BULL. AUSTRAL. MATH. SOC. 46L85, 54C40
VOL. 72 (2005) [197–212]

# ON $C^*$-ALGEBRAS WITH THE APPROXIMATE $n$-TH ROOT PROPERTY

A. CHIGOGIDZE, A. KARASEV, K. KAWAMURA AND V. VALOV

We say that a $C^*$-algebra $X$ has the approximate $n$-th root property ($n \geqslant 2$) if for every $a \in X$ with $\|a\| \leqslant 1$ and every $\varepsilon > 0$ there exits $b \in X$ such that $\|b\| \leqslant 1$ and $\|a - b^n\| < \varepsilon$. Some properties of commutative and non-commutative $C^*$-algebras having the approximate $n$-th root property are investigated. In particular, it is shown that there exists a non-commutative (respectively, commutative) separable unital $C^*$-algebra $X$ such that any other (commutative) separable unital $C^*$-algebra is a quotient of $X$. Also we illustrate a commutative $C^*$-algebra, each element of which has a square root such that its maximal ideal space has infinitely generated first Čech cohomology.

## 1. INTRODUCTION

All topological spaces in this paper are assumed to be (at least) completely regular. A compact Hausdorff space is called a *compactum* for simplicity. By $C^*$-algebra and homomorphisms between $C^*$-algebras, we mean unital $C^*$-algebras and unital $*$-homomorphisms. For a space $X$ and an integer $n \geqslant 2$, we consider the following conditions ($\|\cdot\|$ denotes the supremum norm):

$(*)_n$ For each bounded continuous function $f\colon X \to \mathbb{C}$ and each $\varepsilon > 0$, there exists a continuous function $g\colon X \to \mathbb{C}$ such that $\|f - g^n\| < \varepsilon$.

$(**)_n$ For each bounded continuous function $f\colon X \to \mathbb{C}$ and each $\varepsilon > 0$, there exist bounded continuous functions $g_1, \dots, g_n\colon X \to \mathbb{C}$ such that $f = \prod_{i=1}^{i=n} g_i$ and $\|g_i - g_j\| < \varepsilon$ for each $i, j$.

We say that the space $C^*(X)$ of all bounded complex-valued functions on $X$ has the approximate $n$-th root property if $X$ satisfies condition $(*)_n$. The results in this paper were inspired by the following theorem established by Kawamura and Miura [**10**]:

**THEOREM 1.1.** *Let $X$ be a compactum with* $\dim X \leqslant 1$ *and $n$ a positive integer. Then the following conditions are equivalent.*

Received 3rd March, 2005

The second author was partially supported by his NSERC Grant 257231-04. The paper was started during the third author's visit to Nipissing University in July 2004. The last author was partially supported by his NSERC Grant 261914-03.

(1) *$C(X)$ has the approximate $n$-th root property.*

(2) *$X$ satisfies condition $(**)_n$.*

(3) *the first Čech cohomology $\check{H}^1(X;\mathbb{Z})$ is $n$-divisible, that is, each element of $\check{H}^1(X;\mathbb{Z})$ is divided by $n$.*

Let $\mathcal{A}(n)$ denote the class of all completely regular spaces satisfying condition $(*)_n$ and $\mathcal{A}_1(n)$ is the subclass of $\mathcal{A}(n)$ consisting of spaces $X$ with $\dim X \leqslant 1$.

In Section 2 we investigate some properties of the classes $\mathcal{A}(n)$ and $\mathcal{A}_1(n)$. In particular, the following theorem is established.

**Theorem 1.2.** *Let $n$ be a positive integer and let $\mathcal{K}$ denote one of the classes $\mathcal{A}(n)$ and $\mathcal{A}_1(n)$. Then, for every cardinal $\tau \geqslant \omega$, there exists a compactum $X_\tau \in \mathcal{K}$ of weight $\leqslant \tau$ and a $\mathcal{K}$-invertible map $f_{\mathcal{K}}\colon X_\tau \to \mathbb{I}^\tau$.*

Here, a map $h\colon X \to Y$ is said to be *invertible* for the class $\mathcal{K}$ (or simply, *$\mathcal{K}$-invertible*) if for every map $g\colon Z \to Y$ with $Z \in \mathcal{K}$ there exists a map $\overline{g}\colon Z \to X$ such that $g = h \circ \overline{g}$.

Theorem 1.2 implies the next corollary.

**Corollary 1.3.** *Let $n$ be a positive integer and let $\mathcal{K}$ be one of the classes $\mathcal{A}(n)$ and $\mathcal{A}_1(n)$. Then, for every $\tau \geqslant \omega$, there exists a compactum $X \in \mathcal{K}$ of weight $\tau$ which contains every space from $\mathcal{K}$ of weight $\leqslant \tau$.*

It is easily seen that the modification of condition $(*)_n$, obtained by requiring both $f$ and $g$ to be of norm $\leqslant 1$, is equivalent to $(*)_n$. This observation leads us to consider the following classes of general (non-commutative) $C^*$-algebras. We say that a $C^*$-algebra $X$ satisfies *the approximation $n$-th root property* if for every $a \in X$ with $\|a\| \leqslant 1$ and every $\varepsilon > 0$ there exists $b \in X$ such that $\|b\| \leqslant 1$ and $\|a - b^n\| < \varepsilon$. The class of all $C^*$-algebras with the approximate $n$-th root property is denoted by $\mathcal{AP}(n)$. Let $\mathcal{AP}_1(n)$ be the subclass of $\mathcal{AP}(n)$ consisting of $C^*$-algebras of bounded rank $\leqslant 1$ (recall that bounded rank of $C^*$-algebras is a non-commutative analogue of the covering dimension dim, see [5]). We also consider the class $\mathcal{HP}(n)$ of $C^*$-algebras $X$ with the following property: for every *invertible* element $a \in X$ with $\|a\| \leqslant 1$ and every $\varepsilon > 0$ there exists $b \in X$ such that $\|b\| \leqslant 1$ and $\|a - b^n\| < \varepsilon$.

In the sequel, $\mathcal{AP}(n)_s$ denotes the class of all separable $C^*$-algebras from $\mathcal{AP}(n)$. The notations $\mathcal{AP}_1(n)_s$ and $\mathcal{HP}(n)_s$ have the same meaning.

Recall now the concept of $\mathfrak{R}$-invertibility introduced in [2], where $\mathfrak{R}$ is a given class of $C^*$-algebras. A homomorphism $p\colon X \to Y$ is said to be *$\mathfrak{R}$-invertible* if, for any homomorphism $g\colon X \to Z$ with $Z \in \mathfrak{R}$, there exists a homomorphism $\overline{g}\colon Y \to Z$ such that $g = \overline{g} \circ p$. We also introduce the notion of a *universal* $C^*$-algebra for a given class $\mathfrak{R}$ as a $C^*$-algebra $Y \in \mathfrak{R}$ such that any other $C^*$-algebra from $\mathfrak{R}$ is a quotient of $Y$.

Section 3 is devoted to the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$. The results of this section can be considered as non-commutative counterparts of the results from Section 2. For example, Theorem 1.4 below is a non-commutative version of Theorem 1.2.

**THEOREM 1.4.** *Let $n$ be a positive integer and let $\mathcal{K}$ be one of the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$. Then there exists a $\mathcal{K}$-invertible unital $*$-homomorphism $p\colon C^*(\mathbb{F}_\infty) \to Z_\mathcal{K}$ of $C^*(F_\infty)$ to a separable unital $C^*$- algebra $Z_\mathcal{K} \in \mathcal{K}$, where $C^*(\mathbb{F}_\infty)$ is the group $C^*$-algebra of the free group on countable number of generators.*

It is well-known that every separable $C^*$-algebra is a surjective image of $C^*(\mathbb{F}_\infty)$. Therefore, if $\mathfrak{R}$ is a class of separable $C^*$-algebras and $p\colon C^*(F_\infty) \to Y_\mathfrak{R}$ is a $\mathfrak{R}$-invertible homomorphism with $Y_\mathfrak{R} \in \mathfrak{R}$, then $Y_\mathfrak{R}$ is universal for the class $\mathfrak{R}$. Hence, Theorem 1.4 implies that each of the classes $\mathcal{AP}(n)_s$, $\mathcal{AP}_1(n)_s$ and $\mathcal{HP}(n)_s$ has a universal element.

Let us note that there exists a non-commutative $C^*$-algebra which belongs to any one of the classes $\mathcal{AP}(n)_s$, $\mathcal{AP}_1(n)_s$ and $\mathcal{HP}(n)_s$. Indeed, let $X = M(m)$ be the algebra of all $m \times m$ complex matrixes, where $m \geqslant 2$ is a fixed integer. By [**1**], the bounded rank of any $A \in X$ is 0. Moreover, using the canonical Jordan form representation, one can show that if $A \in X$ and $n \geqslant 2$, then $A$ can be approximated by a matrix $B \in X$ with $C^n = B$ for some $C \in X$. Hence, the class $X$ is a common part of $\mathcal{AP}(n)_s$, $\mathcal{AP}_1(n)_s$ and $\mathcal{HP}(n)_s$. This implies that the universal elements of $\mathcal{AP}(n)_s$, $\mathcal{AP}_1(n)_s$ and $\mathcal{HP}(n)_s$ are also non-commutative.

Section 4 deals with *square root closed compacta*, compacta $X$ such that, for every $f \in C(X)$, there is $g \in C(X)$ with $f = g^2$. It is known that if $X$ is a first-countable connected compactum, then $X$ is square-root closed if and only if $X$ is locally connected, $\dim X \leqslant 1$ and $\check{H}^1(X;\mathbb{Z})$ is trivial, see [**6, 8, 10, 12**]. A topological characterisation of general square root closed compacta is still unknown. Here we show that a square root closed compactum $X$ with $\dim X \leqslant 2$, constructed based on the idea of Cole ([**13**, Chapter 3, Section 19], and Karahanjan [**9**] has infinitely generated first Čech cohomology $\check{H}^1(X;\mathbb{Z})$. This space is the limit of an inverse system $\left(X_\alpha, \pi_\alpha^\beta : \alpha < \omega_1\right)$ starting with the unit disk in the plane and such that each map $\pi_\alpha^\beta\colon X_\beta \to X_\alpha$ is invertible with respect to the class of square root closed compacta. A similar construction yields a one-dimensional such compactum. This illustrates that the topological characterisation of (not necessarily first countable) square root closed compacta would be rather different than the one for first-countable compacta mentioned above. Also, the invertibility of the maps $\pi_\alpha^\beta$ allows us to obtain a universal element for the class of square root closed compacta with arbitrarily fixed weight.

## 2. Some properties of the classes $\mathcal{A}(n)$ and $\mathcal{A}_1(n)$

**LEMMA 2.1.** *Let $X$ be the limit space of an inverse system $\{X_\alpha, p_\alpha^\beta : \alpha, \beta \in A\}$ of compacta. Then, for every $f \in C(X)$ and every $\varepsilon > 0$, there exists $\alpha \in A$ and $g \in C(X_\alpha)$ such that $g \circ p_\alpha$ is $\varepsilon$-close to $f$, where $p_\alpha\colon X \to X_\alpha$ is the $\alpha$-th limit projection.*

PROOF: We take a finite cover $\omega$ of $f(X)$ consisting of open and convex subsets of $\mathbb{C}$ each of diameter $< \varepsilon$. Since $X$ is compact, we can find $\alpha$ and an open cover

$\gamma = \{U_j : j = 1, \dots, m\}$ of $X_\alpha$ such that $p_\alpha^{-1}(\gamma)$ is a star-refinement of the cover $f^{-1}(\omega)$. Without loss of generality, we can assume that each $U_j$ is functionally open in $X_\alpha$, that is, $U_j = h_j^{-1}\big((0,1]\big)$ for some function $h_j\colon X_\alpha \to [0,1]$. For any $j$ we fix a point $x_j \in p_\alpha^{-1}(U_j)$ and the required function $g\colon X_\alpha \to \mathbb{C}$ is defined by $g(y) = \sum_{j=1}^{j=m} h_j(y) f(x_j)$. ∎

**Corollary 2.2.** *Let $\mathcal{K}$ be one of the classes $\mathcal{A}(n)$ and $\mathcal{A}_1(n)$. If $X$ is the limit space of an inverse system $\{X_\alpha, p_\alpha^\beta : \alpha, \beta \in A\}$ of compacta with each $X_\alpha \in \mathcal{K}$, then $X \in \mathcal{K}$.*

Proof: This is a direct application of Lemma 2.1 for the class $\mathcal{A}(n)$. Since the limit space of any inverse system of at most one dimensional compacta is of dimension $\leqslant 1$, the validity of our corollary for $\mathcal{A}(n)$ yields its validity for $\mathcal{A}_1(n)$. ∎

We say that a class of spaces $\mathcal{K}$ is *factorisable* if, for every map $f\colon X \to Y$ of a compactum $X \in \mathcal{K}$, there exists a compactum $Z \in \mathcal{K}$ of weight $w(Z) \leqslant w(Y)$ and maps $\pi\colon X \to Z$ and $p\colon Z \to Y$ such that $f = p \circ \pi$.

**Proposition 2.3.** *Any one of the classes $\mathcal{A}(n)$ and $\mathcal{A}_1(n)$ is factorisable.*

Proof: We consider first the class $\mathcal{A}(n)$. Fix a map $f\colon X \to Y$ of a compactum $X \in \mathcal{A}(n)$ and assume $w(Y) \leqslant \tau$. Obviously, we can assume $X$ is of weight $w(X) > \tau$ and $Y$ is compact. By induction, we construct sequences of compacta $X_k$, dense subsets $M_k \subset C(X_k)$ of cardinality $\leqslant \tau$ and maps $\pi_k\colon X \to X_k$, $p_k^{k+1}\colon X_{k+1} \to X_k$, $k \geqslant 0$, satisfying the following conditions:

(0) $X_0 = Y$, $\pi_0 = f$,

(1) $p_k^{k+1} \circ \pi_{k+1} = \pi_k$, $w(X_k) \leqslant \tau$ and $M_k$ separates points of $X_k$ $(k \geqslant 0)$;

(2) For every $h \in M_k$ and every $\varepsilon > 0$, there exists $g \in M_{k+1}$ such that $\|h \circ p_k^{k+1} - g^n\| < \varepsilon$ $(k \geqslant 0)$.

The weight of the function space $C(Y)$ is $\leqslant \tau$, so $C(Y)$ contains a dense subset $M_0$ of cardinality $\leqslant \tau$, separating points of $Y$. Suppose the spaces $X_i$, the sets $M_i$ and the maps $\pi_i$, $p_{i-1}^i$, $i \leqslant k$, have been constructed for some $k$. Since $X \in \mathcal{A}(n)$, for each $h \in M_k$ and each positive rational number $r \in Q^+$, there exists $g(h,r) \in C(X)$ with $\big\|h \circ \pi_k - g(h,r)^n\big\| < r$. Let $\pi_{k+1}\colon X \to X_k \times (\mathbb{R})^{M_k \times Q^+} \times (\mathbb{R})^{M_k}$ be the diagonal product of $\pi_k$ and all maps $g(h,r)$ and $h \circ \pi_k$, where $h \in M_k$, $r \in Q^+$. Let $X_{k+1} = \pi_{k+1}(X)$ and $p_k^{k+1}\colon X_{k+1} \to X_k$ be the natural projection onto $X_k$. Since $M_k$ separates points of $X_k$ (condition (1)), $\pi_{k+1}$ is an embedding and hence every $g(h,r)$ can be represented as $g_{k+1}(h,r) \circ \pi_{k+1}$ with $g_{k+1}(h,r) \in C(X_{k+1})$. Because $w(X_{k+1}) \leqslant \tau$, $C(X_{k+1})$ contains a dense subset $M_{k+1}$ of cardinality $\leqslant \tau$ containing all $g_{k+1}(h,r)$, $h \in M_k$, $r \in Q^+$ and also separating points of $X_{k+1}$. Obviously, $X_{k+1}$, $M_{k+1}$ and $\pi_{k+1}$ satisfy conditions (1) and (2). Let $Z$ be the limit of the inverse sequence $\{X_k, p_k^{k+1} : k = 1, 2 \dots\}$, $p\colon Z \to Y$ the first limit projection and $\pi\colon X \to Z$ the limit of the maps $\pi_k$. Also let $p_k\colon Z \to X_k$ be

https://doi.org/10.1017/S0004972700035012 Published online by Cambridge University Press

the $k$-th limit projection. By Lemma 2.1, for every $h \in C(Z)$ and every $\varepsilon > 0$, there exists $m$ and $g_m \in C(X_m)$ such that $\|h - g_m \circ p_m\| < \varepsilon/3$. Now, take $h_m \in M_m$ with $\|g_m - h_m\| < \varepsilon/3$. According to our construction, $\|h_m \circ p_m^{m+1} - g^n\| < \varepsilon/3$ for some $g \in M_{m+1}$. Hence, $\|h - (g \circ p_{m+1})^n\| < \varepsilon$. Finally, by Lemma 2.1, we see $Z \in \mathcal{A}(n)$.

For the class $\mathcal{A}_1(n)$ we need the following modifications of the previous proof: all $M_k$, $k \geqslant 0$, are dense subsets of $C(X_k)$ of cardinality $|M_k| \leqslant \tau$ satisfying conditions (1) and (2), where the compactum $X_k$ is of dimension $\leqslant 1$ for each $k \geqslant 1$. It suffices to demonstrate the construction of $X_1$ and $M_1$. Using the above notations, take the diagonal product $q_1 \colon X \to Y \times \mathbb{C}^{M_0 \times Q^+} \times \mathbb{C}^{M_0}$ of $\pi_0 = f$ and all maps $g(h,r)$ and $h \circ \pi_0$, where $h \in M_0$ and $r \in Q^+$. Let also $Z_1 = q_1(X)$ and $q_0 \colon Z_1 \to Y$ be the natural projection. Then, $w(Z_1) \leqslant \tau$ and, by the Mardešič factorisation theorem [**11**], there exists a compactum $X_1$ of weight $\leqslant \tau$ and $\dim X_1 \leqslant 1$, and maps $\pi_1 \colon X \to X_1$ and $q_2 \colon X_1 \to Z_1$ with $q_1 = q_2 \circ \pi_1$. Obviously, every $g(h,r)$ can be represented as $g_1(h,r) \circ \pi_1$ with $g_1(h,r) \in C(X_1)$. We denote $p_0^1 = q_0 \circ q_2$ and choose a dense subset $M_1 \subset C(X_1)$ such that $|M_1| \leqslant \tau$ and $M_1$ contains every $g_1(h,r)$ with $h \in M_0$ and $r \in Q^+$, and separates points of $X_1$. In this way we obtain the spaces $X_k$ with $\dim X_k \leqslant 1$. The last inequalities imply that the limit space $Z$ is also of dimension $\leqslant 1$. Moreover, by Lemma 2.1, $Z$ satisfies $(*)_n$, so $Z \in \mathcal{A}_1(n)$. ∎

**COROLLARY 2.4.** *Let $\mathcal{K}$ be one of the classes $\mathcal{A}(n)$ and $\mathcal{A}_1(n)$. Then every space $X \in \mathcal{K}$ has a compactification $Z \in \mathcal{K}$ with $w(Z) = w(X)$.*

PROOF: Obviously, $X \in \mathcal{K}$ implies $\beta X \in \mathcal{K}$. Let $Y$ be an arbitrary compactification of $X$ with $w(Y) = w(X)$ and let $f \colon \beta X \to Y$ be the extension of the identity on $X$. Then, by Proposition 2.3, there exists a compactum $Z \in \mathcal{K}$ and maps $g \colon \beta X \to Z$ and $h \colon Z \to Y$ with $h \circ g = f$ and $w(Z) = w(X)$. It remains only to observe that $Z$ is a compactification of $X$. ∎

**PROPOSITION 2.5.** *Let $\mathcal{K}$ be one of the classes $\mathcal{A}(n)$ and $\mathcal{A}_1(n)$. Then every compactum $X \in \mathcal{K}$ can be represented as the limit space of an $\omega$-spectrum $\{X_\alpha, p_\alpha^\beta : \alpha, \beta \in A\}$ of metrisable compacta with each $X_\alpha \in \mathcal{K}$.*

PROOF: Because of similarity of the arguments, we consider only the class $\mathcal{A}(n)$. First, represent $X$ as the limit space of an $\omega$-spectrum $\{X_\alpha, p_\alpha^\beta : \alpha, \beta \in \Lambda\}$ and introduce the relation $L$ on $\Lambda^2$ consisting of all $(\alpha, \beta) \in \Lambda^2$ such that $\alpha \leqslant \beta$ and for each $f \in C(X_\alpha)$ and $\varepsilon > 0$ there is $g \in C(X_\beta)$ with $\|f \circ p_\alpha^\beta - g^n\| < \varepsilon$. The relation $L$ has the following properties:

(i) for every $\alpha \in \Lambda$ there exists $\beta \in \Lambda$ with $(\alpha, \beta) \in L$:
(ii) if $(\alpha, \beta) \in L$ and $\beta \leqslant \gamma$, then $(\alpha, \gamma) \in L$;
(iii) if $\{\alpha_k\}$ is a chain in $\Lambda$ with each $(\alpha_k, \beta) \in L$, then $(\alpha, \beta) \in L$, where $\alpha = \sup\{\alpha_k\}$.

Indeed, to show (i), we take a countable dense subset $M_\alpha \subset C(X_\alpha)$ and, as in Proposition 2.3, for every $h \in M_\alpha$ and $r \in Q^+$ choose $g(h,r) \in C(X)$ with $\|h \circ p_\alpha - g(h,r)^n\| < r$. Notice that, for each $f \in C(X)$, there is a $\gamma \in \Lambda$ and $\varphi \in C(X_\gamma)$ such that $f = \varphi \circ p_\gamma$. Applying this to $g(h,r)$, we can find $\beta \in \Lambda$, $\beta > \alpha$, such that for each $(h,r) \in M_\alpha \times Q^+$, we have $g(h,r) = g_\beta(h,r) \circ p_\beta$, where $g_\beta(h,r) \in C(X_\beta)$. Then $(\alpha,\beta) \in L$. Property (ii) follows directly and (iii) follows from Lemma 2.1 and the fact that $X_\alpha$ is the limit space of the inverse sequence generated by $X_{\alpha_k}$ and the projections $p_{\alpha_k}^{\alpha_{k+1}}\colon X_{\alpha_{k+1}} \to X_{\alpha_k}$, $k = 1,\ldots$, because $\alpha$ is supremum of the chain $\{\alpha_k\}$.

By [**3**, Proposition 1.1.29], the set $A = \{\alpha \in \Lambda : (\alpha,\alpha) \in L\}$ is cofinal and $\omega$-closed in $\Lambda$. Obviously, $X_\alpha \in \mathcal{A}(n)$ for each $\alpha \in A$ and $X$ is the limit of the inverse system $\{X_\alpha, p_\alpha^\beta : \alpha,\beta \in A\}$. ∎

PROOF OF THEOREM 1.2: We consider the family of all maps $\{h_\alpha\colon Y_\alpha \to \mathbb{I}^\tau\}_{\alpha\in\Lambda}$ such that each $Y_\alpha$ is a closed subset of $\mathbb{I}^\tau$ with $Y_\alpha \in \mathcal{K}$. Let $Y$ be the disjoint sum of all $Y_\alpha$ and the map $h\colon Y \to \mathbb{I}^\tau$ coincides with $h_\alpha$ on every $Y_\alpha$. We extend $h$ to a map $\overline{h}\colon \beta Y \to \mathbb{I}^\tau$. Since $\beta Y \in \mathcal{K}$, by Proposition 2.3, there exists a compactum $X$ of weight $\leqslant \tau$ and maps $p\colon \beta Y \to X$ and $f\colon X \to \mathbb{I}^\tau$ such that $X \in \mathcal{K}$ and $f \circ p = \overline{h}$.

Let us show that $f$ is $\mathcal{K}$-invertible. Take a space $Z \in \mathcal{K}$ and a map $g\colon Z \to \mathbb{I}^\tau$. Considering $\beta Z$ and the extension $\overline{g}\colon \beta Z \to \mathbb{I}^\tau$ of $g$, we can assume that $Z$ is compact. We also can assume that the weight of $Z$ is $\leqslant \tau$ ( otherwise we apply again Proposition 2.3 to find a compact space $T \in \mathcal{K}$ of weight $\leqslant \tau$ and maps $g_1\colon Z \to T$ and $g_2\colon T \to \mathbb{I}^\tau$ with $g_2 \circ g_1 = g$, and then consider the space $T$ and the map $g_2$ instead, respectively, of $Z$ and $g$). Therefore, without loss of generality, we can assume that $Z$ is a closed subset of $\mathbb{I}^\tau$. According to the definition of $Y$ and the map $h$, there is an index $\alpha \in \Lambda$ such that $Z = Y_\alpha$ and $g = h_\alpha$. The restriction $p \mid Z\colon Z \to X$ is a lifting of $g$, that is, $f \circ (p \mid Z) = g$. ∎

## 3. $C^*$-ALGEBRAS WITH THE APPROXIMATE $n$-TH ROOT PROPERTY

In this Section we investigate the behaviour of the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$ with respect to direct systems and then use the result to prove the existence of universal elements in the classes $\mathcal{AP}(n)_s$, $\mathcal{AP}_1(n)_s$ and $\mathcal{HP}(n)_s$.

When we refer to a unital $C^*$-subalgebra of a unital $C^*$-algebra we always assume that the inclusion is a unital $*$-homomorphism. The product in the category of (unital) $C^*$-algebras, that is, the $\ell^\infty$-direct sum, is denoted by $\prod\{X_t\colon t \in T\}$. For a given set $Y$ and a cardinal number $\tau$, the symbol $\exp_\tau Y$ denotes the partially ordered (by inclusion) set of all subsets of $Y$ of cardinality not exceeding $\tau$.

Recall that a direct system $\mathcal{S} = \{X_\alpha, i_\alpha^\beta, A\}$ of unital $C^*$-algebras consists of a partially ordered directed indexing set $A$, unital $C^*$-algebras $X_\alpha$, $\alpha \in A$, and unital $*$-homomorphisms $i_\alpha^\beta\colon X_\alpha \to X_\beta$, defined for each pair of indexes $\alpha,\beta \in A$ with $\alpha \leqslant \beta$, and satisfying the condition $i_\alpha^\gamma = i_\beta^\gamma \circ i_\alpha^\beta$ for each triple of indexes $\alpha,\beta,\gamma \in A$ with $\alpha \leqslant \beta \leqslant \gamma$.

The (inductive) limit of the above direct system is a unital $C^*$-algebra which is denoted by $\varinjlim \mathcal{S}$. For each $\alpha \in A$ there exists a unital $*$-homomorphism $i_\alpha \colon X_\alpha \to \varinjlim \mathcal{S}$ which will be called the $\alpha$-th limit homomorphism of $\mathcal{S}$.

If $A'$ is a directed subset of the indexing set $A$, then the subsystem $\{X_\alpha, i_\alpha^\beta, A'\}$ of $\mathcal{S}$ is denoted $\mathcal{S} \mid A'$.

Let $\tau \geqslant \omega$ be a cardinal number. A direct system $\mathcal{S} = \{X_\alpha, i_\alpha^\beta, A\}$ of unital $C^*$-algebras $X_\alpha$ and unital $*$-homomorphisms $i_\alpha^\beta \colon X_\alpha \to X_\beta$ is called a *direct $C_\tau^*$-system* [4] if the following conditions are satisfied:

(a) $A$ is a $\tau$-complete set, that is, for each chain $C$ of elements of the directed set $A$ with $|C| \leqslant \tau$, there exists an element $\sup C$ in $A$. See [3] for details.

(b) The density $d(X_\alpha)$ of $X_\alpha$ is at most $\tau$, for each $\alpha \in A$.

(c) The $\alpha$-th limit homomorphism $i_\alpha \colon X_\alpha \to \varinjlim \mathcal{S}$ is an injective $*$-homomorphism for each $\alpha \in A$.

(d) If $B = \{\alpha_t \colon t \in T\}$ is a chain of elements of $A$ with $|T| \leqslant \tau$ and $\alpha = \sup B$, then the limit homomorphism $\varinjlim\{i_{\alpha_t}^\alpha \colon t \in T\} \colon \varinjlim(\mathcal{S} \mid B) \to X_\alpha$ is an isomorphism.

**PROPOSITION 3.1.** ([4, Proposition 3.2]) *Let $\tau$ be an infinite cardinal number. Every unital $C^*$-algebra $X$ can be represented as the limit of a direct $C_\tau^*$-system $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$ where the index set $A = \exp_\tau Y$ for some (any) dense subset $Y$ of $X$ with $|Y| = d(X)$.*

**LEMMA 3.2.** ([4, Lemma 3.3]) *If $\mathcal{S}_X = \{X_\alpha, i_\alpha^\beta, A\}$ is a direct $C_\tau^*$-system, then*

$$\varinjlim \mathcal{S}_X = \cup\{i_\alpha(X_\alpha) \colon \alpha \in A\}.$$

The next proposition is a non-commutative version of Corollary 2.2.

**PROPOSITION 3.3.** *Let $\mathcal{K}$ be one of the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$. If $X$ is the limit of a direct system $\mathcal{S} = \{X_\alpha, i_\alpha^\beta, A\}$ consisting of unital $C^*$-algebras and unital $*$-inclusions with $X_\alpha \in \mathcal{K}$ for each $\alpha$, then $X \in \mathcal{K}$.*

PROOF: We consider first the case $\mathcal{K} = \mathcal{AP}(n)$. Let $a \in X$ with $\|a\| \leqslant 1$ and $\varepsilon > 0$. Since $\cup\{X_\alpha \colon \alpha \in A\}$ is dense in $X$ (we identify each $i_\alpha(X_\alpha)$ with $X_\alpha$), there exist $\alpha$ and $y \in X_\alpha$ with $\|a - y\| < \varepsilon/4$. Then, $\|y\| < \|a\| + \varepsilon/4 \leqslant 1 + \varepsilon/4$, so $\|(y/1 + \varepsilon/4)\| < 1$. Since $X_\alpha \in \mathcal{AP}(n)$, there is $b \in X_\alpha$ with $\|(y/1 + \varepsilon/4) - b^n\| < \varepsilon/2$ and $\|b\| \leqslant 1$. Then $\|a - b^n\| \leqslant \|a - (y/1 + \varepsilon/4)\| + \|(y/1 + \varepsilon/4) - b^n\| < \varepsilon$. Hence, $X \in \mathcal{AP}(n)$. The above arguments work also for the class $\mathcal{HP}(n)$ because of the fact that the set of invertible elements of a $C^*$-algebra is open. Indeed, for an invertible element $a$ of $X$, the above fact allows us to choose $y$ in the above argument as an invertible element of $X$. Consequently, $y/(1 + \varepsilon/4)$ is invertible in $X_\alpha$ and, since $X_\alpha \in \mathcal{HP}(n)$, there is $b \in X_\alpha$ with the required properties. Because the limit of any direct system consisting of $C^*$-algebras with bounded rank $\leqslant 1$ has a bounded rank $\leqslant 1$ [**5**, Proposition 4.1], the above proof remains valid for the class $\mathcal{AP}_1(n)$. ∎

As in the commutative case (see Proposition 2.5), we can establish a decomposition theorem for the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$.

**PROPOSITION 3.4.** *Let $\mathcal{K}$ be one of the classes $\mathcal{AP}(n)$, $\mathcal{AP}_1(n)$ and $\mathcal{HP}(n)$. The following conditions are equivalent for any unital $C^*$-algebra $X$:*

(1) $X \in \mathcal{K}$.

(2) *$X$ can be represented as the direct limit of a direct $C^*_\omega$-system $\{X_\alpha, i^\beta_\alpha, A\}$ satisfying the following properties:*

  (a) *The indexing set $A$ is cofinal and $\omega$-closed in the $\omega$-complete set $\exp_\omega Y$ for some (any) dense subset $Y$ of $X$ such that $|Y| = d(X)$.*

  (b) *$X_\alpha$ is a (separable) $C^*$-subalgebra of $X$ with $X_\alpha \in \mathcal{K}$, $\alpha \in A$.*

PROOF: A similar statement holds for the class of all $C^*$-algebras of bounded rank $\leqslant n$ (see [**5**, Proposition 4.2]). So, it suffices to consider the classes $\mathcal{AP}(n)$ and $\mathcal{HP}(n)$. We suppose $\mathcal{K} = \mathcal{AP}(n)$. The implication (2) $\Longrightarrow$ (1) follows from Proposition 3.3.

In order to prove the implication (1) $\Longrightarrow$ (2) we first consider a direct $C^*_\omega$-system $\mathcal{S}_X = \{X_\alpha, i^\beta_\alpha, \Lambda\}$ with the properties indicated in Proposition 3.1. Each $X_\alpha$ is identified with $i_\alpha(X_\alpha)$. We next introduce the following relation $L \subseteq A^2$:
$(\alpha, \beta) \in \Lambda^2$ if and only if $\alpha \leqslant \beta$ and for each $x \in X_\alpha$ with $\|x\| \leqslant 1$ and each $\varepsilon > 0$ there exists $y \in X_\beta$ such that $\|y\| \leqslant 1$ and $\|x - y^n\| < \varepsilon$.

Let us show that $L$ satisfies the following conditions:

(i) for every $\alpha \in \Lambda$ there exists $\beta \in \Lambda$ with $(\alpha, \beta) \in L$:

(ii) If $(\alpha, \beta) \in L$ and $\beta \leqslant \gamma$, then $(\alpha, \gamma) \in L$;

(iii) if $\{\alpha_k\}$ is a chain in $\Lambda$ with each $(\alpha_k, \beta) \in L$, then $(\alpha, \beta) \in L$, where $\alpha = \sup\{\alpha_k\}$.

To verify (i), we take $\alpha \in \Lambda$ and a countable set $M \subset X_\alpha$ which is dense in the unit ball $B_\alpha = \{x \in X_\alpha : \|x\| \leqslant 1\}$. Since $X \in \mathcal{AP}(n)$, for each $x \in M$ and each $r \in Q^+$, we may take (and fix) $y(x,r) \in X$ with $\|x - y(x,r)^n\| < r$ and $\|y(x,r)\| \leqslant 1$. By Lemma 3.2, every $y(x,r)$ belongs to some $X_{\alpha(x,r)}$. Since $\Lambda$ is $\omega$-complete, according to [**3**, Corollary 1.1.28], there exists $\beta \in \Lambda$ such that $\beta \geqslant \alpha$ and $\beta \geqslant \alpha(x,r)$ for each $x \in M$ and $r \in Q^+$. Then, $X_\beta$ contains all $y(x,r)$ and $(\alpha, \beta) \in L$. Condition (ii) follows directly because $\beta \leqslant \gamma$ implies $X_\beta \subset X_\gamma$. Let us establish condition (iii). If $\alpha$ is the supremum of the countable chain $\{\alpha_k\}$, then $X_\alpha$ is the direct limit of the direct system generated by the $C^*$-subalgebras $X_{\alpha_k}$, $k = 1, 2, ldots$, and the corresponding inclusion homomorphisms. This fact and $(\alpha_k, \beta) \in L$ for all $k$ yield $(\alpha, \beta) \in L$.

Since $L$ satisfies the conditions (i)–(iii), we can apply [**3**, Proposition 1.1.29] to conclude that the set $A = \{\alpha \in \Lambda : (\alpha, \alpha) \in L\}$ is cofinal and $\omega$-closed in $\Lambda$. Note that $(\alpha, \alpha) \in L$ precisely when $X_\alpha \in \mathcal{AP}(n)$. Therefore, we obtain a direct $C^*_\omega$-system

$\mathcal{S}'_X = \{X_\alpha, i_\alpha^\beta, A\}$ consisting of $C^*$-subalgebras $X_\alpha \in \mathcal{AP}(n)$ of $X$. Clearly $\varinjlim \mathcal{S}'_X = X$. This completes the proof for the class $\mathcal{AP}(n)$. The case $\mathcal{K} = \mathcal{AP}(n)$ is similar. ∎

PROOF OF THEOREM 1.4: Let $\mathcal{B} = \{f_t\colon C^*(\mathbb{F}_\infty) \to X_t\colon t \in T\}$ denote the set of all unital $*$-homomorphisms on $C^*(\mathbb{F}_\infty)$ such that $X_t \in \mathcal{K}$. We claim that the product $\prod\{X_t\colon t \in T\}$ belongs to $\mathcal{K}$. This is obviously true if $\mathcal{K}$ is either $\mathcal{AP}(n)$ or $\mathcal{HP}(n)$. Since the bounded rank of this product is $\leqslant 1$ provided each $X_t$ is of bounded rank $\leqslant 1$ [**5**, Proposition 3.16], the claim holds for the class $\mathcal{AP}_1(n)$ as well. The $*$-homomorphisms $f_t$, $t \in T$, define the unital $*$-homomorphism $f\colon C^*(\mathbb{F}_\infty) \to \prod\{X_t\colon t \in T\}$ such that $\pi_t \circ f = f_t$ for each $t \in T$, where $\pi_t\colon \prod\{X_t\colon t \in T\} \to X_t$ denotes the canonical projection $*$-homomorphism onto $X_t$. By Proposition 3.4, $\prod\{X_t\colon t \in T\}$ can be represented as the limit of the $C^*_\omega$-system $\mathcal{S} = \{C_\alpha, i_\alpha^\beta, A\}$ such that $C_\alpha$ is a separable unital $C^*$-algebra with $C_\alpha \in \mathcal{K}$ for each $\alpha \in A$. Suppressing the injective unital $*$-homomorphisms $i_\alpha^\beta\colon C_\alpha \to C_\beta$, we may assume, for notational simplicity, that $C_\alpha$'s are unital $C^*$-subalgebras of $\prod\{X_t\colon t \in T\}$. Let $\{a_k\colon k \in \omega\}$ be a countable dense subset of $C^*(\mathbb{F}_\infty)$. By Lemma 3.2, for each $k \in \omega$ there exists an index $\alpha_k \in A$ such that $f(a_k) \in C_{\alpha_k}$. Since $A$ is $\omega$-complete, there exists an index $\alpha_0 \in A$ such that $\alpha_0 \geqslant \alpha_k$ for each $k \in \omega$. Then $f(a_k) \in C_{\alpha_k} \subseteq C_{\alpha_0}$ for each $k \in \omega$. This observation coupled with the continuity of $f$ guarantees that $f(C^*(\mathbb{F}_\infty)) = f(\mathrm{cl}\{a_k\colon k \in \omega\}) \subseteq \mathrm{cl}\Big\{f(\{a_k\colon k \in \omega\})\Big\} \subseteq \mathrm{cl}\, C_{\alpha_0} = C_{\alpha_0}$.

Let $Z_\mathcal{K} = C_{\alpha_0}$ and define the unital $*$-homomorphism $p\colon C^*(\mathbb{F}_\infty) \to Z_\mathcal{K}$ as $f$, regarded as a homomorphism of $C^*(\mathbb{F}_\infty)$ into $Z_\mathcal{K}$. Note that $f = i \circ p$, where $i\colon Z_\mathcal{K} = C_{\alpha_0} \hookrightarrow \prod\{X_t\colon t \in T\}$ stands for the inclusion.

By construction, we see $Z_\mathcal{K} \in \mathcal{K}$. Let us show that $p\colon C^*(\mathbb{F}_\infty) \to Z_\mathcal{K}$ is $\mathcal{K}$-invertible. For a given unital $*$-homomorphism $g\colon C^*(\mathbb{F}_\infty) \to X$, where $X$ is a separable unital $C^*$-algebra with $X \in \mathcal{K}$, we need to establish the existence of a unital $*$-homomorphism $h\colon Z_\mathcal{K} \to X$ such that $g = h \circ p$. Indeed, by definition of the set $\mathcal{B}$, we conclude that $g = f_t\colon C^*(\mathbb{F}_\infty) \to X_t = X$ for some index $t \in T$. Observe that $g = f_t = \pi_t \circ f = \pi_t \circ i \circ p$. This allows us to define the required unital $*$-homomorphism $h\colon Z_\mathcal{K} \to X$ as the composition $h = \pi_t \circ i$. Hence, $p$ is $\mathcal{K}$-invertible. ∎

## 4. EXAMPLE

In this section, we show that a construction due to B. Cole (see [**13**, Chapter 3, Section 19]) and M. Karahanjan [**9**, Thoerem 5] yields a square root closed compactum $X$ such that $\check{H}^1(X;\mathbb{Z})$ is infinitely generated. In the sequel, we shall omit the coefficient group $\mathbb{Z}$. We shall need the following theorem which is a consequence of [**7**, Theorem 3.2].

**THEOREM 4.1.** *Let $f\colon X \to Y$ be an open surjective map between compacta. Then $f^*\colon \check{H}^1(Y) \to \check{H}^1(X)$ is a monomorphism.*

Now we outline the construction due to B. Cole. This is based on the exposition in [**13**, Chapter 3, Section 19, p. 194–197]. Let $X$ be a compactum and define

$$S_X = \left\{ \left(x, (z_f)_{f \in C(X)}\right) : f(x) = z_f^2 \quad \text{for each } f \in C(X) \right\} \subset X \times \mathbb{C}^{C(X)}$$

Note that $S_X$ is a closed subset of $X \times \prod\{f(X) \mid f \in C(X)\}$ and hence is a compactum. Also, it is easy to see that $S_X$ is a pull-back in the following diagram:

$$\begin{array}{ccc} S_X & \longrightarrow & \mathbb{C}^{C(X)} \\ \big\downarrow & & \big\downarrow S \\ X & \xrightarrow{F} & \mathbb{C}^{C(X)} \end{array}$$

where $F: X \to \mathbb{C}$ is defined by $F(x) = \big(f(x)\big)_{f \in C(X)} (x \in X)$, and $S: \mathbb{C}^{C(X)} \to \mathbb{C}^{C(X)}$ is defined by $S\big((z_f)_{f\in C(X)}\big) = (z_f^2)_{f \in C(X)}$.

Let $\pi: S_X \to X$ be the map defined by $\pi\left[(x, (z_f)_{f \in C(X)})\right] = x$ for all $x \in X$. Then $\pi$ is an open map with zero-dimensional fibers. The critical property of $S_X$ and $\pi$ is the following:

(∗) for any $f \in C(X)$ there exists $g \in C(X)$ such that $f \circ \pi = g^2$.

Indeed, define $g: S_X \to \mathbb{C}$ by $g[(x, (z_f)_{f \in C(X)})] = z_f$.

Note that (∗) implies:

(∗∗) $\pi$ is invertible with respect to the class of square root closed compacta.

Starting with a compactum $X_0$, by transfinite induction we define an inverse spectrum $\{X_\alpha, \pi_\alpha^\beta: X_\beta \to X_\alpha : \alpha \leqslant \beta < \omega_1\}$ as follows. If $\beta = \alpha + 1$ then $X_\beta = S_{X_\alpha}$ and $\pi_\alpha = \pi: X_\beta = S_{X_\alpha} \to X_\alpha$ is the map defined above. If $\beta$ is a limit ordinal, then $X_\beta = \varprojlim(X_\alpha, \pi_\alpha^\gamma: X_\gamma \to X_\alpha : \alpha \leqslant \gamma < \beta)$ and, for $\alpha < \beta$, let $\pi_\alpha^\beta = \varprojlim(\pi_\alpha^\gamma: X_\gamma \to X_\alpha : \gamma < \beta)$.

We let $X_\Omega = \varprojlim X_\alpha$. The $\alpha$-th limit projection is denoted by $\pi_\alpha : X_\Omega \to X_\alpha$. As the length of the above spectrum is $\omega_1$, the spectrum is factorising in the sense that each $f \in C(X_\Omega)$ is represented as $f = f_\alpha \circ \pi_\alpha$ for some $\alpha < \omega_1$ and $f_\alpha \in C(X_\alpha)$. since its length is $\omega_1$. This implies that $C(X_\Omega)$ is square root closed due to the property (∗).

In what follows, the unit disk in the complex plane $\{z \in \mathbb{C} : |z| \leqslant 1\}$ is denoted by $\Delta$.

**Theorem 4.2.** *$C(\Delta_\Omega)$ is square-root closed,* $\dim \Delta_\Omega \leqslant 2$, *$\check{H}^1(\Delta_\Omega)$ is infinitely generated and 2-divisible.*

Notice that for each square root closed compactum $X$, $\check{H}^1(X)$ is 2-divisible. Hence, in view of the discussion above, we need only to show that $\check{H}_1(\Delta_\Omega)$ is infinitely generated. To show this, we need the following.

**THEOREM 4.3.** *$\check{H}^1(S_\Delta)$ is infinitely generated.*

Note that Theorem 4.2 immediately follows from Theorems 4.1 and Theorem 4.3. The proof of Theorem 4.3 is divided into two parts.

STEP 1. If $\check{H}^1(S_\Delta)$ is finitely generated then $\check{H}^1(S_\Delta) = 0$.

STEP 2. $\check{H}^1(S_\Delta) \neq 0$.

Now we shall accomplish Steps 1 and 2.

**PROPOSITION 4.4.** *Let $Y$ be a closed subspace of a compactum $X$ such that there exists a retraction $r\colon X \to Y$. Let also $i\colon Y \hookrightarrow X$ be the inclusion. Then there exist an embedding $\bar{i}\colon S_Y \hookrightarrow S_X$ and a retraction $\bar{r}\colon S_X \to S_Y$ such that the following diagram is commutative.*

$$\begin{array}{ccccc} S_Y & \xrightarrow{\bar{i}} & S_X & \xrightarrow{\bar{r}} & S_Y \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ Y & \xrightarrow{i} & X & \xrightarrow{r} & Y \end{array}$$

PROOF: Define $\bar{i}$ by

$$\bar{i}\Big[\big(y, (\eta_g)_{g\in C(Y)}\big)\Big] = \big(y, (\xi_f)_{f\in C(X)}\big)$$

where $\xi_f = \eta_{f|Y}$ for all $f \in C(X)$. Define $\bar{r}$ by

$$\bar{r}\Big[\big(x, (\xi_f)_{f\in C(X)}\big)\Big] = \big(r(x), (\eta_g)_{g\in C(X)}\big)$$

where $\eta_g = \xi_{g\circ r}$ for all $g \in C(Y)$. ∎

Now we are ready to accomplish Step 1. Let $\Delta_m = \{z \in \mathbb{C}\colon |z| \leqslant 1/m\} \subset \Delta$. Let $r_n\colon \Delta_n \to \Delta_{n+1}$ be the radial retraction and $i_n\colon \Delta_{n+1} \hookrightarrow \Delta_n$ be the inclusion. Consider the following sequence of commutative diagrams.

$$\begin{array}{ccccccccccccc} S_{\Delta_1} & \xleftarrow{\bar{i}_1} & S_{\Delta_2} & \xleftarrow{\bar{i}_2} & \cdots & \longleftarrow & S_{\Delta_n} & \xleftarrow{\bar{i}_n} & S_{\Delta_{n+1}} & \longleftarrow & \cdots & \longleftarrow & \varprojlim S_{\Delta_n} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & & & \downarrow{\scriptstyle \pi_n} & & & & & & \downarrow{\scriptstyle \varprojlim \pi_n = \pi_\infty} \\ \Delta_1 & \xleftarrow{i_1} & \Delta_2 & \xleftarrow{i_2} & \cdots & \longleftarrow & \Delta_n & \xleftarrow{i_n} & \Delta_{n+1} & \longleftarrow & \cdots & \longleftarrow & \{0\} \end{array}$$

It follows easily form the commutativity of the diagram that $\lim_{\longleftarrow} S_{\Delta_n}$ is homeomorphic to the inverse limit of the sequence

$$\pi_1^{-1}(0) \xleftarrow{\bar{i}_1|} \pi_2^{-1}(0) \longleftarrow \cdots \longleftarrow \pi_n^{-1}(0) \xleftarrow{\bar{i}_n|} \pi_{n+1}^{-1}(0) \longleftarrow \cdots$$

https://doi.org/10.1017/S0004972700035012 Published online by Cambridge University Press

Since each fiber $\pi_n^{-1}(0)$ is 0-dimensional, we have $\dim \lim_{\longleftarrow} S_{\Delta_n} = 0$. This implies that $\check{H}^1(\underset{\longleftarrow}{\lim} S_{\Delta_n}) = \underset{\longrightarrow}{\lim} \check{H}^1(S_{\Delta_n}) = 0$, which is equivalent to the following observation.

**PROPOSITION 4.5.** *For each $\alpha \in \check{H}^1(S_{\Delta_1}) = \check{H}^1(S_\Delta)$, there exists an $n$ such that $(\bar{i}_1 \circ \cdots \circ \bar{i}_n)^*(\alpha) = 0$.*

Let $A_n$ be the annulus defined by $A_n = \{z \in \mathbb{C} \mid (1/m+1) \leqslant |z| \leqslant 1/m\}$, so that $\Delta_n = \{0\} \cup (\cup\{A_j \mid j \geqslant n\})$. Let $h\colon \Delta = \Delta_1 \to \Delta_2$ be the homeomorphism which maps $A_j$ to $A_{j+1}$ ($j \geqslant 1$) by "radial homeomorphisms" and such that $h(0) = 0$. Then the following diagram is commutative

$$\begin{array}{ccc} \Delta_n & \xrightarrow{\ h|\ } & \Delta_{n+1} \\ {\scriptstyle i_n}\big\uparrow & & \big\uparrow{\scriptstyle i_{n+1}} \\ \Delta_{n+1} & \xrightarrow{\ h|\ } & \Delta_{n+1} \end{array}$$

Define $h_n\colon S_{\Delta_n} \to S_{\Delta_{n+1}}$ by $h_n\Big[\big(x, (u_f)_{f\in C(\Delta_n)}\big)\Big] = \big(h(x), (v_g)_{g\in C(\Delta_{n+1})}\big)$, where $v_g = u_{g\circ h}$, $g \in C(\Delta_{n+1})$. Note that $h_n$ is a homeomorphism.

**PROPOSITION 4.6.** *The following diagram is commutative.*

$$\begin{array}{ccc} S_{\Delta_{n+1}} & \xrightarrow{\ \bar{i}_n\ } & S_{\Delta_n} \\ {\scriptstyle h_{n+1}}\big\downarrow & & \big\downarrow{\scriptstyle h_n} \\ S_{\Delta_{n+2}} & \xrightarrow{\ \bar{i}_{n+1}\ } & S_{\Delta_{n+1}} \end{array}$$

PROOF: For each $(x_{n+1}, (z_f)_{f\in C(\Delta_{n+1})}) \in S_{\Delta_{n+1}}$ we have

$$\bar{i}_n\Big[\big(x_{n+1}, (z_f)_{f\in C(\Delta_{n+1})}\big)\Big] = \big(x_{n+1}, (u_f)_{f\in C(\Delta_n)}\big)$$

where $u_f = z_{f|\Delta_n} = z_{f\circ i_n}$, $f \in C(\Delta_n)$, and

$$h_n\Big[\big(x_{n+1}, (u_f)_{f\in C(\Delta_n)}\big)\Big] = \big(h(x_{n+1}), (v_f)_{f\in C(\Delta_{n+1})}\big)$$

where $v_f = u_{f\circ h} = z_{(f\circ h)\circ i_n} = z_{f\circ(h\circ i_n)}$. On the other hand,

$$h_{n+1}\Big[\big(x_{n+1}, (z_f)_{f\in C(\Delta_{n+1})}\big)\Big] = \big(h(x_{n+1}), (u_g)_{g\in C(\Delta_{n+1})}\big)$$

where $u_g = z_{g\circ h}$, $g \in C(\Delta_{n+2})$, and

$$\bar{i}_{n+1}\Big[\big(h(x_{n+1}), (u_g)_{g\in C(\Delta_{n+1})}\big)\Big] = \big(h(x_{n+1}), (v_f)_{f\in C(\Delta_{n+1})}\big)$$

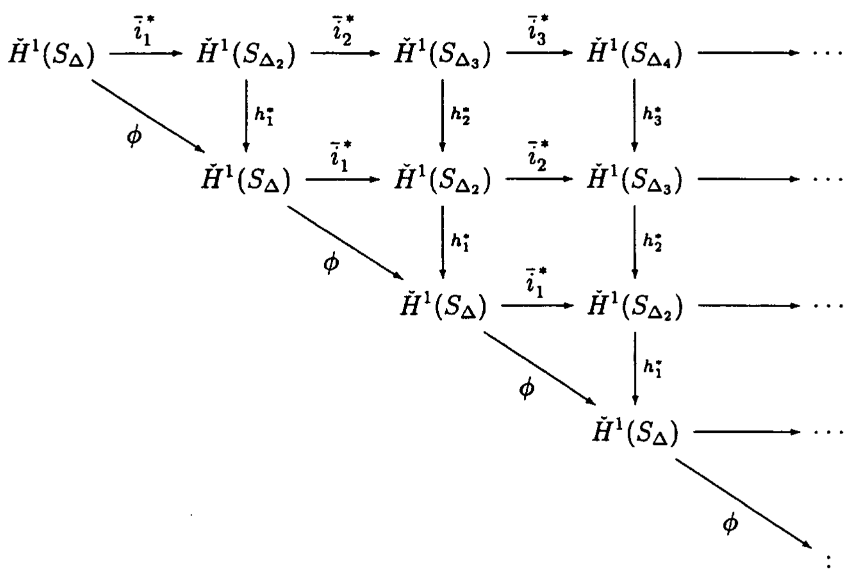
where $v_f = u_{f\circ i_{n+1}} = z_{(f\circ i_{n+1})\circ h} = z_{f\circ(i_{n+1}\circ h)}$. Since $h \circ i_n = i_{n+1} \circ h$, we conclude that the diagram is commutative. ∎

The above lemma provides a commutative diagram in cohomologies:

https://doi.org/10.1017/S0004972700035012 Published online by Cambridge University Press

$$(\dagger)\qquad \begin{array}{ccc} \check{H}^1(S_{\Delta_{n+1}}) & \xleftarrow{\;\bar{i}_n^*\;} & \check{H}^1(S_{\Delta_n}) \\ {\scriptstyle h_{n+1}^*}\Big\uparrow & & \Big\uparrow{\scriptstyle h_n^*} \\ \check{H}^1(S_{\Delta_{n+2}}) & \xleftarrow{\;\bar{i}_{n+1}^*\;} & \check{H}^1(S_{\Delta_{n+1}}) \end{array}$$

Let $\phi = h_1^* \circ i_1^* \colon \check{H}^1(S_\Delta) \to \check{H}^1(S_\Delta)$. Since $\bar{r}_1 \circ \bar{i}_1 = \mathrm{id}_{S_\Delta}$ we have $\bar{i}_1^* \circ \bar{r}_1^* = \mathrm{id}_{\check{H}^1(S_\Delta)}$ and hence $\phi$ is an epimorphism. We use diagram (†) to obtain the following diagram, in which all vertical arrows are isomorphisms.



The above diagram together with Proposition 4.5 imply that, for each $\alpha \in \check{H}^1(S_\Delta)$, there exists $n$ such that $\phi^n(\alpha) = 0$. If $\check{H}^1(S_\Delta)$ were finitely generated, we then would have $\check{H}^1(S_\Delta) = 0$ because of the following observation.

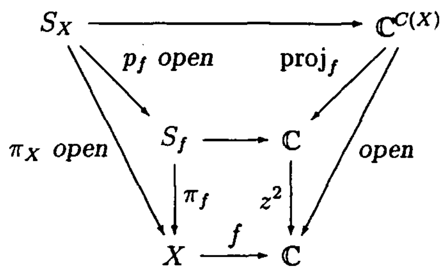
**Proposition 4.7.** *Let $A$ be a finitely generated Abelian group. If there exists an epimorphism $f \colon A \to A$ such that for any $a \in A$ there exists $n$ with $f^n(a) = 0$, then $A$ is trivial.*

Proof: Note that $f \otimes 1_{\mathbb{Q}} \colon A \otimes \mathbb{Q} \to A \otimes \mathbb{Q}$ is an epimorphism of a vector space $A \otimes \mathbb{Q}$, which is finite-dimensional over $\mathbb{Q}$. Hence $f \otimes 1_{\mathbb{Q}}$ is an isomorphism with the property in the hypothesis. This implies $\operatorname{rank} A = 0$ and therefore $A$ is a finite Abelian group. Then $f$ is an isomorphism and therefore $A = 0$. ∎

Thus Step 1 is completed and we proceed to Step 2.

https://doi.org/10.1017/S0004972700035012 Published online by Cambridge University Press

**Proposition 4.8.** *For a continuous function $f \in C(X)$, let $S_f = \{(x,z) : f(x) = z^2 \text{ for each } x \in X\} \subset X \times \mathbb{C}$. Let also $\pi_f \colon S_f \to X$ be the projection. Then the natural map $p_f \colon S_X \to S_f$, $(x, (z_g)_{g \in C(X)}) \mapsto (x, z_f)$ is open. Thus we have the following diagram.*




Proof: Consider $g_1, g_2, \dots, g_n \in C(X)$ and open subset $U_X \subset X$, $V_f, V_{g_1}, \dots, V_{g_n} \subset \mathbb{C}$. It suffices to show that

$$p_f\Big[\Big(U_X \times V_f \times V_{g_1} \times \cdots \times V_{g_n} \times \prod_{g \neq g_1,\dots,g_n,f} \mathbb{C}\Big) \cap S_X\Big]$$

is open in $S_f$. Take a point

$$(x, z_f, (z_{g_i})_{i=1}^n, (z_g)_{g \neq f, g_1,\dots,g_n}) \in U_X \times V_f \times V_{g_1} \times \cdots \times V_{g_n} \times \prod_{g \neq g_1,\dots,g_n,f} \mathbb{C}$$

and choose $\varepsilon > 0$ such that $B(z_f, \varepsilon) = \{w \in \mathbb{C} \colon |w - z_f| < \varepsilon\} \subset V_f$ and $B(z_{g_i}, \varepsilon) \subset V_{g_i}$ for all $i = 1, 2, \dots, n$. Let $a = f(x)$, $a_i = g_i(x)$, $i = 1, 2, \dots, n$. There exists $\delta > 0$ such that if $|b - a| < \delta$ and $|b_i - a_i| < \delta$, $i = 1, \dots, n$, then the equations

$$z^2 - b = 0$$
$$z_i^2 - b_i = 0, i = 1, \dots, n$$

have solutions $z_b$ and $z_{b_i}$ respectively such that $|z_b - z_f| < \varepsilon$, $|z_{b_i} - z_{g_i}| < \varepsilon$. Choose a neighbourhood $N$ of $x$ such that $|f(y) - f(x)| < \delta$ and $|g_i(y) - g_i(x)| < \delta$ for all $y \in N$ and $i = 1, \dots, n$. We claim that

$$N \times B(z_f, \varepsilon) \subset p_f\Big[\Big(U_X \times V_f \times V_{g_1} \times \cdots \times V_{g_n} \times \prod_{g \neq g_1,\dots,g_n,f} \mathbb{C}\Big) \cap S_X\Big]$$

Indeed, for each pint $(y, w) \in N \times B(z_f, \varepsilon) \subset N \times V_f$ we have $|g_i(y) - g_i(x)| < \delta$, $i = 1, 2, \dots, n$ by the choice of $N$. Then we can find $z_i \in B(z_{g_i}, \varepsilon)$ such that $z_i^2 = g_i(y)$. Now for arbitrary choice of $z_g$, where $g \neq f, g_1, g_2, \dots, g_n$ with $z_g^2 = g(x)$, we have

$$(y, w, (z_i)_{i=1}^n, (z_g)) \in U_X \times V_f \times V_{g_1} \times \cdots \times V_{g_n} \times \prod_{g \neq g_1,\cdots,g_n,f} \mathbb{C}$$

https://doi.org/10.1017/S0004972700035012 Published online by Cambridge University Press

and $p_f\Big[\big(y, w, (z_i)_{i=1}^{n}, (z_g)\big)\Big] = (y, w)$. This proves the claim and hence completes the proof of the proposition. ∎

By Proposition 4.8 and Theorem 4.1, the statement of the Step 2 follows from the next observation.

**PROPOSITION 4.9.** *There exists a mapping $f\colon \Delta \to \mathbb{C}$ such that $\check{H}^1(S_f) \neq 0$.*

PROOF: Let $f(x, y) = \big(-2|x| + \sqrt{1-y^2}, y\big)$ for all $(x, y) \in \Delta$. Then $S_f$ is homeomorphic to cylinder $S^1 \times I$. ∎

This completes the proof of Theorem 4.2.

The above construction is carried out word by word for disks of arbitrary dimensions. In particular, applying the above to the one-dimensional disk $[-1, 1]$, we have the following corollary which suggests that a topological characterisation of general square root closed compacta could be rather different than the one for first-countable such compacta by [8] and [**12**].

**COROLLARY 4.10.** *There exists an one-dimensional square root closed compactum $X$ with infinitely generated first Čech cohomology.*

For an infinite cardinal $\tau \geqslant \omega$, we consider $(\mathbb{I}^\tau)_\Omega$ and the limit projection $\pi_\Omega : (\mathbb{I}^\tau)_\Omega \to \mathbb{I}^\tau$. By the invertibility property (**) of $\pi : S_X \to X$ for arbitrary compactum $X$ and the standard spectral argument, it follows easily that $\pi_\Omega$ is also invertible with respect to the class of square root closed compacta. Hence we have

**PROPOSITION 4.11.** *The square root closed compactum $(\mathbb{I}^\tau)_\Omega$ contains every square root closed compactum of weight $\leqslant \tau$.*

## REFERENCES


[1] L.G. Brown and G.K. Pedersen, '$C^*$-algebra of real rank zero', *J. Functional Anal.* **99** (1991), 131–149.

[2] A. Chigogidze, 'Universal $C^*$-algebra of real rank 0', *Infin. Dimens. Anal. Quantum Probab. Related Top.* **3** (2000), 445–452.

[3] A. Chigogidze, *Inverse spectra* (North Holland, Amsterdam, 1996).

[4] A. Chigogidze, 'Uncountable direct systems and a characterization of non-separable projective $C^*$-algebras', *Mat. Stud.* **12** (1999), 171–204.

[5] A. Chigogidze and V. Valov, 'Bounded rank of $C^*$-algebras', *Topology Appl.* **140** (2004), 163–180.

[6] R.S. Countryman, 'On the characterization of compact Hausdorff $X$ for which $C(X)$ is algebraically closed', *Pacific J. Math.* **20** (1967), 433–438.

[7] J. Grispolakis and E.D. Tymchatyn, 'On confluent mappings and essential mappings - a survey', *Rocky Mountain J. Math.* **11** (1981), 131–153.

[8] O. Hatori and T. Miura, 'On a characterization of the maximal ideal spaces commutative $C^*$-algebras in which every element is the square of another', *Proc. Amer. Math. Soc.* **128** (1999), 239–242.

[9] M.I. Karahanjan, 'On some algebraic characterization of the algebra of all continuous functions on a locally connected compactum', (in Russian), *Math. Sb. (N.S.)* **107** (1978), 416–434. English translation **35** (1979), 681–696.

[10] K. Kawamura and T. Miura, 'On the existence of continuous (approximate) roots of algebraic equations', (preprint).

[11] S. Mardešić, 'On covering dimension and inverse limit of compact spaces', *Illinois J. Math.* **4** (1960), 278–291.

[12] T. Miura and K. Niijima, 'On a characterization of the maximal ideal spaces of algebraically closed commutative $C^*$-algebras', *Proc. Amer. Math. Soc.* **131** (2003), 2869–2876.

[13] E.L. Stout, *The theory of uniform algebras* (Bogden and Quigley Inc., New York, 1971).

Department of Mathematical Sciences
University of North Carolina at Greensboro
P.O. Box 26170
Greensboro, NC 27402-6170
United States of America
e-mail: chigogidze@uncg.edu

Department of Computer Science and Mathematics
Nipissing University
P.O. Box 5002
North Bay, ON, P1B 8L7
Canada
e-mail: alexandk@nipissingu.ca

Institute of Mathematics
University of Tsukuba
Tsukuba
Ibaraki 305-8071
Japan
e-mail: kawamura@math.tsukuba.as.jp

Department of Computer Science and Mathematics
Nipissing University
P.O. Box 5002
North Bay, ON, P1B 8L7
Canada
e-mail: veskov@nipissingu.ca