

## SEMICONTINUOUS FUNCTIONS AND CONVEX SETS IN $C(K)$ SPACES

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### Abstract

The stability properties of the family  $\mathcal{M}$  of all intersections of closed balls are investigated in spaces  $C(K)$ , where  $K$  is an arbitrary Hausdorff compact space. We prove that  $\mathcal{M}$  is stable under Minkowski addition if and only if  $K$  is extremally disconnected. In contrast to this, we show that  $\mathcal{M}$  is always ball stable in these spaces. Finally, we present a Banach space (indeed a subspace of  $C[0, 1]$ ) which fails to be ball stable, answering an open question. Our results rest on the study of semicontinuous functions in Hausdorff compact spaces.

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### 1. Introduction

Would you say that the *outer* parallel body of an intersection of closed balls is again an intersection of closed balls? We solve this problem, raised in [5], surprisingly in the negative, in contrast with the result proved in [6] stating that the *inner* parallel body of an intersection of closed balls is always an intersection of closed balls. The family  $\mathcal{M}$  of all intersections of closed balls is one of the most interesting classes of convex sets. The question of whether every closed, convex and bounded set of a normed space is in  $\mathcal{M}$ , a property which is known under the name of the *Mazur intersection property* (MIP), attracted during the past 60 years the attention of many authors. However, the properties of the family  $\mathcal{M}$  when the space fails the MIP have seldom been investigated. Results in this direction can be found, for instance, in [2, 5, 7, 8]. We are interested here in the interplay between Minkowski addition and

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topological properties of  $\mathcal{M}$ , on the one hand, and the geometry of the underlying Banach spaces, on the other.

We say that  $\mathcal{M}$  is (i) *stable* if  $\overline{C + D} \in \mathcal{M}$  when  $C, D \in \mathcal{M}$ ; and (ii) *ball stable* if  $C + \lambda B \in \mathcal{M}$  for every  $C \in \mathcal{M}$  and every  $\lambda > 0$ , where  $B$  is the (closed) unit ball of the space. These definitions are motivated by the natural extension of the notions of Minkowski sum and parallel body, respectively, to the context of infinite dimensional Banach spaces. It was proved in [5] that  $\mathcal{M}$  is stable in  $(C(K), \|\cdot\|_\infty)$  when  $K$  is extremally disconnected. We improve this result by showing that the stability of  $\mathcal{M}$  in  $(C(K), \|\cdot\|_\infty)$  actually characterizes the extremal disconnectedness of  $K$ . In contrast to this, we show that  $\mathcal{M}$  is ball stable in  $(C(K), \|\cdot\|_\infty)$  for every compact Hausdorff space  $K$ . We exhibit the first example of a Banach space where  $\mathcal{M}$  fails to be ball stable, answering a question raised in [5]. Finally, the ball stability of  $\mathcal{M}$  in  $(C(K), \|\cdot\|_\infty)$  is used to prove that, in this space,  $\mathcal{M}$  is (topologically) closed.

The proofs involve the use of some fine properties of semicontinuous functions defined in compact Hausdorff spaces. Those readers familiar with the theory of Riesz spaces will recognize some methods and results from the theory of Banach lattices with strong unit. Indeed, in some cases, the usual techniques from this theory can be used to provide alternative proofs of our results.

## 2. Semicontinuous functions and intersections of closed balls in $C(K)$

Given an arbitrary Hausdorff compact space and  $t \in K$ , denote by  $\delta_t$  the Dirac functional defined as  $\delta_t(f) = f(t)$  for every  $f \in C(K)$ . Given bounded real-valued functions  $f, g$  on  $K$  such that  $f(t) \leq g(t)$  for every  $t \in K$ , we denote

$$[f, g] = \{h \in C(K) : f(t) \leq h(t) \leq g(t) \text{ for all } t \in K\}.$$

This set can also be written as

$$[f, g] = \bigcap_{t \in K} \delta_t^{-1}[f(t), g(t)].$$

The function  $f : K \rightarrow \mathbb{R}$  is called:

- (i) *lower semicontinuous* if  $\liminf_{y \rightarrow x} f(y) \geq f(x)$  for all  $x \in K$ ; and
- (ii) *upper semicontinuous* if  $\limsup_{y \rightarrow x} f(y) \leq f(x)$  for all  $x \in K$ .

The set of points of continuity of a function  $f : K \rightarrow \mathbb{R}$  will be denoted by  $D_f$ . When  $D_f$  is dense in  $K$  we say that  $f$  is *densely continuous*. Semicontinuous functions (upper or lower) on arbitrary topological spaces are always continuous on a residual set [4]. Consequently, when defined on a compact space, they are densely continuous. This is a property that will have a special relevance throughout this paper. We will call  $f, g : K \rightarrow \mathbb{R}$  an *admissible pair* when:

- (a) they are lower and upper semicontinuous, respectively; and
- (b) for every  $x \in K$ ,  $\liminf_{y \rightarrow x} g(y) \geq \limsup_{y \rightarrow x} f(y)$ .

Due to the density of  $D_f$  and  $D_g$ , (b) is equivalent to:

- (b') for every  $x \in K$ ,  $\liminf_{y \rightarrow x, y \in D_g} g(y) \geq \limsup_{y \rightarrow x, y \in D_f} f(y)$ ;

both imply that  $f(x) \leq g(x)$  for every  $x \in K$ .

LEMMA 2.1. *If  $K$  is a compact Hausdorff space and  $f, g: K \rightarrow \mathbb{R}$  form an admissible pair, then there exist functions  $\tilde{f}, \tilde{g}: K \rightarrow \mathbb{R}$  such that  $\tilde{f}, \tilde{g}$  are upper and lower semicontinuous, respectively,  $f \leq \tilde{f} \leq \tilde{g} \leq g$  and  $\emptyset \neq [\tilde{f}, \tilde{g}] = [f, g]$ .*

PROOF. Recall (see [3, page 61]) that a  $T_1$ -space  $X$  is normal if and only if, for every pair of real valued functions  $\psi, \varphi$ , upper and lower semicontinuous, respectively, satisfying  $\psi(x) \leq \varphi(x)$  for every  $x \in X$ , there exists a continuous function  $\phi: X \rightarrow \mathbb{R}$  such that  $\psi(x) \leq \phi(x) \leq \varphi(x)$  for every  $x \in X$ . As a consequence, we just need to find a pair of functions  $\tilde{f}, \tilde{g}: K \rightarrow \mathbb{R}$  upper and lower semicontinuous, respectively, satisfying

$$(2.1) \quad f(x) \leq \tilde{f}(x) \leq \tilde{g}(x) \leq g(x)$$

for every  $x \in K$ , and  $[\tilde{f}, \tilde{g}] = [f, g]$ . Then, the above sandwich result yields the rest of the proof, namely that  $\emptyset \neq [\tilde{f}, \tilde{g}]$ . To this end, we define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in D_f, \\ \limsup_{y \rightarrow x, y \in D_f} f(y) & \text{if } x \notin D_f, \end{cases}$$

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in D_g, \\ \liminf_{y \rightarrow x, y \in D_g} g(y) & \text{if } x \notin D_g. \end{cases}$$

First we prove that  $\tilde{g}$  is lower semicontinuous. Indeed,

$$(2.2) \quad \tilde{g}(x) = \liminf_{y \rightarrow x, y \in D_g} g(y) = \liminf_{y \rightarrow x, y \in D_g} \tilde{g}(y) = \liminf_{y \rightarrow x} \tilde{g}(y)$$

for every  $x \in K$ , where the two first equalities are simply due to the definition of  $\tilde{g}$  and the density of  $D_g$  in  $K$ . With respect to the last equality notice that always  $\liminf_{y \rightarrow x} \tilde{g}(y) \leq \liminf_{y \rightarrow x, y \in D_g} \tilde{g}(y)$ . To prove the converse inequality assume, on the contrary, that

$$(2.3) \quad \alpha = \liminf_{y \rightarrow x} \tilde{g}(y) < \liminf_{y \rightarrow x, y \in D_g} \tilde{g}(y) = \beta.$$

Consider  $\gamma = \alpha + (2/3)(\beta - \alpha)$ . We will show that, for every neighborhood  $G_x$  of  $x$ , there is  $w \in G_x \cap D_g$  satisfying  $\tilde{g}(w) < \gamma$ , hence  $\liminf_{y \rightarrow x, y \in D_g} \tilde{g}(y) \leq \gamma < \beta$ ,

a contradiction. Since  $\alpha = \liminf_{y \rightarrow x} \tilde{g}(y)$ , given  $\varepsilon = (\beta - \alpha)/3$ , there is  $z \in G_x$  such that  $\tilde{g}(z) < \alpha + \varepsilon$ . If  $z \in D_g$ , we are done. If not, recall that  $\tilde{g}(z) = \liminf_{y \rightarrow z, y \in D_g} g(y)$ . Since  $G_x$  is also a neighborhood of  $z$ , and  $D_g$  is dense in  $K$ , the set  $G_x \cap D_g$  is a nonempty neighborhood of  $z$  in the relative topology of  $D_g$ . This implies that there is  $w \in G_x \cap D_g$  such that  $g(w) < \alpha + 2\varepsilon = \gamma$  and contradicts (2.3), as claimed above. An analogous argument shows that  $\tilde{f}$  is upper semicontinuous. Besides,  $\tilde{f}(x) \leq \tilde{g}(x)$  for every  $x \in K$  since  $f, g$  form an admissible pair. Finally, the proof that  $[\tilde{f}, \tilde{g}] = [f, g]$  is straightforward: on the one hand,  $[\tilde{f}, \tilde{g}] \subset [f, g]$  is due to (2.1); on the other hand, to verify the reverse inclusion, consider  $h \in [f, g]$ . Then  $h(x) \geq \limsup_{y \rightarrow x, y \in D_g} f(y) = \tilde{f}(x)$  for every  $x \in K$ . Analogously,  $h(x) \leq \tilde{g}(x)$ , for every  $x \in K$ , hence  $h \in [\tilde{f}, \tilde{g}]$ .  $\square$

The characterization given in [10] of convex sets which are intersections of balls in  $(C(K), \|\cdot\|_\infty)$  can be easily formulated in terms of admissible pairs: *If  $K$  is a compact Hausdorff space, then  $C \subset C(K)$  is a nonempty intersection of closed balls if and only if  $C = [f, g]$ , where  $f, g$  form an admissible pair.*

Let us give a brief proof of this fact, for the sake of completeness. If we consider a family of balls  $\{B_i = h_i + r_i B\}$  satisfying  $C = \bigcap_i B_i \neq \emptyset$ , then  $C = [f, g]$  where  $f(t) = \sup_i \{h_i(t) - r_i\}$  and  $g(t) = \inf_i \{h_i(t) + r_i\}$  for each  $t \in K$ . It is clear that  $f$  is lower semicontinuous and  $g$  is upper semicontinuous. Given  $h \in C$ , we have  $\limsup_{s \rightarrow t} f(s) \leq h(t) \leq \liminf_{s \rightarrow t} g(s)$  for every  $t \in K$ , hence  $f, g$  form an admissible pair.

Conversely, suppose that  $C = \bigcap_{t \in K} \delta_t^{-1}[f(t), g(t)]$  where  $f, g$  are lower and upper semicontinuous, respectively. Consider  $h \in C(K)$  such that  $h \notin C$ . There is  $t_0 \in K$  such that  $h(t_0) \notin [f(t_0), g(t_0)]$ . Assume, for instance, that  $h(t_0) < f(t_0)$ . Notice that  $f$  is the supremum of a family of continuous functions on  $X$ , say  $\{\psi_i\}$ . Indeed, since  $K$  is normal,  $f$  is the pointwise supremum of the set of all continuous functions that are less or equal to  $f$  (which is nonempty, because  $f$  is bounded below). Consequently, there is  $i_0$  such that  $\psi_{i_0}(t_0) - h(t_0) = 2m > 0$ . Take  $M > 0$  satisfying  $\psi_{i_0}(t) + M \geq g(t)$  for every  $t \in K$  ( $g$  is bounded above). Finally, consider the ball  $D$  centered in  $\psi_{i_0} + (M - m)/2$  and having radius  $(M + m)/2$ . Then,  $C \subset D$ , but  $h \notin D$ . As a direct consequence of Lemma 2.1, we have  $C = [f, g] \neq \emptyset$  provided  $f, g$  form an admissible pair.

The above representation will be used repeatedly through this paper since it turns out to be very useful in virtue of the nice properties of semicontinuous functions. For instance, given an admissible pair  $f, g$ , the points of continuity of both functions can be characterized as follows.

**PROPOSITION 2.2.** *When  $f, g$  form an admissible pair,  $g$  is continuous at  $x_0 \in K$  if and only if  $g(x_0) = \sup \delta_{x_0}(C)$ , where  $C = [f, g] \subset C(K)$ . Analogously,  $f$  is*

continuous at  $x_0$  if and only if  $f(x_0) = \inf \delta_{x_0}(C)$ .

PROOF. Necessity was proved in [10, Theorem 4.3]. Let us sketch the idea of this implication, for the sake of completeness. Pick  $t_0 \in K$ , a point of continuity of  $g$ . Given  $\varepsilon > 0$ , our purpose is to find  $h \in [f, g]$  such that  $h(t_0) \geq g(t_0) - \varepsilon$ . To this end, Lemma 2.1 ensures that  $[f, g] \neq \emptyset$  and we begin by choosing an arbitrary function  $\varphi \in C$ . We may assume that  $\varphi(t_0) < g(t_0)$  (since otherwise there is nothing to prove) and  $\varphi(t_0) < g(t_0) - \varepsilon$  (considering a smaller  $\varepsilon$ , if necessary). By continuity of  $g$  and  $\varphi$  at  $t_0$ , there is a neighborhood  $G$  of  $t_0$  such that  $t \in G$  implies  $g(t) > g(t_0) - \varepsilon$  and  $\varphi(t) < g(t_0) - \varepsilon$ . Consider, finally, a Urysohn function  $\psi: K \rightarrow \mathbb{R}$  satisfying  $\psi(t) = 0$  for all  $t \in K \setminus G$ ,  $\psi(t_0) = 1$  and  $0 \leq \psi(t) \leq 1$  for each  $t \in K$ . We now modify  $\varphi$  in order to obtain the desired function  $h$ , as follows:  $h(t) = (1 - \psi(t))\varphi(t) + \psi(t)(g(t_0) - \varepsilon)$  for all  $t \in K$ . The argument for the points of continuity of  $f$  is analogous. To prove sufficiency, consider  $x_0 \in K$  such that  $g(x_0) = \sup \delta_{x_0}(C)$ . Consider the function  $\tilde{g}(x) = \sup_{h \in C} h(x)$ ,  $x \in K$ , which is well defined since  $C \neq \emptyset$ . Moreover,  $\tilde{g}$  is lower semicontinuous and  $\tilde{g}(x) \leq g(x)$ . Therefore we can write

$$g(x_0) = \tilde{g}(x_0) \leq \liminf_{x \rightarrow x_0} \tilde{g}(x) \leq \liminf_{x \rightarrow x_0} g(x),$$

so the upper semicontinuous function  $g$  is lower semicontinuous at  $x_0$  hence is continuous at this point. Again, the argument for  $f$  is completely analogous.  $\square$

To finish this section, we include the following useful result that applies, in particular, to Minkowski sums of sets which are intersections of closed balls in  $C(K)$ . It will be used in the proof of the main result of the next section. It is a special form of the Riesz decomposition property, but, in default of a suitable reference, we give a proof for greater clarity.

PROPOSITION 2.3. Every pair of (not necessarily continuous) functions  $\varphi, \psi : K \rightarrow \mathbb{R}$  with  $\varphi \leq \psi$  and every pair  $f, g \in C(K)$  with  $f \leq g$  satisfy

$$[f, g] + [\varphi, \psi] = [f + \varphi, g + \psi]$$

provided  $[\varphi, \psi] \neq \emptyset$ .

PROOF. For the case  $f = g$ , the conclusion holds even when  $f$  is not continuous:  $f + [\varphi, \psi] = [f + \varphi, f + \psi]$ . We will consider first the following special case:  $\varphi, \psi \in C(K)$ . Since

$$\begin{aligned} [f, g] + [\varphi, \psi] &= (f + [0, g - f]) + (\varphi + [0, \psi - \varphi]) \\ &= (f + \varphi) + [0, g - f] + [0, \psi - \varphi], \end{aligned}$$

we may assume that  $f = 0$  and  $\varphi = 0$ . To prove that  $[0, g] + [0, \psi] = [0, g + \psi]$ , we just need to show that

$$(2.4) \quad [0, g] + [0, \psi] \supset [0, g + \psi],$$

since the reverse inclusion is straightforward. The inclusion (2.4) is a consequence of the classical Riesz decomposition property [9]. For instance, every  $h \in [0, g + \psi]$  can be decomposed as  $h = h_1 + h_2$  where  $h_1 = \min\{h, g\}$  and  $h_2 = h - h_1$  are continuous functions satisfying  $h_1 \in [0, g]$  and  $h_2 \in [0, \psi]$ .

We can proceed now with the general case, namely that  $[f, g] + [\varphi, \psi] = [f + \varphi, g + \psi]$ . Again, the nontrivial inclusion is  $[f, g] + [\varphi, \psi] \supset [f + \varphi, g + \psi]$ . Consider  $h \in [f + \varphi, g + \psi]$ . If we can find a pair of continuous functions  $\tilde{f}, \tilde{g}$  such that  $\varphi \leq \tilde{f} \leq \tilde{g} \leq \psi$  and

$$(2.5) \quad f + \tilde{f} \leq h \leq g + \tilde{g},$$

then we have finished: the problem can be reduced to the special case since (2.5) implies that  $h \in [f, g] + [\tilde{f}, \tilde{g}]$  and, obviously,  $[\tilde{f}, \tilde{g}] \subset [\varphi, \psi]$ . To this end, given a continuous function  $\xi \in [\varphi, \psi]$ , we define

$$\tilde{g} = \max\{\xi, h - g\} \quad \text{and} \quad \tilde{f} = \min\{\tilde{g}, h - f\}.$$

Then,  $\varphi \leq \tilde{g} \leq \psi$  and  $\varphi \leq \tilde{f} \leq \tilde{g}$ . It is not difficult to check that (2.5) holds for this choice of  $\tilde{f}$  and  $\tilde{g}$ . □

The above result can be used to prove that, when  $B$  is the unit ball in  $C(K)$ ,  $C = \cap_i B_i$  is an intersection of balls and  $\lambda \geq 0$ , then

$$(2.6) \quad C + \lambda B = \cap_i B_i + \lambda B = \cap_i (B_i + \lambda B).$$

In particular,  $\mathcal{M}$  is always ball stable in  $(C(K), \|\cdot\|_\infty)$ . The proof of (2.6) follows the lines of [5, Proposition 2.1], where the same equality was proved in the case that  $K$  was extremally disconnected.

### 3. Stability and ball stability of $\mathcal{M}$

The importance of the ball stability of  $\mathcal{M}$  is related, for instance, to the continuity of the *ball hull mapping*, which associates to each bounded set  $D$  its *ball hull*  $\widehat{D}$ , the intersection of all closed balls containing  $D$ , as illustrated in the following remark. The continuity of the ball hull mapping, in turn, implies that  $\mathcal{M}$  is topologically closed (see the argument following Remark 3.2).

REMARK 3.1. The ball hull mapping under the Hausdorff metric is Lipschitz with constant 1 provided  $\mathcal{M}$  is ball stable.

PROOF. Since the ball hull mapping restricted to  $\mathcal{M}$  is the identity, it is clear that the constant must be greater or equal than 1. Now consider two bounded sets  $C, D$  satisfying  $\text{dist}(C, D) < \varepsilon$ . If  $\mathcal{M}$  is ball stable, then  $C \subset D + \varepsilon B \subset \overline{\widehat{D} + \varepsilon B} \in \mathcal{M}$ , and this implies  $\widehat{C} \subset \overline{\widehat{D} + \varepsilon B}$ . Analogously, by a symmetric argument,  $\widehat{D} \subset \overline{\widehat{C} + \varepsilon B}$  and hence  $\text{dist}(\widehat{C}, \widehat{D}) \leq \varepsilon$ .  $\square$

REMARK 3.2.  $\mathcal{M}$  is ball stable in  $(C(K), \|\cdot\|_\infty)$ , for every compact Hausdorff space  $K$ . As a consequence,  $\mathcal{M}$  is (topologically) closed in these spaces.

The first part of Remark 3.2 follows directly from Proposition 2.3. The argument to prove the second part is quite simple. Since  $\mathcal{M}$  is ball stable in  $(C(K), \|\cdot\|_\infty)$ , the ball hull mapping is continuous. Consider a sequence  $\{C_n\} \subset \mathcal{M}$  which converges to  $C \in C(K)$ . Then

$$\begin{array}{ccc} C_n & \longrightarrow & C \\ \parallel & & \\ \widehat{C}_n & \longrightarrow & \widehat{C} \end{array}$$

thus implying that  $C = \widehat{C}$ . Let us mention that we do not know whether the continuity of the ball hull mapping implies the ball stability of  $\mathcal{M}$ . On the other hand, it is tempting to think that the ball stability of  $\mathcal{M}$  is a property shared by every Banach space. Indeed, it is explicitly mentioned in [5] that there are no examples of spaces for which  $\mathcal{M}$  is not ball stable. Here we present one, showing that the conjecture above is false. Recall that the convex body  $\overline{C + \lambda B}$  is called an *outer parallel body* of  $C$ .

THEOREM 3.3. *The outer parallel body of an intersection of closed balls needs not be an intersection of closed balls.*

PROOF. Let  $X = \{f \in C[0, 1] : 2f(1/2) = f(1)\}$ , let  $C = [f, g]$  in  $C[0, 1]$  with  $g(t) = \chi_{[1/2, 1]}(t)$  and  $f(t) = -\chi_{[0, 1/2]}(t)$ . Then  $C' = [f, g] \cap X$  is an intersection of balls in  $X$  but this is not the case for  $D = \overline{C' + B}$ , where  $B$  is the unit ball of  $X$ .

It is not difficult to show that  $C'$  is an intersection of balls. Indeed, consider  $\xi \in X \setminus C'$ . Say, for instance, that there is  $t_0 \in [0, 1]$  such that  $\xi(t_0) > g(t_0)$ . If  $t_0 \in [1/2, 1]$ , then  $\xi \notin B$  and  $C' \subset B$ . Otherwise, if  $t_0 \in [0, 1/2)$ , consider

$$\zeta(x) = \begin{cases} -1 & \text{if } 0 \leq x \leq t_0, \\ \alpha x - \alpha/2 & \text{if } t_0 \leq x \leq 1/2, \\ 0 & \text{if } 1/2 \leq x \leq 1, \end{cases}$$

where  $\alpha = (1/2 - t_0)^{-1}$ . Then  $\zeta \in C'$  and, moreover,  $C' \subset \zeta + B$ , while  $\|\xi - \zeta\| > 1$ , thus showing that  $C'$  is an intersection of closed balls of radius 1. We will show now that the function  $h(t) = 2t$  (which is in  $X \setminus D$ ) belongs to every closed ball in  $X$  containing  $D$ . Indeed, a closed ball in  $X$  is just a set of the form  $\phi + \lambda B$  where  $\phi \in X$  and  $\lambda > 0$ . For every  $t \in [0, 1/2)$ ,  $\phi(t) + \lambda \geq 1$ . Also, for every  $t \in (1/2, 1)$ ,  $\phi(t) + \lambda \geq 2$ . Since  $\phi$  is continuous, the above inequalities are true for  $t \in [0, 1/2]$  and  $[1/2, 1]$ , respectively. Then  $h(t) \leq \phi(t) + \lambda$ , for every  $t \in [0, 1]$ . A similar argument shows that  $\phi(t) - \lambda \leq -2$  if  $t \in [0, 1/2]$  and  $\phi(t) - \lambda \leq -1$  if  $t \in [1/2, 1]$ . Consequently,  $\phi(t) - \lambda \leq h(t) \leq \phi(t) + \lambda$  for every  $t \in [0, 1]$ , so  $\|h - \phi\|_\infty \leq \lambda$ .  $\square$

Given a convex set  $C$  and  $\lambda$ , the set  $C \sim \lambda B = \{x \in C : \text{dist}(x, X \setminus C) \geq \lambda\}$  is called an *inner parallel body* of  $C$ . The above result should be compared with the one obtained in [6]: if  $C$  is an intersection of closed balls then  $C \sim \lambda B$  is again an intersection of closed balls (which can possibly be empty). For a systematic study of parallel bodies (in finite dimensional spaces) the reader is referred to the authoritative book by Schneider [12]. We now come to the main result of this section which characterizes those Hausdorff compacta for which  $\mathcal{M}$  is stable in  $(C(K), \|\cdot\|_\infty)$ . This result contrasts the fact that  $\mathcal{M}$  is always ball stable in  $(C(K), \|\cdot\|_\infty)$ .

**THEOREM 3.4.** *Given a compact Hausdorff space  $K$ ,  $\mathcal{M}$  is stable in  $(C(K), \|\cdot\|_\infty)$  if and only if  $K$  is extremally disconnected.*

**PROOF.** The stability of  $\mathcal{M}$  in  $(C(K), \|\cdot\|_\infty)$  when  $K$  is extremally disconnected, was proved in [5] using the fact that, once we fix an extreme point  $e$  of the unit ball, there is a unique way of making  $C(K)$  into a complete vector lattice such that the order interval  $[-e, e]$  is just the unit ball. Here we outline a direct proof. The main idea is the following: if  $K$  is extremally disconnected and  $C = [f, g] \subset C(K)$  is an intersection of balls, then there exist two continuous functions  $\tilde{f}, \tilde{g} : K \rightarrow \mathbb{R}$  satisfying

$$(3.1) \quad f(x) \leq \tilde{f}(x) \leq \tilde{g}(x) \leq g(x)$$

and also  $[f, g] = [\tilde{f}, \tilde{g}]$ . The result follows by applying Proposition 2.3. In order to find  $\tilde{f}$  and  $\tilde{g}$ , we modify the functions  $f$  and  $g$  in a similar fashion as we did in Lemma 2.1:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in D_f, \\ \liminf_{y \rightarrow x, y \in D_f} f(y) & \text{if } x \notin D_f, \end{cases}$$

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in D_g, \\ \limsup_{y \rightarrow x, y \in D_g} g(y) & \text{if } x \notin D_g. \end{cases}$$



Then,  $\tilde{f}$  and  $\tilde{g}$  still form an admissible pair which satisfies (3.1) and  $[f, g] = [\tilde{f}, \tilde{g}]$ . Let us show that  $\tilde{g}$  is continuous (the proof for  $\tilde{f}$  is analogous). Suppose, on the contrary, that there is  $x_0 \in K$  such that  $\tilde{g}$  is not continuous at  $x_0$ . Then,

$$\liminf_{y \rightarrow x_0, y \in D_g} \tilde{g}(y) = \liminf_{y \rightarrow x_0, y \in D_g} g(y) = \alpha < \tilde{g}(x_0).$$

Let  $\beta = (1/2)(\tilde{g}(x_0) + \alpha)$ . Since  $\tilde{g}$  is upper semicontinuous,  $G = \tilde{g}^{-1}(-\infty, \beta)$  is open. Moreover,  $\overline{G}$  is open because  $K$  is extremally disconnected. Notice that  $x_0 \in \overline{G}$ , since  $\liminf_{y \rightarrow x_0, y \in D_g} \tilde{g}(y) < \beta$ , so there is an open neighborhood  $U_{x_0}$  satisfying  $x_0 \in U_{x_0} \subset \overline{G}$ . However this implies  $\tilde{g}(x_0) = \limsup_{y \rightarrow x_0, y \in D_g} g(y) \leq \beta$ , a contradiction.

To prove the converse, assume that there is an open set  $G \subset K$  such that  $\overline{G}$  is not open (in particular  $\overline{G} \neq K$ ). We construct a pair of sets  $C, D \in \mathcal{M}$  such that  $\overline{C + D}$  is not an intersection of balls. To this end, define  $f, g : K \rightarrow \mathbb{R}$  by

$$(3.2) \quad g(x) = \begin{cases} 1 & \text{if } x \in K \setminus G, \\ 0 & \text{if } x \in G, \end{cases} \quad f(x) = \begin{cases} 0 & \text{if } x \in K \setminus \overline{G}, \\ -1 & \text{if } x \in \overline{G}. \end{cases}$$

It can be readily verified that  $g$  is upper semicontinuous,  $f$  is lower semicontinuous and  $f \leq g$ . The points of continuity of  $g$  and  $f$  are  $G \cup (K \setminus \overline{G})$ . Our strategy is to define  $C = [f, g]$ , then to choose  $D = -C$  and finally to prove that  $\overline{C - C} = C - C \notin \mathcal{M}$ . Consider

$$S = (\overline{G} \setminus G) \cap (\overline{K \setminus \overline{G}})$$

and define  $\tilde{g} = 1 - \chi_S$  and  $\tilde{f} = -\tilde{g}$ . We claim that  $C - C = [\tilde{f}, \tilde{g}]$ . Notice first that  $S \neq \emptyset$  because  $\overline{G}$  is not open. It is clear also that  $C - C \subset [\tilde{f}, \tilde{g}]$ , since  $\varphi(y) = 0$  for every function  $\varphi \in C$  and each  $y \in S$ , hence for every function in  $-C$  as well. To prove the reverse inclusion, consider  $h \in [\tilde{f}, \tilde{g}]$  and the usual decomposition  $h = h^+ + h^-$ , where  $h^+ = \max\{h, 0\}$  and  $h^- = \min\{h, 0\}$ . We can also decompose  $h$  as follows:  $h = h_1 + h_2$ , where

$$h_1(x) = \begin{cases} h^-(x) & \text{if } x \in \overline{G}, \\ h^+(x) & \text{if } x \in K \setminus \overline{G}, \end{cases} \quad h_2(x) = \begin{cases} h^+(x) & \text{if } x \in \overline{G}, \\ h^-(x) & \text{if } x \in K \setminus \overline{G}. \end{cases}$$

Both  $h_1$  and  $h_2$  are continuous since  $h(y) = 0$  for each  $y \in S$  and then the Pasting Lemma can be applied [11]. It is straightforward to check that  $h_1 \in C$  and  $h_2 \in -C$ . On the other hand,  $C - C = [\tilde{f}, \tilde{g}]$  and  $[\tilde{f}, \tilde{g}] = \bigcap_{x \in K} \delta_x^{-1}([\tilde{f}(x), \tilde{g}(x)])$ . Hence, as an intersection of closed sets, the set  $C - C$  must be closed. Let us check that  $C - C$  is not an intersection of balls. Assume, on the contrary, the existence of  $\hat{f}$  and  $\hat{g}$ , lower and upper semicontinuous, respectively, satisfying

$$C - C = [\tilde{f}, \tilde{g}] = [\hat{f}, \hat{g}].$$

When  $x \in K$  is a point of continuity for  $f$  and  $g$  we know, using Lemma 2.2, that  $\sup \delta_x(C - C) = \tilde{g}(x)$  and  $\inf \delta_x(C - C) = \tilde{f}(x)$ . Consequently, at these points,  $\hat{f}(x) \leq \tilde{f}(x)$  and  $\hat{g}(x) \geq \tilde{g}(x)$ . Therefore  $\hat{f}(x) \leq -1$  and  $\hat{g}(x) \geq 1$  for every  $x \in (G \cup (K \setminus \bar{G}))$ . Since  $\hat{f}$  and  $\hat{g}$  are lower and upper semicontinuous, respectively, this implies that  $\hat{f}(x) \leq -1$  and  $\hat{g}(x) \geq 1$  for every  $x \in \bar{G}$ , thus for every  $x \in K$ . Hence  $[\hat{f}, \hat{g}]$  contains the unit ball, which contradicts the fact that  $[\hat{f}, \hat{g}] = C - C \subset \delta_y^{-1}(0)$  for each  $y \in S$ .  $\square$

A closed, convex and bounded set  $C$  is a *Mazur set* provided that for every hyperplane  $H$  such that  $\text{dist}(C, H) > 0$ , there is a ball  $D$  satisfying  $C \subset D$  and  $D \cap H = \emptyset$ . In virtue of the Hahn-Banach theorem, Mazur sets are intersections of balls which simply satisfy a stronger separation property. When every intersection of balls is a Mazur set, we say that the space is a *Mazur space*. This class was introduced in [5], where some of its structural properties were investigated. In particular, it turns out that  $\mathcal{M}$  is always stable in these spaces. On the other hand, it was also proved in [5] that  $(C(K), \|\cdot\|_\infty)$  is a Mazur space when  $K$  is extremally disconnected. These results, together with Theorem 3.4, prove that  $(C(K), \|\cdot\|_\infty)$  is a Mazur space if and only if  $K$  is extremally disconnected.

Some arguments used in the proof of Theorem 3.4 can be used to characterize the extremal disconnectedness of  $K$  in terms of two classic notions from convex geometry. Recall that a closed, bounded and convex set  $C$  in a Banach space has *constant width*  $d > 0$  if, for every  $f \in X^*$  with  $\|f\| = 1$ , we have  $\sup f(C - C) = d$ ; we say that  $C$  is *diametrically maximal* if, for every  $x \notin C$ ,  $\text{diam}(\{x\} \cup C) > \text{diam } C$ . Sets with constant width are always diametrically maximal [1]. The two notions coincide in any two dimensional space, but they fail to coincide in certain 3-dimensional spaces. In the case of  $(C(K), \|\cdot\|_\infty)$ ,  $K$  a compact Hausdorff space, they coincide if and only if  $K$  is extremally disconnected. To prove necessity, assume that  $K$  is extremally disconnected and  $C$  is diametrically maximal. Then  $C$  is an intersection of balls [1] and, as we observed in the proof of Theorem 3.4, there are two continuous functions  $f, g : K \rightarrow \mathbb{R}$  satisfying  $C = [f, g]$ . This fact, together with the characterizations given in [10], proves that  $C$  has constant width. Conversely, if  $K$  is not extremally disconnected, we can consider the functions  $f, g$  defined in (3.2). The set  $[f, g]$  is diametrically maximal, but has no constant width.

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