}_{n'} : \tilde{\Delta}_1(\tilde{\lambda} \pm \varepsilon_i)) + (T^{d-1}_{n'} : \tilde{\Delta}_1(\tilde{\lambda} + \varepsilon_m)) \\ &\quad + (T^{d-1}_{n'} : \tilde{\Delta}^+_1(\tilde{\lambda} - \varepsilon_m)) + (T^{d-1}_{n'} : \tilde{\Delta}^-_1(\tilde{\lambda} - \varepsilon_m)) \\ &= (T^d_n : \tilde{\Delta}_1(\tilde{\lambda})). \end{aligned}$$

Here the second equality uses Lemma 3.22 for the first and the last terms (both in case $d - 1$). The last equality uses Proposition 3.21. □

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COROLLARY 3.23. *Let* $\mathrm{char}(\mathbb{K}) = 0$. *If* $1 \le d \le 2m < m'$, *then*

$$\dim(\mathrm{End}_{\tilde{U}_1}(T_n^d)) = \dim(\mathrm{End}_{U_1'}((T')_{n'}^d)).$$

PROOF. Note that $\dim(\mathrm{End}_{\tilde{\mathbf{U}}_1}(T_n^d)) = \sum_L (V^{\otimes d} : L)^2$ (the sum runs over all simple $\tilde{\mathbf{U}}_1$-modules $L$ that are composition factors of $V^{\otimes d}$). There is a similar formula for $\dim(\mathrm{End}_{\mathbf{U}_1'}((T')_{n'}^d)$ as well. By Proposition 3.19, we can use Lemma 3.22 to see that the dimensions agree. □

We now leave the case $\mathrm{char}(\mathbb{K}) = 0$ and state and prove the main result of this subsection, where we assume only that $\mathrm{char}(\mathbb{K}) \neq 2$.

THEOREM 3.24. *If* $1 \le d \le 2m$ *and* $\delta_p \equiv 2m \bmod p$, *then* $\mathcal{B}_d(\delta) \cong \mathrm{End}_{\tilde{U}_1}(T_n^d)$.

PROOF. By Theorem 3.17, we know that the Schur–Weyl–Brauer homomorphism

$$\Phi_{\mathrm{Br}} : \mathcal{B}_d(\delta') \to \mathrm{End}_{\mathbf{U}_1'}((T')_{n'}^d)$$

is injective for $d \le 2m'$ and surjective for $d < m'$ (where $\delta_p' \equiv 2m' \bmod p$). Hence, for $m' > 2m$, we have that $\dim(\mathcal{B}_d(\delta')) = \dim(\mathrm{End}_{\mathbf{U}_1'}((T')_{n'}^d))$ and so also $\dim(\mathcal{B}_d(\delta')) = \dim(\mathrm{End}_{\tilde{\mathbf{U}}_1}(T_n^d))$, by Corollary 3.23 and Proposition 2.3 (since $(T')_{n'}^d$ is a $\mathbf{U}_1'$-tilting module). Clearly, we have $\mathrm{im}(\Phi_{\mathrm{Br}}) \subset \mathrm{End}_{\tilde{\mathbf{U}}_1}(T_n^d)$. The statement follows, since $\dim(\mathcal{B}_d(\delta')) = \dim(\mathcal{B}_d(\delta))$. □

REMARK 3.25. Note that the whole discussion in this subsection goes through in case $m \le 3$ as well (with the corresponding $\sigma$ from above).

## 4. Some kernels of Schur–Weyl actions

In this section we explicitly describe kernels of the epimorphisms $\Phi_{q\mathrm{SW}}^{\mathbf{A}}$, $\Phi_{\mathrm{wBr}}$ and $\Phi_{\mathrm{Br}}$ from the dualities in Section 3, some of which we use in the proofs of our main theorems.

In the case of $\mathcal{H}_d^{\mathbf{A}}(q)$, all kernels were determined in [22, Theorem 4]. In our setup, we have for $n$ as in Theorem 3.4 and $q = 1$ the *anti-symmetrizer*

$$e_d(n) = \sum_{w \in S_d} (-1)^{l(w)} w \in \ker(\Phi_{q\mathrm{SW}}^{\mathbf{A}}), \tag{4.1}$$

where $l(w)$ is the length of $w$. Clearly, $e_d(n)e_d(n) = d!e_d(n)$. Thus, $e_d(n)$ is a quasi-idempotent (an idempotent up to an invertible scalar) if and only if $\mathrm{char}(K) > d$ or $\mathrm{char}(K) = 0$ and nilpotent otherwise. By Härterich's results, the $\mathbb{K}$-linear span of $e_d(d-1)$ equals $\ker(\Phi_{q\mathrm{SW}}^{\mathbf{A}})$.

For some 'boundary cases' we can explicitly write down the kernels for the other algebras as well, as we aim to show next. Note that this generalizes Härterich's results.

DEFINITION 4.1. Define $e_{r,s}(\delta) \in \mathcal{B}_{r,s}(\delta)$ and $E_d(\delta) \in \mathcal{B}_d(\delta)$ via

$$e_{r,s}(\delta) = \sum_x (-1)^{l(x)} x \in \mathcal{B}_{r,s}(\delta) \quad \text{and} \quad E_d(\delta) = \sum_x (-1)^{l(x)} x \in \mathcal{B}_d(\delta).$$

Here the sums run over all primitive diagrams (see Remarks 3.10 and 3.16).

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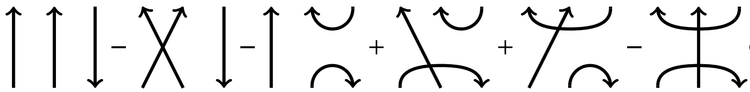
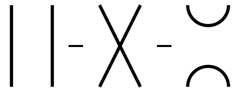
Example 4.2. If $r = 2$, $s = 1$ (walled Brauer case) or if $d = 2$ (Brauer case), then

$e_{2,1}(\delta) =$  $\in \mathcal{B}_{2,1}(\delta)$,

$E_2(\delta) =$  $\in \mathcal{B}_2(\delta)$.

Moreover, if $s = 0$ and $\delta \in \mathbb{K}$ is arbitrary, then $e_{r,0}(\delta)$ are the elements given in (4.1).

### 4.1. The walled Brauer case.

Proposition 4.3. *Let $n = r + s - 1 = \delta_p$. Then the $\mathbb{K}$-linear span of $e_{r,s}(n) \in \mathcal{B}_{r,s}(\delta)$ equals $\ker(\Phi_{\mathrm{wBr}})$. Furthermore, the following holds.*

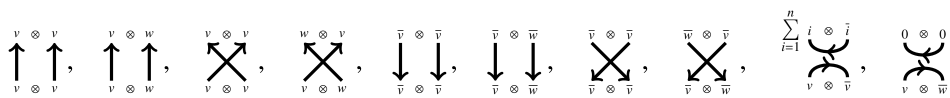
(a) *If $\mathrm{char}(\mathbb{K}) > \max\{r, s\}$ or $\mathrm{char}(\mathbb{K}) = 0$, then $e_{r,s}(n)$ is a quasi-idempotent.*

(b) *If $\mathrm{char}(\mathbb{K}) = p \leq \max\{r, s\}$, then $e_{r,s}(n)$ is nilpotent.*

Proof. The case $r + s = 1$ is clear, so we may assume now that $r + s \geq 2$.

Claim 1. $e_{r,s}(n) \in \ker(\Phi_{\mathrm{wBr}})$.

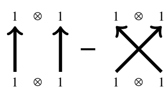
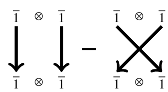
Proof of Claim 1. We want to use the diagrammatic presentation of $\mathcal{B}_{r,s}(\delta)$ from Remark 3.10. If we denote a basis of $V$ by $\{1, \ldots, n\}$ and its dual basis of $V^*$ by $\{\overline{1}, \ldots, \overline{n}\}$ (we assume throughout the proof that vectors of the form $\vec{v}, \vec{w} \in T_n^{r,s}$ that we use below have only tensor factors from either $\{1, \ldots, n\}$ or $\{\overline{1}, \ldots, \overline{n}\}$), then the action of $\mathcal{B}_{r,s}(\delta)$ on $T_n^{r,s}$ can locally be pictured as

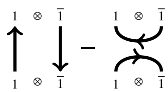


for $v, w \in \{1, \ldots, n\}$ with $v \neq w$. For example, the cap–cup generator sends a basis vector of $V \otimes V^*$ of the form $v \otimes \overline{v}$ to the full sum $\sum_{i=1}^n i \otimes \overline{i}$ and all other basis vectors to zero.

We need to show that an arbitrary basis vector $\vec{v} \in T_n^{r,s}$ is sent to zero by $e_{r,s}(n)$. For this purpose, we argue inductively, where the induction is on the total number $d = r + s$ of strands.

In case $d = 2$, we have $r = 2, s = 0$ or $r = 0, s = 2$ or $r = 1, s = 1$. Moreover, we have $n = 1$ and the only possible basis vectors $\vec{v} \in T_n^{r,s}$ in these cases are $1 \otimes 1$ or $\overline{1} \otimes \overline{1}$ or $1 \otimes \overline{1}$. Then

$e_{2,0}(1)(1 \otimes 1) =$ , $e_{0,2}(1)(\overline{1} \otimes \overline{1}) =$ ,

$e_{1,1}(1)(1 \otimes \overline{1}) =$ 

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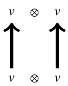
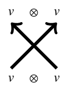
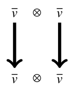
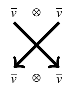
We see that all of these act as zero on a basis of $T_n^{r,s}$. Hence, they are all in the kernel.

Let $d > 2$ and let $\vec{v} = v_1 \otimes \cdots \otimes \overline{v}_{r+s}$. We need to show that $e_{r,s}(n)(\vec{v}) = 0$. We do a case-by-case check depending on the tensor factors of $\vec{v}$. For simplicity of notation, we assume that those tensor factors of $\vec{v}$ that we consider are next to each other (otherwise, we can permute them next to each other) and we only display the relevant part of $\vec{v}$. The cases are:

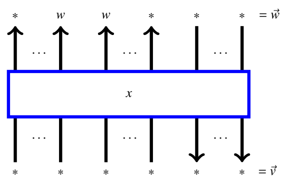
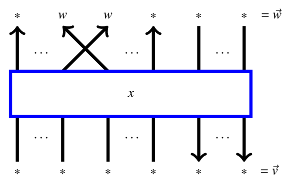
(i) $\vec{v}$ has tensor factors of the form $v \otimes v$ or $\overline{v} \otimes \overline{v}$. Then any primitive diagram $x$ acting nontrivially on $\vec{v}$ is locally of the following form.

Case $v \otimes v$ :  or  ; Case $\overline{v} \otimes \overline{v}$ :  or 

Note that, for each primitive diagram $x \in \mathcal{B}_{r,s}(\delta)$ that is locally as on the left-hand sides above, there is precisely one primitive diagram $\tilde{x} \in \mathcal{B}_{r,s}(\delta)$ that is locally as on the right-hand sides above and otherwise equal to $x$. These appear in $e_{r,s}(n)$ with different signs and their contributions cancel. This shows that $e_{r,s}(n)(\vec{v}) = 0$.

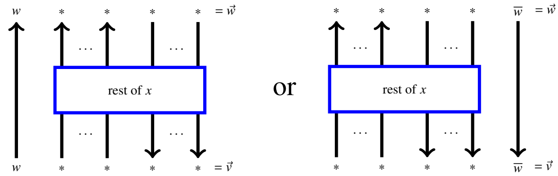
(ii) $\vec{v}$ has no entry pairs of the form $v \otimes v$ or $\overline{v} \otimes \overline{v}$. We fix a primitive diagram $x$ and do another case-by-case check depending on the matrix entry corresponding to a fixed pair $\vec{v}$ and $\vec{w} = x(\vec{v})$. We again assume that the tensor factors of $\vec{w}$ under consideration are next to each other.

– $\vec{w}$ has no tensor factors of the form $w \otimes w$ or $\overline{w} \otimes \overline{w}$. Then $\vec{w}$ (that is the contribution of $x$) is cancelled by a primitive diagram $\tilde{x}$ obtained from $x$ by applying an extra crossing at the corresponding position. Or, in pictures (for brevity, we only display the upwards oriented version, but the other case is completely similar):

$x =$  , $\tilde{x} =$ 

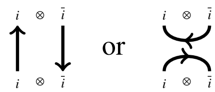
Here the $*$ represent arbitrary tensor factors (which are the same for $x$ and $\tilde{x}$). Since $x$ and $\tilde{x}$ appear with different signs in $e_{r,s}(n)$, these two terms cancel each other.

– There is a tensor factor $w$ (or $\overline{w}$) of $\vec{w}$ that appears isolated, that is, no other tensor factors of $\vec{w}$ are of the form $w$ or $\overline{w}$. Since $\vec{w} = x(\vec{v})$ is nonzero and we are not in case (i), there exists a unique connecting strand in $x$ from a bottom entry $w$ (or $\overline{w}$) to this isolated top entry. In pictures (where we for simplicity assume that this unique strand is on the left, respectively, right):




Here the entries $*$ mean arbitrary tensor factors that are neither $w$ nor $\overline{w}$. Now $e_{r,s}(n)(\vec{v}) = 0$ if and only if $e'_{r,s}(n)(\vec{v}') = 0$, where $\vec{v}'$ is obtained from $\vec{v}$ by removing the two isolated tensor factors and $e'_{r,s}(n)$ is obtained from $e_{r,s}(n)$ by first removing all summands which are not of the form as above and then removing the unique strand. Hence, we can argue now by induction.

– $\vec{w}$ has only entry pairs of the form $w \otimes \overline{w}$. Then the same is true for $\vec{v}$ (otherwise, we are in case (i)). Since $n = r + s - 1$, we know that there is at least one pair $i \otimes \overline{i}$ that appears in both $\vec{v}$ and $\vec{w}$. Similarly to case (i), the primitive diagram $x$ is locally of the following form.




Thus, for each such $x$, there is precisely one $\tilde{x}$ which is locally different from $x$ as illustrated above and identical to $x$ otherwise. Since $x$ and $\tilde{x}$ appear with different signs in $e_{r,s}(n)$, their contributions cancel.

These are all possible cases. Hence, all matrix coefficients of $e_{r,s}(n) \in \mathrm{End}_{\mathrm{U}_1}(T_n^d)$ are trivial and so Claim 1 follows. □

Claim 2. The $\mathbb{K}$-linear span of $e_{r,s}(n)$ equals $\ker(\Phi_{\mathrm{wBr}})$.

Proof of Claim 2. By Theorem 3.12, we see that $\mathcal{B}_{r,s}(\delta)$ surjects onto $\mathrm{End}_{\mathrm{U}_1}(T_n^{r,s})$. The dimension of $\mathrm{End}_{\mathrm{U}_1}(T_n^{r,s})$ is independent of $\mathbb{K}$ by Proposition 2.3. Thus, we may assume that $\mathbb{K} = \mathbb{C}$ to calculate the dimension. Now $\dim(\mathrm{End}_{\mathrm{U}_1}(T_n^{r,s})) = \dim(\mathcal{B}_{r,s}(\delta)) - 1$ by part (c) of Theorem 3.12: for $n = r + s - 1$, only the pair of Young diagrams with maximal numbers of columns is missing in the direct sum decomposition and the missing simple $\mathcal{B}_{r,s}(\delta)$-module has dimension one (in the semisimple case, $D_1^{\lambda,\overline{\mu}}$ has a basis parametrized by so-called up–down tableaux; see, for example, [10, Section 6]). Hence, $\dim(\ker(\Phi_{\mathrm{wBr}})) = 1$, independent of $\mathbb{K}$. Since $0 \neq e_{r,s}(n) \in \ker(\Phi_{\mathrm{wBr}})$ (by Claim 1), its $\mathbb{K}$-linear span equals $\ker(\Phi_{\mathrm{wBr}})$. □

By Claims 1 and 2, we have $e_{r,s}(n) \in \ker(\Phi_{\mathrm{wBr}})$ and $\dim(\ker(\Phi_{\mathrm{wBr}})) = 1$. Thus, $e_{r,s}(n)e_{r,s}(n) = ae_{r,s}(n)$ for some $a \in \mathbb{K}$. A direct computation shows that the scalar in front of the identity diagram of $e_{r,s}(n)e_{r,s}(n)$ is $r!s!$. Thus, we can divide by this value to get an 'honest' idempotent if and only if $\mathrm{char}(\mathbb{K}) > \max\{r, s\}$ or $\mathrm{char}(\mathbb{K}) = 0$. □

Remark 4.4. That $\dim(\ker(\Phi_{\mathrm{wBr}}))$ is independent of $\mathbb{K}$ was already obtained using quite different methods in [12, Corollary 7.2]. The explicit description of $\ker(\Phi_{\mathrm{wBr}})$ from Proposition 4.3 seems to be new, but is already implicitly contained in [10, Section 8].

### 4.2. The Brauer case.

Proposition 4.5. *Let $n = 2d - 2 = -\delta$. Then the $\mathbb{K}$-linear span of $E_d(n) \in \mathcal{B}_d(\delta)$ equals* $\ker(\Phi_{\mathrm{Br}})$. *Furthermore, the following holds.*

(a) *If* $\mathrm{char}(\mathbb{K}) > d$ *or* $\mathrm{char}(\mathbb{K}) = 0$*, then $E_d(n)$ is a quasi-idempotent.*

(b) *If* $\mathrm{char}(\mathbb{K}) = p \leq d$*, then $E_d(n)$ is nilpotent.*

Proof. We can argue *mutatis mutandis* as in the proof of Proposition 4.3. To be more precise, the analogue of Claim 1 works almost word-by-word as in the walled Brauer case. For the analogue of Claim 2, we use part (c) of Theorem 3.17, where the only summand missing in the $(\mathbf{U}_1, \mathcal{B}_d(\delta))$-bimodule decomposition is the one for the unique Young diagram with maximal number of columns (the corresponding simple $\mathcal{B}_d(\delta)$-module is one dimensional, which can be deduced from [20, Section 4]). For the analogue of the proof of (a) and (b), we note that the scalar in front of the identity diagram of $E_d(n)E_d(n)$ can be easily seen to be $d!$. □

Remark 4.6. Lehrer and Zhang described $\ker(\Phi_{\mathrm{Br}})$ for all $d \in \mathbb{Z}_{>0}$ and all $\delta \in \mathbb{Z}$ in the case $\mathbb{K} = \mathbb{C}$ in [35, Theorem 4.3]. In particular, they showed that $\ker(\Phi_{\mathrm{Br}})$ is generated (as an ideal) by an idempotent. They also argued in [35, Proposition 9.2] how their results generalize to the case of arbitrary $\mathbb{K}$, but with a less explicit description than we give above.

**4.3. Application to semisimplicity.** The description of the kernels will be an important tool in the proof of the semisimplicity criteria (see Theorems 6.1 and 7.1), because of the following result.

Proposition 4.7. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be algebras over $\mathbb{K}$ and let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a surjective algebra homomorphism such that* $\ker(\Phi)$ *is spanned as a $\mathbb{K}$-vector space by an idempotent $e$. Then semisimplicity of $\mathcal{A}_2$ implies semisimplicity of $\mathcal{A}_1$.*

Proof. Clearly, $\mathcal{A}_1 / \ker(\Phi) \cong \mathcal{A}_2$ as algebras. Now, for any algebra $\mathcal{A}$ with ideal $I$ such that $\mathcal{A}/I$ is semisimple, we have $I \supset \mathrm{Rad}(\mathcal{A})$ (here $\mathrm{Rad}(\mathcal{A})$ means the Jacobson radical of $\mathcal{A}$). Assuming that $\mathcal{A}_2$ is semisimple, we have $\mathrm{span}_{\mathbb{K}}(\{e\}) = \ker(\Phi) \supset \mathrm{Rad}(\mathcal{A}_1)$. Since $e$ is an idempotent, it follows that $\mathrm{Rad}(\mathcal{A}_1) = 0$. □

## 5. Semisimplicity: the Hecke algebras of types A and B

Theorem 5.1 (Semisimplicity criteria for the Hecke algebras of types **A** and **B**). *$\mathcal{H}_d^{\boldsymbol{A}}(q)$ and $\mathcal{H}_d^{\boldsymbol{B}}(q)$ are semisimple if and only if one of the following conditions holds:*

(1) $\mathrm{char}(\mathbb{K}) > d$ *and* $q = 1$*;*

(2) $\mathrm{char}(\mathbb{K}) = 0$ *and* $q = 1$*;*

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(3) $q \in \mathbb{K}^*, q \neq 1$ *is a root of unity with* $\mathrm{ord}(q^2) > d$*;*
(4) $q \in \mathbb{K}^*, q \neq 1$ *is a nonroot of unity.*

The proof of Theorem 5.1 requires some preparation.

**5.1. The Schur–Weyl dual story.** Let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{gl}_m)$, $V$ and $T_n^d = V^{\otimes d}$ be as before in Theorem 3.4. Note that $V$ corresponds to a Young diagram with precisely one node (via Conventions 3.1). Thus, by Proposition 2.2, we can use the classical Littlewood–Richardson rule to see that a Weyl factor $\Delta_q(\lambda)$ appears in a $\Delta_q$-filtration of $T_n^d$ if and only if $\lambda \in \Lambda^+(d)$ (hence, the Young diagram associated to $\lambda$ has $d$ nodes). Note that $V \in \mathcal{T}$ and, hence, also $T_n^d \in \mathcal{T}$ by Proposition 2.3. Thus, by Lemma 2.4, the semisimplicity of $T_n^d$ is equivalent to the condition that all of its occurring Weyl factors $\Delta_q(\lambda)$ are simple $\mathbf{U}_q$-modules.

Proposition 5.2. *We have the following.*

(a) *Let* $\mathrm{char}(\mathbb{K}) > 0$ *and* $q = 1$*. Then* $T_n^d$ *is a semisimple* $\boldsymbol{U}_1$*-module if and only if* $\mathrm{char}(\mathbb{K}) > d$*.*
(b) *Let* $\mathrm{char}(\mathbb{K}) = 0$ *and* $q = 1$*. Then* $T_n^d$ *is always a semisimple* $\boldsymbol{U}_1$*-module.*
(c) *Let* $q \in \mathbb{K}^*$ *be a root of unity. Then* $T_n^d$ *is a semisimple* $\boldsymbol{U}_q$*-module if and only if* $\mathrm{ord}(q^2) > d$*.*
(d) *Let* $q \in \mathbb{K}^*$ *be a nonroot of unity. Then* $T_n^d$ *is always a semisimple* $\boldsymbol{U}_q$*-module.*

Proof. *'If' of* (a). Let $\mathrm{char}(\mathbb{K}) = p > d$, $q = 1$ and $\lambda \in \Lambda^+(d)$. For the positive roots $\alpha \in \Phi^+$ of the form $\alpha = \varepsilon_i - \varepsilon_j$ with $1 \leq i < j \leq n$,

$$\langle \lambda + \rho, (\varepsilon_i - \varepsilon_j)^\vee \rangle = n - i + \lambda_i - (n - j + \lambda_j) = j - i + \lambda_i - \lambda_j \leq n + d < n + p.$$

Hence, *JSF* from (2.4) for $\Delta_1(\lambda)$ gives

$$\sum_{k' \geq 1} \mathrm{ch}(\Delta_1^{k'}(\lambda)) = -\sum_{i<j} \sum_{k \in \mathbb{Z}_{>0}} \nu_p(kp) \chi(\lambda - kp(\varepsilon_i - \varepsilon_j)), \tag{5.1}$$

where the right-hand sum runs over all $1 \leq i < j \leq n$ and $k \in \mathbb{Z}_{>0}$ such that $kp < j - i + \lambda_i - \lambda_j$. We claim that the sum in (5.1) is zero.

For this purpose, fix $1 \leq i < j \leq n$ and $k \in \mathbb{Z}_{>0}$ and assume that $\chi(\lambda - kp(\varepsilon_i - \varepsilon_j))$ appears on the right-hand side in (5.1). We first note that $\lambda_j = 0$: if $\lambda_j > 0$, then the Young diagram of $\lambda$ contains at least $j - 1 + \lambda_i$ nodes, that is $j - 1 + \lambda_i \leq d$. But then also $j - i + \lambda_i - \lambda_j \leq d < p \leq kp$ and $\chi(\lambda - kp(\varepsilon_i - \varepsilon_j))$ does not occur in (5.1), which gives a contradiction. So we have $kp < j - i + \lambda_i$.

Moreover, $(\lambda + \rho - kp(\varepsilon_i - \varepsilon_j))_i = \lambda_i + n - i - kp$. Note that $i < i - \lambda_i + kp < j$: the left inequality follows from $d < p$, while the right follows from $kp < j - i + \lambda_i$. Furthermore, the $(i' = i - \lambda_i + kp)$th coordinate of $\lambda + \rho - kp(\varepsilon_i - \varepsilon_j)$ is $n - i + \lambda_i - kp$ (note that $\lambda_{i'} = 0$: as above, $\lambda_{i'} > 0$ would imply $i' - 1 + \lambda_i = i - 1 + kp \leq d < p \leq kp$, which is clearly impossible). Thus, it equals the $i$th coordinate of $\lambda + \rho - kp(\varepsilon_i - \varepsilon_j)$. Set $\mu = \lambda + \rho - kp(\varepsilon_i - \varepsilon_j)$. Then

$$\mu = (\mu_1, \ldots, \mu_{i-1}, \boxed{\lambda_i + n - i + kp}, \mu_{i+1}, \ldots, \mu_{i'-1}, \boxed{\lambda_i + n - i + kp}, \mu_{i'+1}, \ldots, \mu_j, \ldots, \mu_n).$$

Thus, $\mu$ is a singular $\mathbf{U}_q$-weight. This, by (2.1), implies that $\chi(\lambda - kp(\varepsilon_i - \varepsilon_j)) = 0$.

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Altogether, we have proved that the right-hand side of (5.1) is zero. Hence, $\Delta_1(\lambda)$ is a simple $\mathbf{U}_1$-module by Theorem 2.9 (for all $\lambda \in \Lambda^+(d)$), which shows the 'if' part of (a).

*'Only if' of* (a). By the above observation, $T_n^d$ has Weyl factors which are of the form $\Delta_1(d\varepsilon_1)$ and $\Delta_1((d-1)\varepsilon_1 + \varepsilon_2)$. If we have $\mathrm{char}(\mathbb{K}) = p \leq d$ and $q = 1$, then either $\Delta_1(d\varepsilon_1)$ or $\Delta_1((d-1)\varepsilon_1 + \varepsilon_2)$ is a nonsimple $\mathbf{U}_1$-module. To see this, we use *JSF* and calculate

$$d\varepsilon_1 + \rho - p(\varepsilon_1 - \varepsilon_2) = (\boxed{d+n-1-p}, \boxed{n-2+p}, n-3, \ldots, 2, 1, 0).$$

Since $p \leq d$ implies $d+n-1-p \neq n-j$ for $j \geq 2$ and clearly $n-2+p \neq n-j$, this gives a nontrivial contribution (in the case where $d+n-1-p \neq n-2+p$; otherwise take $(d-1)\varepsilon_1 + \varepsilon_2$ instead of $d\varepsilon_1$) due to the fact that cancellations do not occur in type $\mathbf{A}_{m-1}$; see Remark 2.11. Alternatively, one can use $\mathfrak{sl}_2$-theory, where the combinatorics in the $\mathfrak{sl}_2$ case is as in [6, Proposition 2.20]. Thus, we see that the 'only if' part holds true in (a).

(b) This follows from the fact that the category of $\mathbf{U}_1$-modules is semisimple in the classical case; see, for example, Remark 2.1.

(c) *Mutatis mutandis* as in the proof of (a): we use *JSF* from (2.2) or (2.3) (replacing $p$ by $\mathrm{ord}(q^2) = \ell$) and then the same arguments as in (a) work.

(d) This follows again directly from the semisimplicity of the corresponding categories of $\mathbf{U}_q$-modules; see, for example, Remark 2.1.

We have proved the proposition. □

### 5.2. Proof of the semisimplicity criterion for $\mathcal{H}_d^{\mathbf{A}}(q)$ and $\mathcal{H}_d^{\mathbf{B}}(q)$.

Proof of Theorem 5.1. *Case* $\mathcal{H}_d^{\mathbf{A}}(q)$. We choose $n \geq d$ and the conclusion follows from Theorem 3.4 together with Corollary 2.6 and Proposition 5.2.

*Case* $\mathcal{H}_d^{\mathbf{B}}(q)$. By Theorem 3.6, we choose $\frac{1}{2}n \geq d$. We consider the Schur–Weyl dual situation with $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{gl}_m \oplus \mathfrak{gl}_m)$ acting on $T_n^d$. Note that $V = \Delta_q(\omega_1, 0) \oplus \Delta_q(0, \omega_1)$. Hence, $\Delta_q(\lambda, \mu)$ is a Weyl factor of $T_n^d$ if and only if $(\lambda, \mu) \in \Lambda^+(d_1) \times \Lambda^+(d_2)$ with $d_1 + d_2 = d$. Moreover, $\Delta_q(\lambda, \mu)$ is a simple $\mathbf{U}_q$-module if and only if $\Delta_q(\lambda)$ and $\Delta_q(\mu)$ are simple $\mathbf{U}_q(\mathfrak{gl}_m)$-modules. Thus, by Corollary 2.6, $\mathcal{H}_d^{\mathbf{B}}(q)$ is semisimple if and only if $\Delta_q(\lambda)$ is a simple $\mathbf{U}_q(\mathfrak{gl}_m)$-module for all $\lambda \in \Lambda^+(d')$ with $1 \leq d' \leq d$. By Theorem 5.1, this is precisely the case when $\mathcal{H}_{d'}^{\mathbf{A}}(q)$ is semisimple for all $1 \leq d' \leq d$, which, by Theorem 5.1 again, is equivalent to the semisimplicity of $\mathcal{H}_d^{\mathbf{A}}(q)$.

The criterion follows. □

Remark 5.3. These semisimplicity criteria are not new, but were found using different methods: for $\mathcal{H}_d^{\mathbf{A}}(q)$ it can be deduced from the work of Gyoja and Uno [21] (they worked over $\mathbb{C}$, but their arguments can be generalized to any field $\mathbb{K}$; see also [39, page 12, Exercise 10]). For $\mathcal{H}_d^{\mathbf{B}}(q)$, it was first found in [13, Theorem 5.5].

Remark 5.4. Similar as in the case of $\mathcal{H}_d^{\mathbf{B}}(q)$, one could also prove semisimplicity criteria for Ariki–Koike algebras using the Schur–Weyl dualities mentioned in

Remark 3.7 (of course, these criteria are known; see, for example, [7, Main Theorem], but they again fit into the same framework). For brevity and to avoid some technicalities, we do not discuss this in more detail here. We point out that the *JSF* (in the related, but slightly different framework of cyclotomic $q$-Schur algebras) was already successfully applied in [37] in the study of blocks of Ariki–Koike algebras.

REMARK 5.5. Our methods also apply to tensor products of arbitrary fundamental representations. For example, given $\vec{k} = (k_1, \ldots, k_d)$ with $k_i \in \{1, \ldots, n-1\}$, we could consider algebras of the form $\mathrm{End}_{\mathbf{U}_q}(T_n^{\vec{k}}) = \mathrm{End}_{\mathbf{U}_q}(\Delta_q(\omega_{k_1}) \otimes \cdots \otimes \Delta_q(\omega_{k_d}))$. These algebras are known as *spider algebras* in the sense of Kuperberg [33]. The semisimplicity criterion of $\mathrm{End}_{\mathbf{U}_q}(T_n^{\vec{k}})$ is not known, but it should be possible to deduce it from our setup.

## 6. Semisimplicity: the walled Brauer algebra

For the whole section let $r, s \in \mathbb{Z}_{\geq 0}$, not both zero. Choose $\delta$ and $\delta_p$ (recalling that $\delta_0 = |\delta|$) in accordance with Conventions 3.11.

THEOREM 6.1 (Semisimplicity criterion for the walled Brauer algebra). *$\mathcal{B}_{r,s}(\delta)$ is semisimple if and only if one of the following conditions holds.*

(1) $\delta_p \neq 0$, $\mathrm{char}(\mathbb{K}) = p$ *and* $r + s \leq \min\{\delta_p + 1, p - \delta_p + 1\}$.
(2) $\delta_0 \neq 0$, $\mathrm{char}(\mathbb{K}) = 0$ *and* $r + s \leq \delta_0 + 1$.
(3) $\delta_p = 0$, $\mathrm{char}(\mathbb{K}) = p \geq 5$ *and* $(r, s) \in \{(2, 1), (1, 2), (3, 1), (1, 3)\} \cup \{(a, 0), (0, a) \mid a < p\}$.
(4) $\delta_3 = 0$, $\mathrm{char}(\mathbb{K}) = 3$ *and* $(r, s) \in \{(2, 1), (1, 2)\} \cup \{(a, 0), (0, a) \mid a < 3\}$.
(5) $\delta_2 = 0$, $\mathrm{char}(\mathbb{K}) = 2$ *and* $(r, s) \in \{(1, 0), (0, 1)\}$.
(6) $\delta_0 = 0$, $\mathrm{char}(\mathbb{K}) = 0$ *and* $(r, s) \in \{(2, 1), (1, 2), (3, 1), (1, 3)\} \cup \{(a, 0), (0, a) \mid a \in \mathbb{Z}_{>0}\}$.

The proof of Theorem 6.1 again requires some preparation and is split into several lemmas.

### 6.1. The Schur–Weyl dual story: from $(r, s)$ to $(r + 1, s + 1)$.

Let $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{gl}_m)$, $V$ and $T_n^{r,s}$ be as in Theorem 3.12. As before, $V, V^* \in \mathcal{T}$ and so is $T_n^{r,s}$ by Proposition 2.3. Recall that we can calculate the Weyl factors of $T_n^{r,s}$ as in the classical case.

PROPOSITION 6.2. *If $T_n^{r,s}$ is a nonsemisimple $\mathbf{U}_1$-module, then so is $T_n^{r+1,s+1}$.*

PROOF. A direct computation shows that $\Delta_1(0) \cong \mathbb{K}$ is a Weyl factor of $T_n^{1,1}$. Because of this and $T_n^{r+1,s+1} \cong T_n^{r,s} \otimes T_n^{1,1}$, we have that any Weyl factor of $T_n^{r,s}$ is also a Weyl factor of $T_n^{r+1,s+1}$. The conclusion follows then from Lemma 2.4. □

COROLLARY 6.3. *Let* $\mathrm{char}(\mathbb{K}) = p$. *Then $\mathcal{B}_{r,s}(\delta)$ is semisimple if and only if $\mathcal{B}_{s,r}(\delta)$ is semisimple.*

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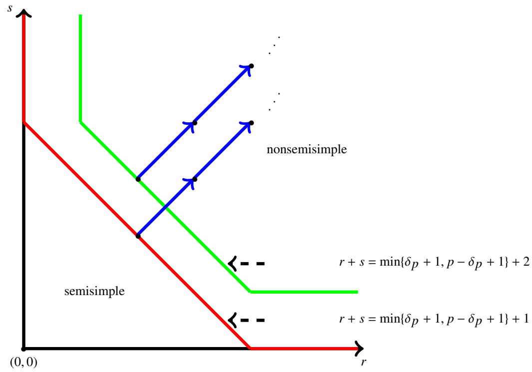
PROOF. Note that $(T_n^{r,s})^* \cong T_n^{s,r}$ as $\mathbf{U}_1$-modules. Thus, $T_n^{r,s}$ is a semisimple $\mathbf{U}_1$-module if and only if $T_n^{s,r}$ is a semisimple $\mathbf{U}_1$-module. Choose $n \geq r+s+2$ with $n \equiv \delta_p \bmod p$. Then the statement follows directly from Theorem 3.12, Proposition 6.2 and Theorem 2.5. □

COROLLARY 6.4. *Let* $\mathrm{char}(\mathbb{K}) = p$. *If* $\mathcal{B}_{r,s}(\delta)$ *is nonsemisimple, then so is* $\mathcal{B}_{r+1,s+1}(\delta)$.

PROOF. We can then proceed similarly as in the proof of Corollary 6.3. □

Corollary 6.4 fails for $\mathrm{char}(\mathbb{K}) = 0$, because we cannot choose $n$ 'big enough'. But for $\mathrm{char}(\mathbb{K}) = p$ this allows us to prove nonsemisimplicity of $\mathcal{B}_{r,s}(\delta)$ by proving nonsemisimplicity for certain 'boundary values'. For example, if $\delta_p \neq 0$, then this can be illustrated as




The 'boundary line' (bottom) where the semisimplicity fails is illustrated above. We also displayed the passage from $(r, s)$ to $(r+1, s+1)$ provided by Corollary 6.4. Note that we have to check additionally points on a line 'above the boundary line' (top).

**6.2. The $\delta_p \neq 0$ case.** We assume in this subsection that $\mathrm{char}(\mathbb{K}) = p$ and $\delta_p \neq 0$.

LEMMA 6.5. *If* $r+s < \delta_p + 1$ *and* $r+s \leq p-\delta_p+1$, *then* $\mathcal{B}_{r,s}(\delta)$ *is semisimple.*

PROOF. We consider $T_n^{r,s}$ for $n = \delta_p$. Any Weyl factor $\Delta_1(\lambda)$ of $T_n^{r,s}$ satisfies

$$\langle \lambda+\rho, (\varepsilon_i - \varepsilon_j)^\vee \rangle \leq \langle r\varepsilon_1 - s\varepsilon_n + \rho, (\varepsilon_1 - \varepsilon_n)^\vee \rangle \leq \delta_p - 1 + r + s \leq p.$$

Here the last inequality follows from the assumption that $r+s \leq p-\delta_p+1$. This means that all Weyl factors of $T_n^{r,s}$ are simple $\mathbf{U}_1$-modules, since there is no positive root $\alpha \in \Phi^+$ which gives a contribution to *JSF*. As usual, the statement follows from Theorem 3.12 (note that we have $r+s < \delta_p+1$ and the needed isomorphism holds) and Corollary 2.6. □

LEMMA 6.6. *If* $r+s > p-\delta_p+1$ *with* $r, s \geq 1$, *then* $\mathcal{B}_{r,s}(\delta)$ *is nonsemisimple.*

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Proof. Set $n = \delta_p$ and consider again $T_n^{r,s}$. Let first $s = 1$ and assume that $r > p - \delta_p$ with $r < p$. As before, we only need to give one Weyl factor $\Delta_1(\lambda)$ which is a nonsimple $\mathbf{U}_1$-module. We take $\lambda = (p - \delta_p + 1)\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{r-p+\delta_p} - \varepsilon_n$. Then $\alpha = \varepsilon_1 - \varepsilon_n$ contributes to *JSF* because

$$\lambda + \rho - p(\varepsilon_1 - \varepsilon_n) = (\boxed{0}, \delta_p - 1, \delta_p - 2, \ldots, p - r, p - r - 2, \ldots, 2, 1, \boxed{p-1})$$

(note that $r < p$). This is a regular $\mathbf{U}_1$-weight because $\delta_p < p$. Again, cancellation cannot occur; see Remark 2.11. Thus, $T_n^{r,s}$ is a nonsemisimple $\mathbf{U}_1$-module.

We now verify nonsemisimplicity for the 'boundary values'. Assume that $r, s > 1$. Set

$$\flat_1 = \{(r, s) \mid r + s = (p - \delta_p + 1) + 1\}, \quad \flat_2 = \{(r, s) \mid r + s = (p - \delta_p + 1) + 2\}.$$

A direct computation using *JSF* shows that $\lambda = r\varepsilon_1 - s\varepsilon_n$ is a Weyl factor $\Delta_1(\lambda)$ of $T_n^{r,s}$ that is a nonsimple $\mathbf{U}_1$-module for all pairs $(r, s) \in \flat_1 \cup \flat_2$ (the positive root making *JSF* nonzero is $\alpha = \varepsilon_1 - \varepsilon_n$). For those $(r, s)$, we have that $T_n^{r,s}$ is a nonsemisimple $\mathbf{U}_1$-module.

By Theorem 3.12 and Corollary 2.6, we see that $\mathcal{B}_{r,s}(\delta)$ is nonsemisimple under the same conditions (surjectivity in Theorem 3.12 suffices since semisimple algebras have semisimple quotients). By Lemma 3.9, we additionally see that $\mathcal{B}_{r,1}(\delta)$ is nonsemisimple for $r \geq p$. Thus, the statement follows from Corollaries 6.4 and 6.3. □

Lemma 6.7. *If $r + s > \delta_p + 1$ with $r, s \geq 1$, then $\mathcal{B}_{r,s}(\delta)$ is nonsemisimple.*

Proof. Very similar to the proof of Lemma 6.6. This time we take $n = p + \delta_p$ and we consider $T_n^{r,s}$. The 'boundary values' for which we need to check nonsemisimplicity are

$$\flat_1 = \{(r, 1) \mid \delta_p + 1 \leq r < p\},$$
$$\flat_2 = \{(r, s) \mid r + s = (\delta_p + 1) + 1,\ s \geq 2\}, \quad \flat_3 = \{(r, s) \mid r + s = (\delta_p + 1) + 2,\ s \geq 2\}.$$

For these we directly verify, using again *JSF*, that $T_n^{r,s}$ is a nonsemisimple $\mathbf{U}_1$-module and the statement follows similarly as before. Since the arguments are straightforward, we only list a Weyl factor $\Delta_1(\lambda)$ that is a nonsimple $\mathbf{U}_1$-module for each case (together with the positive roots giving a nonzero contribution to *JSF*).

For $\flat_1$: $\lambda = (r - \delta_p)\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{\delta_p+1} - \varepsilon_n$, positive root: $\alpha = \varepsilon_{\delta_p+1} - \varepsilon_n$,
For $\flat_2$: $\lambda = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r - \varepsilon_{n-s+1} - \cdots - 2\varepsilon_n$, positive root: $\alpha = \varepsilon_r - \varepsilon_{n-s+1}$,
For $\flat_3$: $\lambda = 2\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{r-1} - \varepsilon_{n-s+1} - \cdots - 2\varepsilon_n$, positive root: $\alpha = \varepsilon_{r-1} - \varepsilon_{n-s+1}$.

Again, no cancellations occur by Remark 2.11 and the statement follows as usual. □

**6.3. The $\delta_p = 0$ case.** We assume in this subsection that $\mathrm{char}(\mathbb{K}) = p$ and $\delta_p = 0$.

Lemma 6.8. *Let $p \geq 3$. Then $\mathcal{B}_{2,1}(\delta)$ is semisimple. If $p \geq 5$, then $\mathcal{B}_{3,1}(\delta)$ is also semisimple.*

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Proof. Set $n = p$ and consider $T_n^{r,s}$ for $r = 2, 3$ and $s = 1$. By Theorem 3.12 and Corollary 2.6, it suffices to check that $T_n^{r,s}$ has only Weyl factors $\Delta_1(\lambda)$ which are simple $\mathbf{U}_1$-modules.

*Case* $r = 2$. The Weyl factors of $T_n^{r,s}$ are $\Delta_1(2\varepsilon_1 - \varepsilon_p)$, $\Delta_1(\varepsilon_1 + \varepsilon_2 - \varepsilon_p)$ and $\Delta_1(\varepsilon_1)$. The third is clearly a simple $\mathbf{U}_1$-module and it remains to verify the same for the other two factors. As before, we want to use *JSF*.

- If $\lambda = 2\varepsilon_1 - \varepsilon_p$, then the only possible positive root $\alpha \in \Phi^+$ that contributes to the corresponding *JSF* is $\alpha = \varepsilon_1 - \varepsilon_p$ (and it contributes only once). But
$$\lambda + \rho - p(\varepsilon_1 - \varepsilon_p) = (1, p-2, p-3, \ldots, 2, 1, p-1)$$
is a singular $\mathbf{U}_1$-weight (because $p \geq 3$). Thus, *JSF* of $\Delta_1(2\varepsilon_1 - \varepsilon_p)$ is zero.
- Similarly, if $\lambda = \varepsilon_1 + \varepsilon_2 - \varepsilon_p$, then, as before, the only positive root $\alpha \in \Phi^+$ we need to consider is $\alpha = \varepsilon_1 - \varepsilon_p$. But
$$\lambda + \rho - p(\varepsilon_1 - \varepsilon_p) = (0, p-1, p-3, \ldots, 2, 1, p-1)$$
is a singular $\mathbf{U}_1$-weight. Hence, *JSF* of $\Delta_1(\varepsilon_1 + \varepsilon_2 - \varepsilon_p)$ is zero.

Thus, $T_n^{r,s}$ is a semisimple $\mathbf{U}_1$-module.

*Case* $r = 3$. We get the Weyl factors $\Delta_1(3\varepsilon_1 - \varepsilon_p)$, $\Delta_1(2\varepsilon_1 + \varepsilon_2 - \varepsilon_p)$, $\Delta_1(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_p)$, $\Delta_1(\varepsilon_1 + \varepsilon_2)$ and $\Delta_1(2\varepsilon_1)$. We proceed as above. For $\lambda = 3\varepsilon_1 - \varepsilon_p$, the only positive roots $\alpha \in \Phi^+$ we need to consider for *JSF* are $\alpha = \varepsilon_1 - \varepsilon_p$ and $\alpha = \varepsilon_1 - \varepsilon_{p-1}$. Both contribute only one term and
$$\lambda + \rho - p(\varepsilon_1 - \varepsilon_p) = (2, p-2, p-3, \ldots, 2, 1, p-1),$$
$$\lambda + \rho - p(\varepsilon_1 - \varepsilon_{p-1}) = (2, p-1, p-3, \ldots, 2, p+1, -1).$$
Since $p \geq 5$, both are singular $\mathbf{U}_1$-weights. Similar for the remaining Weyl factors and we omit the calculation for brevity. We only note that one has to consider $\alpha = \varepsilon_1 - \varepsilon_p$ for $\lambda = 2\varepsilon_1 + \varepsilon_2 - \varepsilon_p$, $\lambda = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_p$ and $\lambda = 2\varepsilon_1$, while for $\lambda = \varepsilon_1 + \varepsilon_2$ no positive root $\alpha \in \Phi^+$ needs to be considered.

The lemma follows. □

Lemma 6.9. *Assume that* $r, s \geq 1$.

(a) *If* $p \geq 5$, *then* $\mathcal{B}_{r,s}(\delta)$ *is semisimple if and only if* $(r, s) \in \{(2, 1), (1, 2), (3, 1), (1, 3)\}$.
(b) *If* $p = 3$, *then* $\mathcal{B}_{r,s}(\delta)$ *is semisimple if and only if* $(r, s) \in \{(2, 1), (1, 2)\}$.
(c) *If* $p = 2$, *then* $\mathcal{B}_{r,s}(\delta)$ *is never semisimple.*

Proof. We only prove (a) and leave the other (completely similar) cases to the reader.

Because of Corollaries 6.4 and 6.3, it suffices to check that $\mathcal{B}_{r,s}(\delta)$ is nonsemisimple for $(r, s) = (1, 1)$ (difference 0), $(r, s) = (3, 2)$ (difference 1), $(r, s) = (4, 2)$ (difference 2) and $(r, s) = (r, 1)$ for $4 \leq r$ (difference $\geq 3$).

As before, let $n = p$ and consider $T_n^{r,s}$. Hence, it remains to find a Weyl factor $\Delta_1(\lambda)$ of $T_n^{r,s}$ which is a nonsimple $\mathbf{U}_1$-module. We list such factors in the following. Since cancellations do not occur, see Remark 2.11, this suffices to show that the corresponding *JSF* is nonzero.

- The Weyl factor $\Delta_1(\varepsilon_1 - \varepsilon_p)$ of $T_n^{r,s}$ for $(r, s) = (1, 1)$ is a nonsimple $\mathbf{U}_1$-module:

$$\varepsilon_1 - \varepsilon_p + \rho - p(\varepsilon_1 - \varepsilon_p) = (\boxed{0}, p-2, p-3, \ldots, 2, 1, \boxed{p-1}).$$

- The Weyl factor $\Delta_1(2\varepsilon_1 + \varepsilon_2 - 2\varepsilon_p)$ of $T_n^{r,s}$ for $(r, s) = (3, 2)$ is a nonsimple $\mathbf{U}_1$-module:

$$2\varepsilon_1 + \varepsilon_2 - 2\varepsilon_p + \rho - p(\varepsilon_2 - \varepsilon_p) = (p+1, \boxed{-1}, p-3, \ldots, 2, 1, \boxed{p-2}).$$

- The Weyl factor $\Delta_1(3\varepsilon_1 + \varepsilon_2 - 2\varepsilon_p)$ of $T_n^{r,s}$ for $(r, s) = (4, 2)$ is a nonsimple $\mathbf{U}_1$-module:

$$3\varepsilon_1 + \varepsilon_2 - 2\varepsilon_p + \rho - p(\varepsilon_2 - \varepsilon_p) = (p+2, \boxed{-1}, p-3, \ldots, 2, 1, \boxed{p-2}).$$

- The Weyl factor $\Delta_1((r-2)\varepsilon_1 + 2\varepsilon_2 - \varepsilon_p)$ of $T_n^{r,s}$ for $(r, s) = (r, 1)$ with $4 \leq r$ is a nonsimple $\mathbf{U}_1$-module:

$$(r-2)\varepsilon_1 + 2\varepsilon_2 - \varepsilon_p + \rho - p(\varepsilon_2 - \varepsilon_p) = (p+r-3, \boxed{0}, p-3, \ldots, 2, 1, \boxed{p-1}).$$

  Note that $4 \leq r$ ensures that $(r-2)\varepsilon_1 + 2\varepsilon_2 - \varepsilon_p$ occurs in $T_n^{r,s}$ (the 2 in front of $\varepsilon_2$ is needed for $\alpha = \varepsilon_2 - \varepsilon_p$ to give a contribution to *JSF*).

As in the proof of Lemma 3.9, semisimple algebras have semisimple quotients. Hence, surjectivity in Theorem 3.12 and Corollary 2.6 provides the 'only if' part of (a). Thus, we have proven the statement, because the 'if' part of (a) follows from Lemma 6.8 and Corollary 6.3. □

### 6.4. Proof of the semisimplicity criterion for $\mathcal{B}_{r,s}(\delta)$.

Proof of Theorem 6.1. (1) The 'only if' part of (a) follows from Lemmas 3.9, 6.6 and 6.7

By Lemma 6.5, the only missing case for the 'if' part is the case $r + s = \delta_p + 1$ and $p > \max\{r, s\}$, since in this case the corresponding Schur–Weyl duality gives only a surjection. As in the proof of Lemma 6.5, we see that $T_n^{r,s}$ for $n + 1 = r + s = \delta_p + 1$ is a semisimple $\mathbf{U}_1$-module. Thus, by Theorems 3.12 and 2.5, we have that the algebra $\mathcal{B}_{r,s}(\delta)$ has $\mathrm{End}_{\mathbf{U}_1}(T_n^{r,s})$ as a semisimple quotient. We have calculated $\ker(\Phi_{\mathrm{wBr}})$ in Proposition 4.3 from above: $\ker(\Phi_{\mathrm{wBr}})$ is one dimensional and spanned by the idempotent $e_{r,s}(n)$. The conclusion follows from Proposition 4.7.

(2) We can use Theorem A.3. That is, the statement in (2) can be obtained from the statement in (1) by 'taking the limit $p \to \infty$'.

(3), (4) and (5) Directly from Lemma 6.9 and Theorem 5.1 (for the cases where we have either $r = 0$ or $s = 0$).

(6) Analogous to (2) by Theorem A.3, but using (3) instead of (1).

This finishes the proof. □

Remark 6.10. The semisimplicity criterion for $\mathcal{B}_{r,s}(\delta)$ is again of course not new: it was already discussed in [11, Theorem 6.3], but using a very different approach.

Remark 6.11. Our approach works perfectly fine for the quantized walled Brauer algebras as well. The only difference is that one has to consider the versions (2.2) and (2.3) of *JSF* instead of (2.4). For brevity, we do not discuss the details here.

## 7. Semisimplicity: the Brauer algebra

Let $d \in \mathbb{Z}_{>0}$. Choose $\delta$ and $\delta_p$ (recalling that $\delta_0 = |\delta|$) again as in Conventions 3.11.

Theorem 7.1 (Semisimplicity criterion for the Brauer algebra). *$\mathcal{B}_d(\delta)$ is semisimple if and only if one of the following conditions holds:*

(1) $\delta_p \neq 0$ *odd,* $\mathrm{char}(\mathbb{K}) = p > 2$ *and* $d \leq \min\{\delta_p + 1, \frac{1}{2}(p - \delta_p + 2)\}$*;*
(2) $\delta_p \neq 0$ *even,* $\mathrm{char}(\mathbb{K}) = p > 2$ *and* $d \leq \min\{\delta_p + 1, p - \delta_p + 3, p - 1\}$*;*
(3) $\delta_0 \neq 0$, $\mathrm{char}(\mathbb{K}) = 0$ *and* $d \leq \delta_0 + 1$*;*
(4) $\delta_p = 0$, $\mathrm{char}(\mathbb{K}) = p > 2$, $d \in \{1, 3, 5\}$ *and* $d < p$*;*
(5) $\delta_0 = 0$, $\mathrm{char}(\mathbb{K}) = 0$ *and* $d \in \{1, 3, 5\}$*;*
(6) $\mathrm{char}(\mathbb{K}) = 2$ *and* $d = 1$.

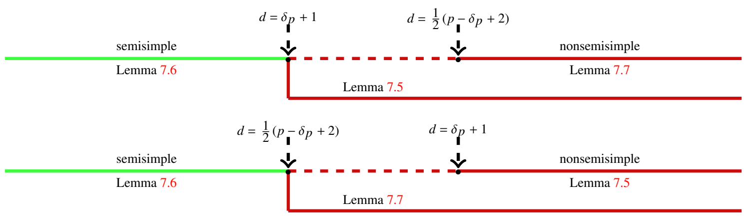
We split the proof of Theorem 7.1 into several lemmas.

Remark 7.2. Note the difference between (1) and (2): the restriction $d \leq \frac{1}{2}(p - \delta_p + 2)$ in the odd case is in general stronger than the restriction $d \leq p - \delta_p + 3$ in the even case. The reason for this is that odd $\delta$ corresponds via Schur–Weyl–Brauer duality to $\mathbf{U}_1(\mathfrak{so}_{2m+1})$ whereas even $\delta$ corresponds to $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{so}_{2m})$. In the latter case the Brauer algebra $\mathcal{B}_d(\delta)$ does not control $\mathrm{End}_{\mathbf{U}_1}(T_n^d)$ well enough since there is a nontrivial automorphism of the Dynkin diagram; see Section 3 (which we use below in the proof of Lemma 7.8). In particular, the semisimplicity in the even case is much harder to prove than in the odd case.

**7.1. A summary of the proof.** The proof of Theorem 7.1 is slightly involved. For the convenience of the reader we summarize its proof. We note that the proof itself is mostly smooth—except for a short list of special cases coming from the fact that the types $\mathbf{B}_m$, $\mathbf{C}_m$ and $\mathbf{D}_m$ are 'special' for small $m$ (we tend to omit the calculations for these for brevity).

First we assume that $\mathrm{char}(\mathbb{K}) = p > 2$, where we use Lemma 3.15 to further assume that $p > d$. We can deduce the case $\mathrm{char}(\mathbb{K}) = 0$ from it by trace form arguments; see Appendix A. In the remaining case, $\mathrm{char}(\mathbb{K}) = 2$ and $d = 1$, semisimplicity of $\mathcal{B}_d(\delta)$ is immediate.

In the situation $\mathrm{char}(\mathbb{K}) = p > 2$, we start by deducing a general argument that enables us to go from $d$ to $d + 2$ (similarly as in the walled Brauer case). We then separate three cases: $\delta_p \neq 0$ odd, $\delta_p \neq 0$ even and $\delta_p = 0$. In all three cases there is a $d_0$ such that $\mathcal{B}_d(\delta)$ is semisimple for $d < d_0$ and nonsemisimple for $d \geq d_0$. We verify these cases separately. For example, our argument in the first case ($\delta_p \neq 0$ odd) can be illustrated as follows.

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Here the top case is $\delta_p + 1 \leq \frac{1}{2}(p - \delta_p + 2)$, while the bottom is $\frac{1}{2}(p - \delta_p + 2) \leq \delta_p + 1$. We have illustrated the 'boundary value' where $\mathcal{B}_d(\delta)$ stops to be semisimple and what lemmas we use to deduce (non)semisimplicity. Note that, as in the walled Brauer case, one 'boundary case' remains to be verified. We do this, as before, by using our explicit description of the kernel of the Schur–Weyl–Brauer action from Proposition 4.5. Similarly in the other cases.

**7.2. The Schur–Weyl–Brauer dual story: from $d$ to $d + 2$.** Let $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{g})$ and $\mathfrak{g}$ be $\mathfrak{so}_{2m+1}$, $\mathfrak{sp}_{2m}$ or $\mathfrak{so}_{2m}$ (types $\mathbf{B}_m, \mathbf{C}_m$ and $\mathbf{D}_m$, respectively). We set $n = 2m + 1$ for $\mathfrak{g} = \mathfrak{so}_{2m+1}$ and $n = 2m$ otherwise. Moreover, let $V$ and $T_n^d$ be as in Theorem 3.17. Again, $V, T_n^d \in \mathcal{T}$ by Proposition 2.3. As usual, we can calculate the Weyl factors of $T_n^d$ as in the classical case.

Proposition 7.3. *If $T_n^d$ is a nonsemisimple $\mathbf{U}_1$-module, then so is $T_n^{d+2}$.*

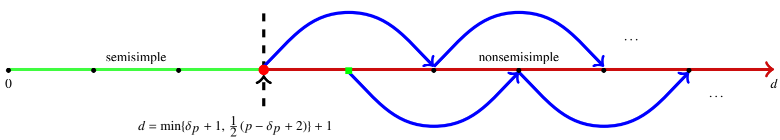
Proof. Note that $\Delta_1(0) \cong \mathbb{K}$ is a Weyl factor of $T_n^2$. As in the proof of Proposition 6.2, any Weyl factor of $T_n^d$ is also a Weyl factor of $T_n^{d+2}$. The conclusion follows from Lemma 2.4. □

Corollary 7.4. *Let* $\operatorname{char}(\mathbb{K}) = p$. *If $\mathcal{B}_d(\delta)$ is nonsemisimple, then so is $\mathcal{B}_{d+2}(\delta)$.*

Proof. Choose $n \geq 2d$ with $n \equiv \delta_p \bmod p$. Then the statement follows directly from Theorem 3.17, Proposition 7.3 and Theorem 2.5. □

Corollary 7.4 again fails in general for $\operatorname{char}(\mathbb{K}) = 0$ for the same reasons as in Corollary 6.4.

Analogously to the case of walled Brauer algebras, we use Corollary 7.4 to check certain boundary values. For example, if $\delta_p \neq 0$ is odd, then this can be illustrated as




Again, there are two boundary values (displayed as a dot and a box).

We want to point out that the relation of $\mathcal{B}_d(\delta)$ and $\mathcal{B}_{d+2}(\delta)$ underlying Corollary 7.4 was already observed in [38, Section 1.3] and [16, Section 5], while the 'trick' to add $p$, used in Corollary 7.4 and below, appeared in [15, Section 5], but both in a different setting.

**7.3. The case $\delta_p \neq 0$ is odd or even.** Let $\operatorname{char}(\mathbb{K}) = p > 2$, $p > d$ and $\delta_p \neq 0$ be odd or even.

Lemma 7.5. *If $d > \delta_p + 1$, then $\mathcal{B}_d(\delta)$ is nonsemisimple.*

Proof. By Corollary 7.4, it suffices to check the boundary values $d = \delta_p + 2$ and $d = \delta_p + 3$. Consider $T_n^d$ for $n = p + \delta_p$. First let us assume that $\delta_p$ is even

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(type $\mathbf{B}_m$ with $m = \frac{1}{2}(p + \delta_p - 1)$). Consider $\lambda = 2\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{\delta_p+1}$ (for $d = \delta_p + 2$) and $\mu = 3\varepsilon_1 + 2\varepsilon_2 + \cdots + \varepsilon_{\delta_p}$ (for $d = \delta_p + 3$). We get

$$\begin{aligned}
&\lambda + \rho - p(\varepsilon_1 + \varepsilon_{\delta_p-1}) \\
&\quad = \tfrac{1}{2} \cdot (\boxed{-p + \delta_p + 2}, p + \delta_p - 2, \ldots, \boxed{-p - \delta_p}, p - \delta_p - 4, \ldots, 3, 1), \\
&\mu + \rho - p(\varepsilon_2 + \varepsilon_{\delta_p}) \\
&\quad = \tfrac{1}{2} \cdot (p + \delta_p + 4, \boxed{-p + \delta_p}, p + \delta_p - 4, \ldots, \boxed{-p - \delta_p + 2}, p - \delta_p - 2, \ldots, 3, 1),
\end{aligned}$$

for the two boundary values. Because we assume that $d < p$, we have that $\delta_p \leq p - 3$. Thus, the maximal $k$ in *JSF* is $k = 1$ and a direct computation verifies that the contributions of the positive roots $\alpha \in \Phi^+$ of the forms $\alpha = \varepsilon_1 + \varepsilon_{\delta_p-1}$ and $\alpha = \varepsilon_2 + \varepsilon_{\delta_p}$ to the *JSF* of $\Delta_1(\lambda)$ and $\Delta_1(\mu)$ from above are not cancelled. Hence, *JSF* of $\Delta_1(\lambda)$ and of $\Delta_1(\mu)$ are nonzero.

Now assume that $\delta_p$ is odd. We take again the same $\lambda$ and $\mu$ and the same reasoning as above works (which takes place in type $\mathbf{D}_m$ for $m = \frac{1}{2}(p + \delta_p)$ now). The surjectivity in Theorem 3.17 and Corollary 2.6 provides the statement, since semisimple algebras have semisimple quotients. Note that the surjectivity fails in type $\mathbf{D}_m$ for the cases $\delta_p = p - 4$ and $d = \delta_p + 2 = p - 2$ or $d = \delta_p + 3 = p - 1$, or $\delta_p = p - 6$ and $d = \delta_p + 3 = p - 3$ (note that $m \geq 4$ in the remaining cases). These have to be $p$-shifted twice to $\mathbf{B}_m$ by taking $m = \frac{1}{2}(2p + \delta_p - 1)$, but the argument is again similar (that is, using *JSF*), but slightly involved due to the 'size' of the numbers in question and omitted for brevity. □

### 7.4. The case $\delta_p \neq 0$ is odd.

Let $\mathrm{char}(\mathbb{K}) = p > 2$, $p > d$ and $\delta_p \neq 0$ be odd.

Lemma 7.6. *If $d \leq \delta_p + 1$ and $d < \frac{1}{2}(p - \delta_p + 2)$, then $\mathcal{B}_d(\delta)$ is semisimple.*

Proof. Note that $p$ is odd and $p > \delta_p$. Hence, $n = p - \delta_p \geq 2d$ is a positive, even number. Consider $T_n^d$ (type $\mathbf{C}_m$ with $m = \frac{1}{2}(p - \delta_p)$). Then Theorem 3.17 gives

$$\mathcal{B}_d(\delta) \cong \mathcal{B}_d(\delta_p) \cong \mathcal{B}_d(\delta_p - p) \cong \mathrm{End}_{\mathbf{U}_1}(T_n^d).$$

Since $d \leq \delta_p + 1$, all Weyl factors $\Delta_1(\lambda)$ of $T_n^d$ satisfy

$$\langle \lambda + \rho, \alpha^\vee \rangle \leq \langle d\varepsilon_1 + \rho, (\varepsilon_1 + \varepsilon_2)^\vee \rangle = d + p - \delta_p - 1 \leq p,$$

for all $\alpha \in \Phi^+$. Hence, the corresponding *JSF*s are all zero and the statement follows from Corollary 2.6 as long as $m \geq 3$. The case $m = 1$ only occurs if $\delta_p = p - 2$ and, thus, $d < \frac{1}{2}(p - \delta_p + 2)$ gives $d < 2$, where semisimplicity is clear. The case $m = 2$ only occurs if $\delta_p = p - 4$ for $p \geq 5$ and, thus, $d < \frac{1}{2}(p - \delta_p + 2)$ gives $d < 3$. Semisimplicity of $\mathcal{B}_d(\delta)$ for $d = 2$ and $p \geq 5$ follows because the following pairwise-orthogonal, primitive idempotents

form a basis of $\mathcal{B}_2(\delta)$. Hence, $\mathcal{B}_2(\delta) \cong \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$. □

Lemma 7.7. *If $d > \frac{1}{2}(p - \delta_p + 2)$, then $\mathcal{B}_d(\delta)$ is nonsemisimple.*

Proof. As above, by Corollary 7.4 it remains to verify nonsemisimplicity for the boundary values $d = \frac{1}{2}(p - \delta_p + 2) + 1$ and $d = \frac{1}{2}(p - \delta_p + 2) + 2$. We first assume that $\delta_p \geq 3$ and we take $T_n^d$ for $n = p + \delta_p$ (thus, we are in type $\mathbf{D}_m$ with $m = \frac{1}{2}(p + \delta_p) \geq 4$). By surjectivity in Theorem 3.17 and by Corollary 2.6, it remains to find Weyl factors $\Delta_1(\lambda)$ for both $T_n^d$ which are nonsimple $\mathbf{U}_1$-modules. We take $\lambda = d\varepsilon_1$ and $\alpha = \varepsilon_1 + \varepsilon_{m-1}$ (for $d = \frac{1}{2}(p - \delta_p + 2) + 1$) and $\alpha = \varepsilon_1 + \varepsilon_{m-2}$ (for $d = \frac{1}{2}(p - \delta_p + 2) + 2$):

$$\lambda + \rho - p(\varepsilon_1 + \varepsilon_{m-1}) = (\boxed{1}, \tfrac{1}{2}(p + \delta_p) - 2, \tfrac{1}{2}(p + \delta_p) - 3, \ldots, 3, 2, \boxed{p+1}, 0),$$
$$\lambda + \rho - (\varepsilon_1 + \varepsilon_{m-2}) = (\boxed{2}, \tfrac{1}{2}(p + \delta_p) - 2, \tfrac{1}{2}(p + \delta_p) - 3, \ldots, 3, \boxed{p+2}, 1, 0).$$

These are regular $\mathbf{U}_1$-weights since $\delta_p < p$, which are not cancelled in *JSF*: only positive roots $\alpha \in \Phi^+$ of the form $\alpha = \varepsilon_1 + \varepsilon_j$ for $j \neq 1$ can contribute to *JSF* (and at most once) and all of these yield singular $\mathbf{U}_1$-weights. Thus, *JSF*s of these $\Delta_1(\lambda)$ are nonzero.

It remains to verify the case $\delta_p = 1$. First note that we do not have to consider $p = 3$, since $d > \frac{1}{2}(p - \delta_p + 2) = 3 = p$ by assumption. For the remaining cases, first assume that $p \geq 7$. We take $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{sp}_{2m})$ with $m = \frac{1}{2}(p - \delta_p)$ (hence, $m \geq 3$) and $T_n^d$ with $n = p - \delta_p$. Then we proceed as before, but in type $\mathbf{C}_m$ and with $\alpha = \varepsilon_1 + \varepsilon_m$ instead of $\alpha = \varepsilon_1 + \varepsilon_{m-1}$ and $\alpha = \varepsilon_1 + \varepsilon_{m-1}$ instead of $\alpha = \varepsilon_1 + \varepsilon_{m-2}$. The remaining case $p = 5, \delta_p = 1$ and $d = 4$ can be done by going to type $\mathbf{B}_m$ with $m = 5$. □

**7.5. The case $\delta_p \neq 0$ is even.** Let $\mathrm{char}(\mathbb{K}) = p > 2$, $p > d$ and $\delta_p \neq 0$ be even.

Lemma 7.8. *If $d \leq \min\{\delta_p, p - \delta_p + 3\}$, then $\mathcal{B}_d(\delta)$ is semisimple.*

Proof. We take the Schur–Weyl–Brauer data as in Theorem 3.24, that is, $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{so}_{2m})$ with $n = \delta_p$ (type $\mathbf{D}_m$ with $m = \frac{1}{2}\delta_p$), and the $d$-fold tensor product $T_n^d$ of its vector representation (we note that our arguments go through in case $m \leq 3$ as well; see Remark 3.25).

We claim that $T_n^d$ is a semisimple $\mathbf{U}_1$-module: it, as usual, remains to check that all Weyl factors $\Delta_1(\lambda)$ of $T_n^d$ have zero as their *JSF*. This follows almost directly, since, for all positive roots $\alpha \in \Phi^+$, we have (recall that $d \leq p - \delta_p + 3$)

$$\langle \lambda + \rho, \alpha^\vee \rangle \leq \langle d\varepsilon_1 + \rho, (\varepsilon_1 + \varepsilon_2)^\vee \rangle = d + 2m - 3 = d + \delta_p - 3 \leq p.$$

Thus, $T_n^d$ is a semisimple $\mathbf{U}_1$-module and, hence, a direct sum of simple Weyl modules. By Proposition 3.19 and Lemma 3.20 (which is valid in this specific $\mathrm{char}(\mathbb{K}) = p$ case since $T_n^d$ is a direct sum of simple Weyl modules), we have that $T_n^d$ is also a semisimple $\tilde{\mathbf{U}}_1$-module. Hence, $\mathrm{End}_{\tilde{\mathbf{U}}_1}(T_n^d)$ is semisimple. Since we assume that $d \leq \delta_p$, we have $\mathcal{B}_d(\delta) \cong \mathrm{End}_{\tilde{\mathbf{U}}_1}(T_n^d)$ by Theorem 3.24. The statement follows. □

Lemma 7.9. *If $d \leq \min\{\delta_p + 1, p - \delta_p + 3\}$, then $\mathcal{B}_d(\delta)$ is semisimple.*

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Proof. By Lemma 7.8, it suffices to check that $\mathcal{B}_d(\delta)$ is semisimple for $d = \delta_p + 1 < p - \delta_p + 3$.

In order to do so, we fix $n = p + \delta_p$ and the statement follows by Theorem 3.17 if we show that $T_n^d$ is a semisimple $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{so}_{2m+1})$-module (type $\mathbf{B}_m$ with $m = \frac{1}{2}(p + \delta_p - 1)$).

Our argument below uses $p \geq 7$. Since $\delta_p + 1 < p - \delta_p + 3$ gives $\delta_p < \frac{1}{2}p + 1$, only $\delta_p = 2$ and $p = 3, 5$ are the cases with $p < 7$ for which we need to check semisimplicity of $\mathcal{B}_d(\delta)$. In case $p = 3$, we would have to check that $d = \delta_p + 1 = 3$. Since we assume that $p > d$, this case does not occur. The case $p = 5$ and $d = 3$ can be verified as usual using *JSF* and is in particular very similar to the case $p \geq 7$ discussed below (the stated inequalities below are not true any more and there are a few extra cases to check). We leave the details to the reader.

Assume now that $p \geq 7$. Following our usual recipe, we have to show that all Weyl factors $\Delta_1(\lambda)$ of $T_n^d$ have zero *JSF*. Note now that such $\lambda$ satisfy $\lambda \in \Lambda^+(d - 2i')$ for some $i' = 0, \ldots, \lfloor \frac{1}{2}d \rfloor$. It turns out that there are two different cases: the first case is $\lambda_t = 0$ and the second is $\lambda_t = 1$ (for $t = \frac{1}{2}(d + 1)$). Note that these are all cases since $\lambda_t > 1$ cannot occur for $\lambda \in \Lambda^+(d - 2i')$ (in particular, we always have $\lambda_{t'} \in \{0, 1\}$ for all $t \leq t' \leq d - 2i'$).

Fix now $\lambda \in \Lambda^+(d - 2i')$. A direct verification shows that positive roots $\alpha \in \Phi^+$ of the form $\alpha = \varepsilon_i - \varepsilon_j$ (for $1 \leq i < j \leq m$) will never yield contributions to *JSF* of $\Delta_1(\lambda)$ (this can be seen similarly as in the proof of Theorem 5.1). Thus, we only need to check positive roots $\alpha \in \Phi^+$ of the form $\alpha = \varepsilon_i + \varepsilon_j$ (for $1 \leq i < j \leq m$) or of the form $\alpha = \varepsilon_i$ (for $i = 1, \ldots, m$). Note now that $d = \delta_p + 1$, $\delta_p < \frac{1}{2}p + 1$ and $p \geq 7$ give

$$\langle \lambda + \rho, (\varepsilon_i + \varepsilon_j)^\vee \rangle \leq \langle d\varepsilon_1 + \rho, (\varepsilon_1 + \varepsilon_2)^\vee \rangle = p + \delta_p + d - 3 = p + 2\delta_p - 2 < 2p,$$
$$\langle \lambda + \rho, (\varepsilon_i)^\vee \rangle \leq \langle d\varepsilon_1 + \rho, (\varepsilon_1)^\vee \rangle = p + \delta_p + 2d - 2 = p + 3\delta_p < \tfrac{5}{2}p + 3 \leq 3p.$$

Thus, it suffices to consider $k = 1$ or $k = 1, 2$ in *JSF* of $\Delta_1(\lambda)$.

*Case* $\lambda_t = 0$. There is a 'tail' in $\lambda + \rho$: every value of the form $\frac{1}{2}(2k' + 1)$ (for $k' \in \mathbb{Z}_{\geq 0}$) appears after $(\lambda + \rho)_t = \rho_t = \frac{1}{2}p - 1$. Assume that $p < \langle \lambda + \rho, (\varepsilon_i)^\vee \rangle < 3p$ for some $i = 1, \ldots, m$ (it follows that $i \leq t$). Thus, $\frac{1}{2}p < \lambda_i + m - \frac{1}{2} - i < \frac{3}{2}p$. Then $(\lambda + \rho - p\varepsilon_i)_i$ will always be in the 'tail' and, hence, $\lambda + \rho - kp\varepsilon_i$ is a singular $\mathbf{U}_1$-weight for $k = 1$ and such $\lambda$:

$$0 < |(\lambda + \rho - p\varepsilon_i)_i| = |\lambda_i + m + \tfrac{1}{2} - i - p| \leq \tfrac{1}{2}p - 1 = (\lambda + \rho)_t = \rho_t.$$

Similarly, $\lambda + \rho - p(\varepsilon_i + \varepsilon_j)$ (for $1 \leq i < j \leq m$) is a singular $\mathbf{U}_1$-weight except if we have $(\lambda + \rho - p\varepsilon_i)_i = (\lambda + \rho)_j$. The latter occurs if and only if $2p < \langle \lambda + \rho, (\varepsilon_i)^\vee \rangle < 3p$. These $\mathbf{U}_1$-weights give contributions to *JSF* of $\Delta_1(\lambda)$, but are cancelled by the contribution for $\varepsilon_i$ and $k = 2$.

In summary, *JSF* of $\Delta_1(\lambda)$ is zero: $1 \leq i < j \leq m$ does not have any contributions for $\alpha = \varepsilon_i + \varepsilon_j$ or $\alpha = \varepsilon_i$ (in the case $0 < \langle \lambda + \rho, (\varepsilon_i)^\vee \rangle \leq p$) or only singular $\mathbf{U}_1$-weights appear (this happens in the case $p < \langle \lambda + \rho, (\varepsilon_i)^\vee \rangle \leq 2p$) or there will be cancellations (this happens in the case $2p < \langle \lambda + \rho, (\varepsilon_i)^\vee \rangle < 3p$).

*Case* $\lambda_t = 1$. Similarly as before. We omit the details for brevity and only note that the assumption $\lambda_t = 1$ ensures that there will be only one ‘gap in the tail’. Hence, the same argument as above goes through with the extra case that $(\lambda + \rho - p\varepsilon_i)_i$ can precisely land in this gap. In this case $\lambda + \rho - p\varepsilon_i$ is a regular $\mathbf{U}_1$-weight, but it is again cancelled in *JSF* of $\Delta_1(\lambda)$ (this time by $\lambda + \rho - p(\varepsilon_i + \varepsilon_j)$, where $j$ is the entry of the gap).

Thus, $T_n^d$ is a semisimple $\mathbf{U}_1$-module, which shows the statement. □

Remark 7.10. The boundary case in Lemma 7.9 could also be done by analyzing the kernel of the Schur–Weyl–Brauer action $\Phi_{\mathrm{Br}}$ as in the proof of Theorem 6.1 and as in the proof of Theorem 7.1 below. But this would require going to the reductive group $O_{2m}$ (Brauer already observed in [8, page 870] that surjectivity of $\Phi_{\mathrm{Br}}$ fails in general for $SO_{2m}$). In order to keep the paper reasonably self-contained, we avoid using the reductive group setting here.

Lemma 7.11. *If $d > p - \delta_p + 3$, then $\mathcal{B}_d(\delta)$ is nonsemisimple.*

Proof. We take $n = p + \delta_p$ (type $\mathbf{B}_m$ with $m = \frac{1}{2}(p + \delta_p - 1)$ again). As usual, by the surjectivity in Theorem 3.17 and Corollary 2.6, it suffices to give a Weyl factor of $T_n^d$ that is a nonsimple $\mathbf{U}_1$-module. By Corollary 7.4, it remains to give such factors in the cases $d = (p - \delta_p + 3) + 1$ and $d = (p - \delta_p + 3) + 2$. Take $\lambda = (d-1)\varepsilon_1 + \varepsilon_2$ and $\mu = (d-2)\varepsilon_1 + 2\varepsilon_2$:

$$\begin{aligned}
&\lambda + \rho - p(\varepsilon_1 + \varepsilon_{\delta_p - 2}) \\
&\quad = \tfrac{1}{2}(\boxed{p - \delta_p + 4}, p + \delta_p - 2, p + \delta_p - 6, \ldots, \boxed{-p - \delta_p + 4}, \ldots, 3, 1), \\
&\mu + \rho - p(\varepsilon_1 + \varepsilon_{\delta_p - 2}) \\
&\quad = \tfrac{1}{2}(\boxed{p - \delta_p + 4}, p + \delta_p, p + \delta_p - 6, \ldots, \boxed{-p - \delta_p + 4}, \ldots, 3, 1).
\end{aligned}$$

Here we assume that $\delta_p \geq 4$ (for $\delta_p = 2$, we have $d > p$ and we can use Lemma 3.15). Only the positive roots $\alpha \in \Phi^+$ with $\alpha = \varepsilon_i \pm \varepsilon_j$ or $\alpha = \varepsilon_i$ for $i = 1$ can give other nonzero contributions to *JSF*s. These remaining positive roots $\alpha \in \Phi^+$ do not cancel the contributions above ($k \leq 1$ for all of these). Hence, *JSF*s for $\Delta_1(\lambda)$ and $\Delta_1(\mu)$ are nonzero. □

**7.6. The $\delta_p = 0$ case.** Let $\mathrm{char}(\mathbb{K}) = p > 2$, $p > d$ and $\delta_p = 0$.

Lemma 7.12. *Let $p \geq 5$. Then $\mathcal{B}_3(\delta)$ is semisimple. If $p \geq 7$, then $\mathcal{B}_5(\delta)$ is also semisimple.*

Proof. Again, we check the corresponding *JSF*. To this end, we consider $T_n^d$ for $n = p$ (we are in type $\mathbf{B}_m$ with $m = \frac{1}{2}(p - 1)$) and $d = 3$. Then $T_n^d$ has only Weyl factors which are simple $\mathbf{U}_1$-modules: its Weyl factors are $\Delta_1(3\varepsilon_1)$, $\Delta_1(2\varepsilon_1 + \varepsilon_2)$, $\Delta_1(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ and $\Delta_1(\varepsilon_1)$, all of which have zero *JSF*. This can be seen as usual and we only do the first case explicitly here.

A direct computation shows that $\langle 3\varepsilon_1 + \rho, \alpha^\vee \rangle \leq p$ for all positive roots $\alpha \in \Phi^+$ except for $\alpha = \varepsilon_1$, where $\langle 3\varepsilon_1 + \rho, \varepsilon_1^\vee \rangle = p + 4 < 2p$ (recall that $p \geq 5$). Then, because $p \geq 5$,

$$3\varepsilon_1 + \rho - p\varepsilon_1 = \tfrac{1}{2} \cdot (4 - p, p - 4, p - 6, \ldots, 3, 1)$$

is a singular $\mathbf{U}_1$-weight. Thus, *JSF* of $\Delta_1(3\varepsilon_1)$ is zero.

Similarly for $d = 5$, $T_n^d$ has only Weyl factors that are simple $\mathbf{U}_1$-modules:

$$\Delta_1(5\varepsilon_1), \quad \Delta_1(4\varepsilon_1 + \varepsilon_2), \quad \Delta_1(3\varepsilon_1 + 2\varepsilon_2), \quad \Delta_1(3\varepsilon_1 + \varepsilon_2 + \varepsilon_3),$$
$$\Delta_1(2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3), \quad \Delta_1(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4), \quad \Delta_1(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5), \quad \Delta_1(3\varepsilon_1),$$
$$\Delta_1(2\varepsilon_1 + \varepsilon_2), \quad \Delta_1(\varepsilon_1 + \varepsilon_2 + \varepsilon_3), \quad \Delta_1(\varepsilon_1),$$

all of which have zero as their *JSF* when $p \geq 7$ (we omit the calculation, which works as above). The statement follows from Theorem 3.17 (note that $n = p > d$) and Corollary 2.6. □

LEMMA 7.13. *$\mathcal{B}_d(\delta)$ is nonsemisimple for $d \in \{2, 4, 6, 8, \ldots\} \cup \{7, 9, 11, 13, \ldots\}$.*

PROOF. By Corollary 7.4, it remains to verify the boundary values $d = 2$ and $d = 7$.

Assume first that $d = 2$ and $p \geq 5$. We consider $T_n^d$ for $n = p$ (we are in type $\mathbf{B}_m$ with $m = \frac{1}{2}(p - 1)$). We have that $T_n^d$ has a Weyl factor of the form $\Delta_1(2\varepsilon_1)$. We calculate

$$2\varepsilon_1 + \rho - p\varepsilon_1 = \tfrac{1}{2} \cdot (2 - p, p - 4, p - 6, \ldots, 3, 1).$$

Thus, the positive root $\alpha \in \Phi^+$ of the form $\alpha = \varepsilon_1$ gives a nonzero contribution to *JSF* of $\Delta_1(2\varepsilon_1)$ (no other positive root $\alpha \in \Phi^+$ contributes and, hence, we have no cancellations). As before, $\mathcal{B}_d(\delta)$ is nonsemisimple by Theorem 3.17 and Corollary 2.6.

Assume now that $d = 2$ and $p = 3$. This case can be done analogously by considering $T_n^d$ for $n = 9 = 3p$ (type $\mathbf{B}_m$ with $m = 4$). We leave the details to the reader.

Next, let $d = 7$ and $p \geq 7$. Then we proceed similarly as above: we take $T_n^d$ for $n = p$ and the Weyl factor $\Delta_1(4\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3)$. Hence, we only need to check positive roots $\alpha \in \Phi^+$ of the form $\alpha = \varepsilon_1$, $\alpha = \varepsilon_1 + \varepsilon_2$ and $\alpha = \varepsilon_1 + \varepsilon_3$. All of them contribute at most once. We get (for $\lambda = 4\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$)

$$\lambda + \rho - p\varepsilon_1 = \tfrac{1}{2} \cdot (6 - p, p, p - 4, p - 8, p - 10, \ldots, 3, 1),$$
$$\lambda + \rho - p(\varepsilon_1 + \varepsilon_2) = \tfrac{1}{2} \cdot (6 - p, -p, p - 4, p - 8, p - 10, \ldots, 3, 1),$$
$$\lambda + \rho - p(\varepsilon_1 + \varepsilon_3) = \tfrac{1}{2} \cdot (6 - p, p, -p - 4, p - 8, p - 10, \ldots, 3, 1).$$

All of these are regular $\mathbf{U}_1$-weights and the first two cancel each other, but the last remains. Thus, *JSF* of $\Delta_1(4\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3)$ is nonzero. The conclusion that $\mathcal{B}_7(\delta)$ is nonsemisimple follows again from Theorem 3.17 and Corollary 2.6. □

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### 7.7. Proof of the semisimplicity criterion for $\mathcal{B}_d(\delta)$.

Proof of Theorem 7.1. (1) By Lemma 7.6, only the case $d = \frac{1}{2}(p - \delta_p + 2) \leq \delta_p + 1$ is missing for the 'if' part, because in this case the corresponding Schur–Weyl–Brauer duality gives only a surjection. As in the proof of Theorem 6.1, we can use Propositions 4.5 and 4.7 to handle this missing case. The 'only if' part follows from Lemmas 7.5 and 7.7.

(2) This follows from Lemma 7.9 and from Lemmas 7.5 and 7.11.

(3) As in the proof of Theorem 6.1, we can again use Theorem A.3.

(4) Directly from Lemmas 7.12 and 7.13, since $\mathcal{B}_1(\delta)$ is always semisimple.

(5) Again by Theorem A.3.

(6) $\mathcal{B}_1(\delta)$ is clearly semisimple, while $\mathcal{B}_d(\delta)$ for $d \geq 2$ is not because of Lemma 3.15. The theorem follows. □

Remark 7.14. Of course, the semisimplicity criterion from Theorem 7.1 was already observed before. In particular, the case $\mathbb{K} = \mathbb{C}$ and $\delta \in \mathbb{Z}$ goes back to a paper of Brown [9, Theorem 8D] and, in case $\delta$ is not an integer or $\mathbb{K}$ is an arbitrary field of characteristic zero and arbitrary $\delta \in \mathbb{K}$, to work of Wenzl [54, Corollary 3.3]. The case for $\mathbb{K}$ being a field of arbitrary characteristic is treated by Rui in [45, Theorem 1.2]. To see that Rui's criterion matches ours, we note that a slight reformulation of Rui's criterion was given by Rui together with Si later in [46, Corollary 2.5]. The latter is easily seen to coincide with the one we obtain.

Remark 7.15. Using our approach, we could re-prove the semisimplicity criterion for the BMW algebra found in [47, Theorem 5.9], but decided to stay in the $q = 1$ case for the sake of brevity.

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## Appendix A. From positive characteristic to characteristic zero

Here we recall some algebraic notions which we use to transfer our results from positive characteristic to characteristic zero. To this end, given a $\mathbb{Z}$-algebra $\mathcal{A}^{\mathbb{Z}}$ and any fixed field $\mathbb{K}$, we denote by $\mathcal{A}^{\mathbb{K}} = \mathcal{A}^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$ the *scalar extension* of $\mathcal{A}$. Moreover, we assume throughout that all $\mathbb{Z}$-algebras are finitely generated and free.

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Fix a $\mathbb{Z}$-algebra $\mathcal{A}^{\mathbb{Z}}$. Recall that there is a *trace form* $\langle\cdot,\cdot\rangle : \mathcal{A}^{\mathbb{Z}} \otimes \mathcal{A}^{\mathbb{Z}} \to \mathbb{Z}$ on $\mathcal{A}^{\mathbb{Z}}$ given as follows. Denote by $R_a \in \mathrm{End}_{\mathbb{Z}}(\mathcal{A}^{\mathbb{Z}})$ the right multiplication with $a \in \mathcal{A}^{\mathbb{Z}}$. By choosing a basis, we can identify $R_a$ with a matrix in $M_{\dim(\mathcal{A}^{\mathbb{Z}})}(\mathbb{Z})$ and define

$$\langle a, b\rangle = \mathrm{tr}(R_b \circ R_a) \in \mathbb{Z}.$$

One can easily show that this assignment is independent of the choice of basis.

PROPOSITION A.1. *Let $\{a_1, \ldots, a_{\dim(\mathcal{A}^{\mathbb{Z}})}\}$ be any basis of $\mathcal{A}^{\mathbb{Z}}$. Then $\mathcal{A}^{\mathbb{K}}$ is semisimple if and only if*

$$\det(M_{\mathbb{Z}}) \in \mathbb{K} - \{0\}, \quad \textit{where } M_{\mathbb{Z}} = (\langle a_i, a_j\rangle)_{i,j=1}^{\dim(\mathcal{A}^{\mathbb{Z}})}.$$

PROOF. This is proven in [31, Proposition 4.46]: the vanishing of the determinant as above is equivalent to the degeneracy of the trace form. □

Recall that we denote by $\mathbb{F}_p$ the finite field with $p$ elements.

PROPOSITION A.2. *Let $\mathbb{K}$ be a field with* $\mathrm{char}(\mathbb{K}) = 0$. *Then $\mathcal{A}^{\mathbb{K}}$ is semisimple if and only if $\mathcal{A}^{\mathbb{F}_p}$ is semisimple for infinitely many primes $p$.*

PROOF. Assume that there is a prime $p$ such that the algebra $\mathcal{A}^{\mathbb{F}_p}$ is semisimple. Then, by Proposition A.1, we have that $p$ does not divide $\det(M_{\mathbb{Z}})$. In particular, $\det(M_{\mathbb{Z}}) \in \mathbb{Z} - \{0\} \subset \mathbb{K} - \{0\}$ and $\mathcal{A}^{\mathbb{K}}$ is therefore semisimple according to the same proposition.

If $\mathcal{A}^{\mathbb{K}}$ is semisimple, then $\det(M_{\mathbb{Z}}) \in \mathbb{K} - \{0\}$ by Proposition A.1. By choosing a $\mathbb{Z}$-basis of $\mathcal{A}^{\mathbb{K}}$, this in turn implies that $\det(M_{\mathbb{Z}}) \in \mathbb{Z} - \{0\}$. Hence, by Proposition A.1, $\mathcal{A}^{\mathbb{F}_p}$ is semisimple for all primes $p > |\det(M_{\mathbb{Z}})|$.

The proposition follows. □

THEOREM A.3. *Let* $\mathrm{char}(\mathbb{K}) = 0$ *and $\delta \in \mathbb{Z}$.*

(a) *$\mathcal{B}_{r,s}(\delta)$ is semisimple over $\mathbb{K}$ if and only if $\mathcal{B}_{r,s}(\delta)$ is semisimple over $\mathbb{F}_p$ for infinitely many primes $p$.*

(b) *$\mathcal{B}_d(\delta)$ is semisimple over $\mathbb{K}$ if and only if $\mathcal{B}_d(\delta)$ is semisimple over $\mathbb{F}_p$ for infinitely many primes $p$.*

PROOF. Note that $\mathcal{B}_{r,s}(\delta)$ and $\mathcal{B}_d(\delta)$ given in Definitions 3.8 and 3.14 and considered over $\mathbb{K}$ are obtained from integral versions $\mathcal{B}^{\mathbb{Z}}_{r,s}(\delta)$ and $\mathcal{B}^{\mathbb{Z}}_d(\delta)$ via scalar extension (the integral versions of these algebras can be found in [12, Section 2] and [20, Section 4], respectively). Hence, the two statements follow from Proposition A.2. □

## Appendix B. Root systems of types $\mathbf{A}_{m-1}$, $\mathbf{B}_m$, $\mathbf{C}_m$ and $\mathbf{D}_m$

For the convenience of the reader we list here the root and weight data of type $\mathbf{A}_{m-1}$ attached to $\mathfrak{g} = \mathfrak{gl}_m$, of type $\mathbf{B}_m$ attached to $\mathfrak{g} = \mathfrak{so}_{2m+1}$ for $m \geq 2$, of type $\mathbf{C}_m$ attached to $\mathfrak{g} = \mathfrak{sp}_{2m}$ for $m \geq 3$ and of type $\mathbf{D}_m$ attached to $\mathfrak{g} = \mathfrak{so}_{2m}$ for $m \geq 4$. We assume that $1 \leq i, j \leq m$, $i \neq j$ and $1 \leq i' \leq m-1$. The root system and its dual are realized inside the Euclidean space $E = \mathbb{R}^m$ with standard basis $\varepsilon_1, \ldots, \varepsilon_m$ and inner product determined by $\langle \varepsilon_i, \varepsilon_j\rangle = \delta_{i,j}$. Then the data is given as follows:

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| | $\mathbf{A}_{m-1}$ | $\mathbf{B}_m$ |
|---|---|---|
| $\Phi$ | $\{\varepsilon_i - \varepsilon_j \mid 1 \le i \neq j \le m\}$ | $\{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid 1 \le i \neq j \le m\}$ |
| $\Phi^+$ | $\{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le m\}$ | $\{\varepsilon_i \pm \varepsilon_j, \varepsilon_i \mid 1 \le i < j \le m\}$ |
| $\Pi$ | $\{\alpha_{i'} = \varepsilon_{i'} - \varepsilon_{i'+1}\}$ | $\{\alpha_{i'} = \varepsilon_{i'} - \varepsilon_{i'+1}\} \cup \{\alpha_m = \varepsilon_m\}$ |
| $\Pi^\vee$ | $\{\alpha^\vee_{i'} = \varepsilon_{i'} - \varepsilon_{i'+1}\}$ | $\{\alpha^\vee_{i'} = \varepsilon_{i'} - \varepsilon_{i'+1}\} \cup \{\alpha^\vee_m = 2\varepsilon_m\}$ |
| $\rho$ | $(m-1, m-2, \dots, 1, 0)$ | $(m-\frac{1}{2}, m-\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2})$ $= \frac{1}{2}\cdot(2m-1, 2m-3, \dots, 3, 1)$ |
| $X$ | $\{\lambda = \sum_{i=1}^m \lambda_i\varepsilon_i \in \mathbb{R}^m \mid \lambda_i \in \mathbb{Z}\}$ | $\{\lambda = \sum_{i=1}^m \lambda_i\varepsilon_i \in \mathbb{R}^m \mid \lambda_i \in \frac{1}{2}\mathbb{Z},\ \lambda_i - \lambda_j \in \mathbb{Z}\}$ |
| $X^+$ | $\{\lambda \in X \mid \lambda_1 \ge \dots \ge \lambda_m\}$ | $\{\lambda \in X \mid \lambda_1 \ge \dots \ge \lambda_m \ge 0\}$ |
| $\omega_i$ | $\omega_{i'} = \sum_{j=1}^{i'} \varepsilon_j$ | $\omega_{i'} = \sum_{j=1}^{i'} \varepsilon_j, \quad \omega_m = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_m)$ |
| $W$ | $S_m$ | $S_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$ |

| | $\mathbf{C}_m$ | $\mathbf{D}_m$ |
|---|---|---|
| $\Phi$ | $\{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid 1 \le i \neq j \le m\}$ | $\{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i \neq j \le m\}$ |
| $\Phi^+$ | $\{\varepsilon_i \pm \varepsilon_j, 2\varepsilon_i \mid 1 \le i < j \le m\}$ | $\{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le m\}$ |
| $\Pi$ | $\{\alpha_{i'} = \varepsilon_{i'} - \varepsilon_{i'+1}\} \cup \{\alpha_m = 2\varepsilon_m\}$ | $\{\alpha_{i'} = \varepsilon_{i'} - \varepsilon_{i'+1}\} \cup \{\alpha_m = \varepsilon_{m-1} + \varepsilon_m\}$ |
| $\Pi^\vee$ | $\{\alpha^\vee_{i'} = \varepsilon_{i'} - \varepsilon_{i'+1}\} \cup \{\alpha^\vee_m = \varepsilon_m\}$ | $\{\alpha^\vee_{i'} = \varepsilon_{i'} - \varepsilon_{i'+1}\} \cup \{\alpha^\vee_m = \varepsilon_{m-1} + \varepsilon_m\}$ |
| $\rho$ | $(m, m-1, \dots, 2, 1)$ | $(m-1, m-2, \dots, 1, 0)$ |
| $X$ | $\{\lambda = \sum_{i=1}^m \lambda_i\varepsilon_i \in \mathbb{R}^m \mid \lambda_i \in \mathbb{Z}\}$ | $\{\lambda = \sum_{i=1}^m \lambda_i\varepsilon_i \in \mathbb{R}^m \mid \lambda_i \in \frac{1}{2}\mathbb{Z},\ \lambda_i - \lambda_j \in \mathbb{Z}\}$ |
| $X^+$ | $\{\lambda \in X \mid \lambda_1 \ge \dots \ge \lambda_m \ge 0\}$ | $\{\lambda \in X \mid \lambda_1 \ge \dots \ge \lambda_{m-1} \ge \lvert\lambda_m\rvert \ge 0\}$ |
| $\omega_i$ | $\omega_i = \sum_{j=1}^{i} \varepsilon_j$ | $\omega_{i''} = \sum_{j=1}^{i''} \varepsilon_{i'}, \quad (1 \le i'' \le m-2)$, $\omega_{m-1} = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_{m-1} - \varepsilon_m)$, $\omega_m = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_{m-1} + \varepsilon_m)$ |
| $W$ | $S_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$ | $S_m \ltimes (\mathbb{Z}/2\mathbb{Z})^{m-1}$ |

In type $\mathbf{A}_{m-1}$, the simple transpositions $s_r$ of $S_m = W$ act on $X$ via permutation. The (dot-)singular type $\mathbf{A}_{m-1}$ weights in the sense of Definition 2.7 and Convention 2.8 are

$$\lambda \in X \text{ is dot-singular} \ \Leftrightarrow\ \text{there exist } i \neq j \text{ such that } (\lambda+\rho)_i = (\lambda+\rho)_j,$$
$$\lambda \in X \text{ is singular} \ \Leftrightarrow\ \text{there exist } i \neq j \text{ such that } \lambda_i = \lambda_j.$$

For types $\mathbf{B}_m$ and $\mathbf{C}_m$ and $i = 1, \dots, m-1$, the elements $s_i \in S_m$ act as in type $\mathbf{A}_{m-1}$, while $s_m : (\lambda_1, \dots, \lambda_m) \mapsto (\lambda_1, \dots, -\lambda_m)$. The (dot-)singular type $\mathbf{B}_m$ and $\mathbf{C}_m$ weights are

$$\lambda \in X \text{ is dot-singular} \Leftrightarrow \text{there exist } i \neq j \text{ such that } (\lambda+\rho)_i = \pm(\lambda+\rho)_j \text{ or } (\lambda+\rho)_i = 0,$$
$$\lambda \in X \text{ is singular} \Leftrightarrow \text{there exist } i \neq j \text{ such that } \lambda_i = \pm\lambda_j \text{ or } \lambda_i = 0.$$

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For type $\mathbf{D}_m$, the action of $W$ on $X$ is as in types $\mathbf{B}_m$ and $\mathbf{C}_m$, but $s_m$ changes two signs instead of one. The (dot-)singular type $\mathbf{D}_m$ weights are given as in types $\mathbf{B}_m$ and $\mathbf{C}_m$, but with an even number of sign-changed entries.

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