Canad. Math. Bull.Vol. **35** (1), 1992 pp. 116–125

FINITE LATTICES OF PROJECTIONS IN FACTORS AND APPROXIMATELY FINITE C^*-ALGEBRAS

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ABSTRACT. A unique factorisation theorem is obtained for tensor products of finite lattices of commuting projections in a factor. This leads to unique tensor product factorisations for reflexive subalgebras of the hyperfinite II_1 factor which have irreducible finite commutative invariant projection lattices. It is shown that the finite refinement property fails for simple approximately finite C*-algebras, and this implies that there is no analogous general result for finite lattice subalgebras in this context.

Recently we have made use of a refinement theory for antisymmetric connected reflexive relations in the analysis of isomorphism classes of tensor products of triangular operator algebras [7]. In the present note we give an account of a similar analysis for the simplest nontriangular context, namely for subalgebras of factors determined by a finite lattice of commuting projections in the factor. In the case of the hyperfinite II_1 factor such subalgebras are in bijective correspondence with directed finite graphs with vertex set supporting a probability measure induced by the trace. The unique factorisation theory of the direct product of such structures leads to a normal form for their subalgebras in terms of tensor indecomposable factors (Theorem 2.1).

In contrast we show that in the class of simple approximately finite (AF)C*-algebras indecomposable tensor product factorisation need not be unique. Specifically there are simple AF C*-algebras A_1, A_2, A_3, A_4 with $K_0(A_i) = \mathbb{Z}^2$, $1 \leq i \leq 4$, such that each algebra is $\otimes$-indecomposable, $A_1 \otimes A_2 = A_3 \otimes A_4$, and yet A_1 is not isomorphic to A_3 or A_4. Thus we cannot expect unique factorisation for multiple tensor products of elementary non-self-adjoint subalgebras determined by a finite projection lattice; uniqueness already fails for trivial subalgebras associated with the trivial lattice $\{0, 1\}$.

The direct or cardinal product of two reflexive binary relations R and S on the sets X and Y respectively is the relation $R \times S$ on $X \times Y$ such that $(x_1 y_1)\ (R \times S)\ (x_2, y_2)$ if and only if $x_1\ R\ x_2$ and $y_1\ S\ y_2$. The set theorists Chang, Jonsson and Tarskii have developed (in [2]) an important refinement and unique factorisation theory for binary relations in which, in particular, it is shown that if R is a connected antisymmetric reflexive binary relation with two indecomposable factorisations $R_1 \times \cdots \times R_n$ and $S_1 \times \cdots \times S_m$, then $n = m$ and for some permutation π, and R_i is isomorphic to $S_{\pi(i)}$ for all i. In the finite case there is naturally associated with such a relation R a subalgebra $A(R)$ of the complex matrix algebra $M_n(\mathbb{C})$, where n is the cardinality of the underlying set X, and furthermore $A(R_1 \times R_2)$ coincides with the tensor product $A(R_1) \otimes A(R_2)$. We see then that there is

Received by the editors October 16, 1989; revised April 4, 1990.
AMS subject classification: 47D25, 46L05.

a direct connection between the product structure of binary relations and the classification of tensor products of operator algebras. This connection, and its generalisations, were exploited in [7] to obtain classifications of finite and infinite tensor products of triangular operator algebras. An earlier analysis of this type was given by Arveson in [1, section 3] for certain reflexive operator algebras with commutative subspace lattices that were infinite tensor products of finite join-irreducible lattices.

In the first section we discuss the representation theory for finite unital distributive lattices in factors and distinguish the notions of external (spatial) factorisations $\mathcal{L}_1 \otimes \mathcal{L}_1$ and internal factorisations $\mathcal{L}_1 \otimes_f \mathcal{L}_2$. The unique factorisation theorem for irreducible lattices in a II_1 factor is obtained rather easily from the binary relation theory mentioned above, and in section two this is applied in the hyperfinite context to obtain a unique factorisation theory for reflexive subalgebras with finite invariant projection lattice.

1. **Refinement for Representations of Finite Distributive Lattices.** A *faithful representation*, or, more briefly, a *realisation* of a finite distributive lattice $\mathcal{M}$ is a lattice $\mathcal{L}$ of commuting self-adjoint projections in a unital operator algebra M, together with a faithful lattice isomorphism $\mathcal{M} \longrightarrow \mathcal{L}$. We assume always that $\mathcal{M}$ is unital in the sense that it contains a greatest element 1 and a least element 0, and we assume that representations $\mathcal{M} \longrightarrow \mathcal{L}$ are unital, with $0 \longrightarrow 0$ and $1 \longrightarrow I$. For convenience we often suppress the underlying isomorphism and speak of $\mathcal{L}$ as a realisation of $\mathcal{M}$. The lattice operations for $\mathcal{L}$ are given by $P \vee Q = P + Q - PQ$, $P \wedge Q = PQ$. The commuting projection lattices $\mathcal{L}$ are also called *finite commutative subspace lattices*, since they are associated with reflexive algebras with commutative invariant subspace lattice. (See [3], Part IV).

Considering the set X of lattice homomorphisms from $\mathcal{M}$ to the trivial lattice $\{0, 1\}$ we can view $\mathcal{M}$ as a lattice of subsets of X, and this will be convenient. Thus we consider a finite set X with an antisymmetric partial order $x < y$, which we also write as $x\ R\ y$, and consider $\mathcal{M}$ as the lattice of all *decreasing* subsets E of X. Recall that E is decreasing if x belongs to E whenever $x < y$ and y belongs to E.

Suppose that $\mathcal{L}$ is a realisation of $\mathcal{M}$ in the unital operator algebra M. An *interval projection* of $\mathcal{L}$ is a non zero projection of the form $L_1 - L_2$ with $L_1 \geq L_2$, projections in $\mathcal{L}$. An atomic interval, or *atom*, is an interval projection which dominates no other interval projection. Clearly the underlying set $X = X(\mathcal{M})$ for $\mathcal{M}$ can be identified with the set of atoms of $\mathcal{L}$ and furthermore $x < y$ if and only if the corresponding atoms $Q(x)$, $Q(y)$ have the propery that $LQ(x) = Q(x)$ whenever $LQ(y) = Q(y)$. Thus if $L(X, R)$ denotes the lattice $\mathcal{M}$ associated with X, R then the set of all realisations of $\mathcal{M}$ in the operator algebra M corresponds simply to the set of paritionings of the identity by non zero projections $Q(x)$, x in X. We say that two *realisations* $\mathcal{L}$ and $\mathcal{L}'$ in M are *equivalent* if there is a unitary operator u in M such that $(\mathrm{Ad}\, u)(Q(x)) = Q'(x)$, where $Q(x)$ and $Q'(x)$, $x \in X$, are the atoms of $\mathcal{L}$ and $\mathcal{L}'$ respectively. The following lemma is immediate.

LEMMA 1.1. *Let $\mathcal{L}$ and $\mathcal{L}'$ be realisations of a finite distributive lattice in the unital operator algebra M. Then $\mathcal{L}$ and $\mathcal{L}'$ are equivalent if and only if the associated atomic projections $Q(x)$ and $Q'(x)$ are equivalent for each x.*

https://doi.org/10.4153/CMB-1992-017-9 Published online by Cambridge University Press

We also say that the finite CSL lattices $\mathcal{L}$ and $\mathcal{L}'$ are *unitarily equivalent* in M if there is a unitary v in M such that $(\mathrm{Ad}\, v)(\mathcal{L}) = \mathcal{L}'$. Clearly two realisations $\mathcal{M} \longrightarrow \mathcal{L}$ and $M \longrightarrow \mathcal{L}'$ are equivalent realisations if and only if the image lattices $\mathcal{L}$ and $\mathcal{L}'$ are unitarily equivalent finite CSL lattices.

It follows from the comparison theory for projections that if M is a II_1 factor then the equivalence classes of realisations of the lattice $L(X,R)$ are in bijective correspondence with faithful probability measures μ on X, the correspondence being given by $\mu(\{x\}) = \tau\big(Q(x)\big)$ where τ is the normalised trace on M. Similarly, the equivalence classes of realisations of $L(X,R)$ in factors of type I and type II_∞ are in correspondence with the appropriate class of positive faithful extended real valued measures on X. In the type III case all realisations are unitarily equivalent.

Define the tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$ of two unital finite distributive lattices $\mathcal{M}_1 = L(X,R)$, $\mathcal{M}_2 = L(Y,S)$ as the lattice $L(X\times Y, R\times S)$. Similarly, if $\mathcal{L}_1 \otimes \mathcal{L}_2$ are realisations of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, then define $\mathcal{L}_1 \otimes \mathcal{L}_2$ to be the associated realisation of $\mathcal{M}_1 \otimes \mathcal{M}_2$ in $M_1 \otimes M_2$ which is generated by the lattices $\mathcal{L}_1 = \{L_1 \otimes I : L_1 \in \mathcal{L}_1\}$, $\mathcal{L}_2 = \{I \otimes L_2 : L_2 \in \mathcal{L}_2\}$. The realisation $\mathcal{L}$ in M is said to admit an *external* (or spatial or tensor) factorisation if there are realisations $\mathcal{L}_i$ in M_i, for $i = 1,2$ such that $M = M_1 \otimes M_2$ and $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$.

On the other hand, there is a simpler notion of *internal* factorisation for a realisation $\mathcal{L}$ in an infinite-dimensional factor M. Write $\mathcal{L} = \mathcal{L}_1 \otimes_f \mathcal{L}_2$ if $\mathcal{L}$ has the complete invariant (X,R,μ) and if there is a factorisation $R = R_1 \times R_2$ with respect to coordinates $X = X_1 \times X_2$ such that μ is a product measure $\mu_1 \times \mu_2$, and (X_1,R_1,μ_1) and (X_2,R_2,μ_2) are the invariants for $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. For the internal factorisation the factor lattices are only determined up to unitary equivalence in M. In the II_1 case the probability measures μ_1 and μ_2 can be recovered from $\mu_1 \times \mu_2$ as marginal measures and so $\mathcal{L}_1$ and $\mathcal{L}_2$ can be regarded as independent sublattices of $\mathcal{L}$. In general, however, and in the I_∞ case for example, the lattice $\mathcal{L}_1 \otimes_f \mathcal{L}_2$ need not have sublattices with the same order multiplicity type as $\mathcal{L}_1$ or $\mathcal{L}_2$. Of course this remark also applies to external tensor products.

In view of the stability of the hyperfinite II_1 factor under spatial tensor product the external and internal factorisation of finite CSL lattices can be identified. There are similarly close connections between these factorisations in the case of II_∞ hyperfinite factors. In the type III case factorisation theory is purely lattice theoretic.

We now formulate the fundamental refinement theorem for connected reflexive antisymmetric binary relations obtained by Chang, Jonsson and Tarskii [2]. See also the simplified development in [7, section 2].

Let $R_i \subset X_i \times X_i$ be a (not necessarily finite) reflexive binary relation on the set X_i, $i = 1,2$. Then the *direct* or *cardinal product* $R_1 \times R_2$ is the reflexive binary relation on $X_1 \times X_2$ consisting of the pairs $\big((x_1,x_2),(x_1',x_2')\big)$ such that $(x_i,x_i') \in R_i$ for $i = 1,2$. We also write $x\ R\ y$ to indicate $(x,y) \in R$. A relation R on the set X is *connected* if for every pair of points x,y in X there is a sequence $x_1,\ldots,x_r$ such that $x = x_1$, $y = x_r$ and either $x_i\ R\ x_{i+1}$ or $x_{i+1}\ R\ x_i$ for each $i = 1,\ldots,r-1$. Write $R \approx S$ to indicate that the binary relations R on X and S on Y are isomorphic (by means of a bijection $X \longrightarrow Y$). A family

$\mathcal{F}$ of reflexive binary relations, which is closed under direct products, is said to have the *finite refinement property*, if whenever $R_1 \times \cdots \times R_n \approx S_1 \times \cdots \times S_m$, with all factor relations in $\mathcal{F}$, it follows that there exist relations T_{ij} in $\mathcal{F}$ such that $S_i \approx T_{i1} \times \cdots \times T_{in}$, $1 \leq i \leq m$, and $R_j \approx T_{1j} \times \cdots \times T_{mj}$, $1 \leq j \leq n$.

A stronger property is the *strict refinement property* for $\mathcal{F}$ which ensures that whenever $\theta : R_1 \times \cdots \times R_n \longrightarrow S_1 \times \cdots \times S_m$ is a binary relation isomorphism, as before, then the product relations have a common refinement, compatible with θ, in the following sense; there exist direct factorisations $X_i = X_{i1} \times \cdots \times X_{in}$ and $Y_j = Y_{1j} \times \cdots \times Y_{mj}$, and relations T_{ij} on X_{ij} and U_{ij} on Y_{ij}, such that $R_i = T_{i1} \times \cdots \times T_{in}$, and $S_j = U_{ij} \times \cdots \times U_{mj}$, for $1 \leq i \leq m$, $1 \leq j \leq m$, and θ admits a factorisation $\theta = \prod_{i=1}^{m} \prod_{j=1}^{n} \theta_{ij}$ where θ_{ij} maps T_{ij} onto U_{ij}.

It is easy to see, for example, that equivalence relations have the refinement property, but not the strict refinement property.

THEOREM 1.2. *The family of connected antisymmetric reflexive binary relations has the strict finite refinement property.*

An immediate consequence of Theorem 1.2 is that if a connected antisymmetric reflexive binary relation is written as a finite product of directly indecomposable relations then this representation is unique up to order and isomorphism of the factor relations. For finite relations of course the existence of such factorisations is immediate and it is this unique factorisation property that we exploit for representations of finite distributive lattices in the next theorem.

The unique factorisation will be used in the following form: If $\pi : X_1 \times \cdots \times X_n \longrightarrow Y_1 \times \cdots \times Y_m$ is a bijection inducing an isomorphism between $R = R_1 \times \cdots \times R_n$ and $S = S_1 \times \cdots \times S_m$, with the factor relations finite connected reflexive antisymmetric and directly indecomposable, then $n = m$ and there are isomorphisms π_i from (X_i, R_i) to $(Y_{\sigma(i)}, S_{\sigma(i)})$, for some permutation σ, such that π factors as $\pi = \pi_1 \times \cdots \times \pi_n$ (with the range coordinates reordered).

A representation $\mathcal{M} \longrightarrow \mathcal{L}$ with $\mathcal{L}$ a unital commutative projection lattice in the operator algebra M is said to be *indecomposable* (or M-indecomposable) if whenever $\mathcal{L} = \mathcal{L}_1 \otimes_f \mathcal{L}_2$ (up to unitary equivalence) it follows that either $\mathcal{L}_1$ or $\mathcal{L}_2$ is the trivial lattice $\{0, I\}$. Similarly, we say that the finite distributative unital lattice $\mathcal{M}$ is indecomposable if it admits no nontrivial tensor factorisation. If $\mathcal{M}$ is indecomposable, then certainly every realisation of $\mathcal{M}$ is indecomposable, but simple examples show that $\mathcal{M}$ may be decomposable and have indecomposable representations. On the other hand $\mathcal{M}$ is said to be *irreducible* if, in the standard representation $\mathcal{M} = L(X, R)$, the finite relation R is connected, and this is equivalent to the irreducibility of any (and hence all) realisations of $\mathcal{M}$. Here a realisation $\mathcal{L}$ is said to be *irreducible* if whenever $I = L_1 + L_2$ with L_1 and L_2 orthogonal projections in $\mathcal{L}$, it follows that $L_1 = I$ or $L_2 = I$.

A considerably stronger notion than the irreducibility of the finite unital distributive lattice $\mathcal{M}$ is *join-irreducibility* in which the identity 1 cannot be the join of *any* two proper lattice elements. Arveson [1] has obtained the following unique factorisation theorem (and refinement property) for this particular class of lattices by direct methods.

THEOREM 1.3. *Let $\mathcal{M}$ be an irreducible finite unital distributive lattice. Then $\mathcal{M}$ admits an indecomposable tensor product factorisation $\mathcal{M} = \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$, and such a representation is unique up to order and isomorphism of the factor lattices.*

PROOF. This is immediate from the unique factorisation property of connected reflexive antisymmetric binary relations and the definition of tensor products of finite distributive lattices. ∎

There is also, of course, a refinement theorem for these lattices, but it is the above result together with the strict form of the uniqueness of factorisations, which is used in the following proof.

THEOREM 1.4. *Let $\mathcal{L}$ be a nontrivial irreducible finite unital commuting projection lattice in the II_1 factor M. Then there is an internal factorisation $\mathcal{L} = \mathcal{L}_1 \otimes_f \cdots \otimes_f \mathcal{L}_n$, where $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are indecomposable nontrivial commuting projection lattices in $\mathcal{M}$, and furthermore such factorisations are unique up to the order and the unitary equivalence classes of the factor lattices.*

PROOF. Let $\mathcal{L} = \mathcal{L}_1 \otimes_f \cdots \otimes_f \mathcal{L}_n$, $\mathcal{L} = \mathcal{N}_1 \otimes_f \cdots \otimes_f \mathcal{N}_m$ be two indecomposable internal factorisations of $\mathcal{L}$ as in the statement of the theorem. (The existence of such factorisations is clear.) The associated coordinate sets $X_i = X(\mathcal{L}_i)$ and $Y_j = X(\mathcal{N}_j)$ are finite sets, which are not singletons, carrying the induced relations R_i and S_j and the probability measures μ_i and ν_j respectively, for $1 \leq i \leq n$, $1 \leq j \leq m$. Thus there is a bijection $\tilde{\pi}$ from $X_1 \times \cdots \times X_n$ to $Y_1 \times \cdots \times Y_m$ which identifies the product measures $\mu_1 \times \cdots \times \mu_n$ and $\nu_1 \times \cdots \times \nu_m$ and effects an isomorphism between the relations $R = R_1 \times \cdots \times R_n$ and $S = S_1 \times \cdots \times S_m$. Each R_i and each S_i is finite, and so there exist indecomposable direct factorisations

$$R_i = R_{i,1} \times \cdots \times R_{i,n_i}, \quad 1 \leq i \leq n,$$
$$S_j = S_{j,1} \times \cdots \times S_{j,m_j}, \quad 1 \leq j \leq m,$$

together with underlying factorisations $X_i = X_{i,1} \times \cdots \times X_{i,n_i}$, $Y_j = Y_{j,1} \times \cdots \times Y_{j,n_j}$. Since $\mathcal{L}_1$ is indecomposable the measure μ_1 does not admit a factorisation $m_1 \times m_2$ with respect to any proper partitioning of the n_i coordinates into two sets.

Since $\mathcal{L}$ is irreducible its underlying relation R is connected and so by Theorem 1.2, and the discussion following it, the families

$$A = \{X_{i,k} : 1 \leq k \leq n_i,\ 1 \leq i \leq n\},$$
$$B = \{Y_{j,k} : 1 \leq k \leq m_j,\ 1 \leq j \leq m\},$$

have the same cardinality, there is a bijection $\pi : A \longrightarrow B$ and there are bijections $\pi_{i,k}$: $X_{i,k} \longrightarrow \pi(X_{i,k})$ inducing isomorphisms between $R_{i,k}$ and $S_{j,\ell}$, where $Y_{j,\ell} = \pi(X_{i,k})$. Furthermore, it follows from the strict refinement theorem that these maps can be chosen so that the automorphism

$$\prod_{i,k} \pi_{i,k}$$

https://doi.org/10.4153/CMB-1992-017-9 Published online by Cambridge University Press

is the identity map on $X(\mathcal{L})$.

We assert that for each $1 \leq i \leq n$ there is a $j = \sigma(i)$ such that $m_j = n_i$ and $\{\pi(X_{i,k}) \colon 1 \leq k \leq n_i\} = \{Y_{j,k} \colon 1 \leq k \leq m_j\}$. If this were not so then, relabelling, we can assume that $\pi(X_{1,i}) = Y_{1,i}$ for $1 \leq i \leq t$, and $\pi(X_{1,i}) = Y_{j_i k_i}$ with $j_i \neq 1$ for $t < i \leq n_1$, where $1 \leq t \leq n_i$. Let Σ_1 be the algebra of elementary cylinder sets associated with coordinates in $X_{1,1}, \ldots, X_{1,t}$, let Σ_2 be the corresponding algebra for the coordinates in $X_{1,t+1}, \ldots, X_{1,n_1}$ and let $\Sigma = \Sigma_1 \times \Sigma_2$ be the generated algebra of sets of the form $E \times X_2 \times \cdots \times X_n$. The restriction of μ to Σ is identifiable with μ_1. Since $\mu = \nu_1 \times \cdots \times \nu_m$, however, it follows that μ_1 admits a factorisation $m_1 \times m_2$ with respect to $\Sigma = \Sigma_1 \times \Sigma_2$, with m_1 the restriction of $\nu_2 \times \cdots \times \nu_m$. This contradiction of the indecomposability of $\mathcal{L}_1$ establishes the assertion.

We have now shown the existence of isomorphisms π_i from (X_i, R_i) to $(Y_{\sigma(i)}, S_{\sigma(i)})$ such that the identity map factors as $\pi_1 \times \cdots \times \pi_n$. Since the factors of a probability measure can be recovered as marginal measures (when these factor measures are themselves probability measures) and since the identity map is measure preserving, it follows that π_i identifies μ_i with $\nu_{\sigma(i)}$ for each i, and this completes the argument.

REMARK. The last theorem can be interpreted graph theoretically, by viewing the complete invariant (R, μ) as a directed finite graph supporting the probability measure μ. The methods above show that this class of connected probability graphs has the finite refinement property and unique factorisation.

2. **Subalgebras of the hyperfinite II_1 factor.** Let R be the hyperfinite II_1 factor and let $\mathcal{L}$ be a finite lattice of commuting projections in R. Then $A = \operatorname{Alg} \mathcal{L}$ is the algebra of elements a with $pap = ap$ for all p in $\mathcal{L}$. It is easy to see that A is relatively reflexive in the sense that $A = \operatorname{Alg}(\operatorname{Lat} A)$ where $\operatorname{Lat} A$ is the lattice of projections in R which are invariant for A. For convenience we call such algebras FCSL subalgebras of R. An FCSL subalgebra A is irreducible if $C^*(A) = R$ and is said to be *essentially indecomposable* if whenever $A = A_1 \otimes A_2$, with A_1, A_2 algebras of the same class, then $A_1 = A$ or $A_2 = A$. Note that we always have $A = A \otimes R$. Two FCSL subalgebras of R are said to be *conjugate* if they are conjugated by a unitary operator in R. In a similar way we define FCSL subalgebras of $M_n(\mathbb{C})$, and the notions of conjugacy, irreducibility, and indecomposability.

By way of a concrete example consider the lattice $\mathcal{L}$ of projections $p_k = q_1 + \cdots + q_k$, $1 \leq k \leq n$, where $q_1 + \cdots + q_n$ is a partition of the identity in R. If the normalised trace τ has rational values on the projections q_i then $\operatorname{Alg} \mathcal{L}$ has the form $R \otimes A$ where A is the FCSL subalgebra of $M_m(C)$ for some m, associated with the lattice of projections $P_k = Q_1 + \cdots + Q_k$, $1 \leq k \leq n$, where $\tau(q_i)m = \operatorname{rank} Q_i$, $1 \leq i \leq n$. Clearly we can choose m minimally so that A is indecomposable. On the other hand if $\tau(q_i)$ is irrational for some i then $\operatorname{Alg} \mathcal{L}$ is essentially indecomposable.

THEOREM 2.1. *Let A be a proper irreducible* FCSL *subalgebra of the hyperfinite* II_1 *factor R. Then*

$$A = R \otimes A_1 \otimes \cdots \otimes A_r$$

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where each of the algebras $A_1, \ldots, A_r$ is a properly non-self-adjoint essentially indecomposable FCSL *subalgebra of R. Furthermore these algebras are uniquely determined up to conjugacy.*

If $B_1, \ldots, B_s$ are FCSL subalgebras of R then each probability measure μ on $\{1, \ldots, s\}$ gives rise to the FCSL subalgebra $(B_1 \oplus \cdots \oplus B_s, \mu)$ for which the trace is normalised so that $\tau\big((0, \ldots, 1_j, 0, \ldots, 0)\big) = \mu(\{j\})$. Combining this with the theorem it follows that we can obtain a normal form for a general FCSL subalgebra A of R, namely,

$$A = R \otimes \Big(\big((A_{1,1} \otimes \cdots \otimes A_{1,r_1}) \oplus \cdots \oplus (A_{s,1} \otimes \cdots \otimes A_{s,r_s})\big), \mu\Big),$$

where the probability measure μ on $\{1, \ldots, s\}$ and the factor algebras are uniquely determined (up to the appropriate permutation).

PROOF. If A_1 and A_2 are FCSL subalgebras of R or $M_n(\mathbb{C})$, for some n, then $\mathrm{Lat}(A_1 \otimes A_2) = (\mathrm{Lat}\, A_1) \otimes (\mathrm{Lat}\, A_2)$. For the hyperfinite factor we identify the external and internal factorisation of projection lattices. The lattice $\mathcal{L} = \mathrm{Lat}\, A$ of the irreducible FCSL subalgebra A is an irreducible projection lattice and so by Theorem 1.4 there is an indecomposable external factorisation

$$\mathcal{L} = \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_r$$

which leads to the factorisation $A = B_1 \otimes \cdots \otimes B_r$ where $B_1 = \mathrm{Alg}\, \mathcal{E}_i$, $1 \leq i \leq r$. Either B_i is essentially indecomposable, or, as in the example preceding Theorem 2.1, $B_i = R \otimes A_i$ where A_i is an indecomposable FCSL subalgebra of $M_m(\mathbb{C})$ for some m. The uniqueness of the representation follows from the uniqueness of the factorisation of $\mathcal{L}$. ∎

REMARK 1. There is a simple analogue of Theorem 2.1 for irreducible FCSL subalgebras of a UHF C*-algebra which can be obtained by the same argements. Each such algebra admits a unique representation of the form $B \otimes A_1 \otimes \cdots \otimes A_r$ where B is a UHF C*-algebra and $A_1, \ldots, A_r$ are irreducible indecomposable FCSL subalgebras of complex matrix algebras.

REMARK 2. The strict finite refinement property holds for reflexive connected antisymmetric relations on countable sets. Consequently, as above, in the case of factorisations of finite length, we can obtain the uniqueness of indecomposable factorisations of connected countable probability graphs, together with a corresponding result for the associated subalgebras of the hyperfinite II_1 factor. These are in fact the irreducible subalgebras which have purely atomic relative invariant projection lattices. There is also a similar generalisation for UHF C*-algebras, and this time there are interesting indecomposable factor algebras. We remark that in these contexts the existence of infinite indecomposable factorisations presents new difficulties and is not guaranteed. (See Remark 2.10 in [7] where nonexistence of such factorisations is shown using [6].)

3. **Nonrefinement for simple** AF **C^*-algebras.** Let $\mathcal{C}$ be a class of (isomorphism equivalence classes of) norm-closed unital operator algebras which contains the trivial algebra $\mathbb{C}$ and is closed under the formation of spatial tensor products $A_1 \otimes A_2$. For definiteness let us say that we are considering isomorphisms which are completely isometric. Thus $\mathcal{C}$ is a unital abelian semigroup of isomorphism types. An element A of $\mathcal{C}$ is $\otimes$-indecomposable if whenever $A = A_1 \otimes A_2$ with A_1, A_2 in $\mathcal{C}$ it follows that $A_1 = \mathbb{C}$ or $A_2 = \mathbb{C}$. We say that $\mathcal{C}$ has the *finite refinement property* if whenever $A_1 \otimes \cdots \otimes A_r = B_1 \otimes \cdots \otimes B_s$ in $\mathcal{C}$ it follows that there are algebras C_{ij} in $\mathcal{C}$, for $1 \le i \le r$, $1 \le j \le s$, such that $A_i = C_{i1} \otimes \cdots \otimes C_{is}$, $1 \le i \le r$, and $B_j = C_{1j} \otimes \cdots \otimes C_{rj}$ for $1 \le j \le s$. Clearly this property implies the uniqueness, up to permutation, of $\otimes$-indecomposable factorisations.

If $\mathcal{C} =$ UHF, the class of Glimm algebras, then $\mathcal{C}$ has the finite refinement property (in fact the infinite refinement property) as an elementary conseqequence of Glimm's theorem. If $\mathcal{C}$ is the class of finite-dimensional C^*-algebras then the refinement property fails for rather elementary reasons (cf. Remark 2.9 of [7]). In view of these remarks it is natural to consider classes of simple C^*-algebras, or in the non-self-adjoint context, classes of algebras A for which $C^*(A)$ is simple. However we have the following.

THEOREM 3.1. *The class of simple approximately finite* C^**-algebras does not have the unique factorisation property.*

Our counterexample is based on AF C^*-algebras whose ordered K_0 groups have the form $(\mathbb{Z}^2, P_\alpha)$, where α is a positive irrational determining the positive cone $P_\alpha = \{(n,m) : \alpha n + m \ge 0\}$. Every countable totally ordered group is an ordered K_0 group of an AF C^*-algebra by [5, Theorem 2.2], and in the case of $(\mathbb{Z}^2, P_\alpha)$, the continued fraction expansion algorithm for α can be used to specify an AF C^*-algebra A_α with $K_0(A_\alpha) = (\mathbb{Z}^2, P_\alpha)$ (See [5, Theorem 2.2], and [4].)

PROPOSITION 3.2. *Let A_α, A_β be approximately finite* C^**-algebras with $K_0(A_\alpha) = (\mathbb{Z}^2, P_\alpha)$, $K_0(A_\beta) = (\mathbb{Z}^2, P_\beta)$, as ordered groups. Then $K_0(A_\alpha \otimes A_\beta) = (\mathbb{Z}^4, Q)$ where $(n_1, n_2, n_3, n_4) \in Q$ if and only if $\alpha\beta\ n_1 + \alpha n_2 + \beta n_3 + n_4 \ge 0$.*

PROOF. Recall first that the tensor product of two ordered abelian groups is the group tensor product together with the positive cone generated by elementary tensors $g_1 \otimes g_2$ with g_1, g_2 in the positive cones of the factor groups. It follows that if G_1, G_2 are two subgroups of $\mathbb{R}$, with the relative ordering, then this ordered group tensor product is simply their product in $\mathbb{R}$ together with the relative order. It now follows that $(\mathbb{Z}^2, P_\alpha) \otimes (\mathbb{Z}^2, P_\beta)$ is isomorphic to $(\mathbb{Z}^4, Q)$. It is elementary to check that $K_0(A \otimes B) = K_0(A) \otimes K_0(B)$, as ordered groups, when A, B are AF C^*-algebras (cf. [5, section 5.1]), and so the proposition follows. ∎

Let $\underline{\gamma} \in \mathbb{R}^k$ and let $Q(\underline{\gamma}) = \{\underline{n} \in \mathbb{Z}^k : \langle \underline{n}, \underline{\gamma} \rangle \ge 0\}$. Let $X \in \mathrm{GL}(k, \mathbb{Z})$, let $\theta : \underline{n} \longrightarrow (X^T)^{-1}\underline{n}$ be the associated group automorphism of $\mathbb{Z}^k$, and let $\underline{\delta} = X_{\underline{\gamma}}$. Then $\theta : (\mathbb{Z}^k, Q(\underline{\gamma})) \longrightarrow (\mathbb{Z}^k, Q(\underline{\delta}))$ is an isomorphism of ordered groups, since $0 \le \langle \underline{n}, \underline{\gamma} \rangle$ if and only if $0 \le \langle \underline{n}, X^{-1}\underline{\delta} \rangle = \langle \theta(\underline{n}), \underline{\gamma} \rangle$. Moreover, if $\underline{u}$ is an order unit for $(\mathbb{Z}^k, Q(\underline{\gamma}))$

then $\theta(\underline{u})$ is an order unit for $(\mathbb{Z}^k, Q(\underline{\delta}))$ and θ implements an isomorphism between the scaled order groups with these units.

Roughly speaking there is enough freedom through such isomorphisms of $(\mathbb{Z}^4, Q(\underline{\gamma}))$ to obtain an isomorphism between $(\mathbb{Z}^2, Q(\alpha_1, \alpha_2)) \otimes (\mathbb{Z}^2, Q(\beta_1, \beta_2))$ and $(\mathbb{Z}^2, Q(\alpha_n', \alpha_2'))$ $\otimes$ $(\mathbb{Z}^2, Q(\beta_1', \beta_2'))$ in certain cases where $(\mathbb{Z}^2, Q(\alpha_1, \alpha_2))$ is not isomorphic to either factor group of the second tensor product. The following example illustrates this.

PROPOSITION 3.3. *The scaled ordered dimension groups*

$$G_1 = \big(\mathbb{Z}^2, Q(1, \sqrt{2}), (1,1)\big) \otimes \big(\mathbb{Z}^2, Q(1, \sqrt{3}), (1,0)\big)$$

and

$$G_2 = \big(\mathbb{Z}^2, Q(1, \sqrt{3} + \sqrt{6}), (1,0)\big) \otimes \big(\mathbb{Z}^2, Q(1, \sqrt{2}), (1,1)\big)$$

are isomorphic.

PROOF. Let $\underline{\gamma} = (1, \sqrt{2}, \sqrt{3}, \sqrt{6})$, $\underline{\delta} = (1, \sqrt{3} + \sqrt{6}, \sqrt{2}, (\sqrt{3} + \sqrt{6})\sqrt{2})$ so that by the last proposition $G_1 = \big(\mathbb{Z}^4, Q(\underline{\gamma})\big)$, $G_2 = \big(\mathbb{Z}^4, Q(\underline{\delta})\big)$.

Let

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

and observe that $X \in \mathrm{GL}(4, \mathbb{Z})$ and $X\underline{\gamma} = \underline{\delta}$. Moreover the order units for G_1 and G_2 are $\underline{u} = (1,1,0,0)$ and $\underline{v} = (1,0,1,0)$, respectively, and $(X^T)^{-1}\underline{u} = \underline{v}$. By our earlier remarks $\theta : \underline{n} \longrightarrow (X^T)^{-1}(\underline{n})$ establishes the desired isomorphism.

PROOF OF THEOREM 3.1. Let $A_1 \otimes A_2$, $A_3 \otimes A_4$ be the AF C*-algebras associated with G_1 and G_2 respectively, where, as a scaled ordered dimension group, $K_0(A_1) = \big(\mathbb{Z}^2, Q(1, \sqrt{2}), (1,1)\big)$ etc. Thus $A_1 \otimes A_2 = A_3 \otimes A_4$. Furthermore these are indecomposable factorisations. Indeed if $A_1 = C \otimes D$, with C and D AF C*-algebras, then, $\mathbb{Z}^2 = K_0(A_1) = \lim_n K_0(C_n \otimes D_n) = \big(\lim_n K_0(C_n)\big) \otimes \big(\lim_n K_0(D_n)\big) = H_1 \otimes H_2$ say, for some abelian groups H_1, H_2. Thus H_1 or H_2 is $\mathbb{Z}$ and so $A_1 = M_k \otimes A$ for some AF algebra A and some integer k which divides the order unit $(1,1)$ of A_1. Since this order unit is minimal in $Q(1, \sqrt{2})$ it follows that $k = 1$. Similarly, A_2, A_3 and A_4 are $\otimes$-indecomposable.

Finally, note that $\sqrt{3} + \sqrt{6}$ is not equivalent to $\alpha = \sqrt{2}$ or $\alpha = \sqrt{3}$, in the sense that

$$\sqrt{3} + \sqrt{6} = \frac{p\alpha + q}{r\alpha + s}$$

for some element $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ in $\mathrm{GL}(2, \mathbb{Z})$. Consequently A_3 is not isomorphic to A_1 or A_2. ∎

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