BULL. AUSTRAL. MATH. SOC. 46E40, 46B03
VOL. 53 (1996) [131–134]

ON THE DUNFORD-PETTIS PROPERTY IN SPACES OF VECTOR-VALUED BOUNDED FUNCTIONS

MANUEL D. CONTRERAS AND SANTIAGO DÍAZ

We show that $L^\infty(\mu, X)$ has the Dunford-Pettis property for some classical Banach spaces including $L^1(\mu)$, $C(K)$, the disc algebra A and H^∞.

A Banach space X is said to have the Dunford-Pettis property if every weakly compact operator from X into an arbitrary Banach space is completely continuous, or equivalently, if given sequences (x_n) in X and (x_n^*) in X^*, both weakly convergent to zero, then $\langle x_n^*, x_n\rangle$ tends to zero. A detailed exposition about this property can be found in [**6**]. In this reference, the following problem is posed [**6**, p.55]: assume that $L^\infty(\mu, X)$ denotes the Banach space of (equivalence classes of) essentially bounded, measurable and X-valued functions defined over a finite measure space (Ω, Σ, μ). When does $L^\infty(\mu, X)$ have the Dunford-Pettis property? In general, this property does not lift from X to $L^\infty(\mu, X)$ [**6**, p.56]. On the other hand, the only non-trivial positive result, as far as we know, is that $L^\infty\big(\mu, L^1(\nu)\big)$ has the property when μ is purely atomic [**3**, Theorem 1]. The aim of this note is to provide some new positive examples. Namely, we show that $L^\infty(\mu, X)$ has the Dunford-Pettis property for every arbitrary finite measure μ, whenever X is either any $\mathcal{L}^1$-space or any $\mathcal{L}^\infty$-space or the disc algebra or the space H^∞ of bounded analytic functions on the disc.

To avoid trivial situations, we always assume that there exists a pairwise disjoint sequence (C_m) in Σ such that $\mu(C_m) > 0$. The notation is standard except, perhaps, the following one: if (A_m) is a sequence of pairwise disjoint Σ-measurable subsets of non-zero measure, we write

$$[A_m] := \left\{ \sum_{m=1}^{\infty} \chi_{A_m}(\cdot) x_m, \quad (x_m) \in \ell^\infty(X) \right\}.$$

It is well-known that $[A_m]$ is a complemented subspace of $L^\infty(\mu, X)$ isometrically isomorphic to $\ell^\infty(X)$. In particular, if $L^\infty(\mu, X)$ has the Dunford-Pettis property,

Received 28th March, 1995
This research has been partially supported by La Consejería de Educación y Ciencia de la Junta de Andalucía.

then $\ell^\infty(X)$ has it and our main aim (Theorem 1 below) is to prove that the converse holds. Similar ideas have been used in [4].

We refer the reader for the terminology used here to the monographs of Diestel and Uhl [**7**], Lindenstrauss and Tzafriri [**8**] and Wojtaszczyk [**10**].

Theorem 1. *$L^\infty(\mu, X)$ has the Dunford-Pettis property if (and only if) $\ell^\infty(X)$ has it.*

Proof: Suppose that (f_n) and (f_n^*) are sequences in $L^\infty(\mu, X)$ and $L^\infty(\mu, X)^*$ respectively, both weakly convergent to zero. We have to prove that $\langle f_n^*, f_n\rangle$ tends to zero.

The proof of Pettis's Measurability Theorem, as can be seen in [**7**, Chapter 2, Theorem 2], shows that, for each $n \in \mathbb{N}$, there are a bounded sequence $(x_m(n))_m$ in X and a sequence $(A_m(n))_m$ of pairwise disjoint Σ-measurable subsets with positive measure covering Ω, such that if

$$g_n(\cdot) := \sum_{m=1}^{\infty} x_m(n)\chi_{A_m(n)}(\cdot) \in L^\infty(\mu, X),$$

we have $\|f_n - g_n\| \leqslant 1/n$. Hence, it is enough to prove that $\langle f_n^*, g_n\rangle$ tends to zero.

Denote by X_n the finite-dimensional subspace of $L^\infty(\mu, X)$ spanned by $g_1, \dots, g_n$. Furthermore, for every $n \in \mathbb{N}$, consider the following family of pairwise disjoint Σ-measurable subsets of Ω:

$$\mathcal{A}^n := \{A_{m_1}(1) \cap A_{m_2}(2) \cap \dots \cap A_{m_n}(n) : \quad m_1, m_2, \dots, m_n \in \mathbb{N}\}.$$

Arrange $\mathcal{A}^n$ in a sequence $(B_m^n)_m$. It is easy to see that X_n is included in $Y_n = [B_m^n]$. Since (Y_n) is increasing, the closure Y of $\bigcup_n Y_n$ is a closed subspace of $L^\infty(\mu, X)$ and g_n is a weakly null sequence in Y. On the other hand, the restrictions $f^*_{n|Y}$ also form a weakly null sequence in Y^*. To finish the proof, we only have to show that Y has the Dunford-Petis property. According to a result due to Bourgain [**3**, Proposition 2], it is enough to prove that $(\oplus_n Y_n)_\infty$ has the Dunford-Pettis property. But, this follows by combining the hypothesis that $\ell^\infty(X)$ has the Dunford-Pettis property with the following topological isomorphisms:

$$(\oplus_n Y_n)_\infty \cong (\oplus_n \ell^\infty(X))_\infty \cong \ell^\infty(X). \qquad \square$$

We need later the following variant of Theorem 1 which is of independent interest.

Theorem 2. *Assume that given a sequence of finite-dimensional subspaces (X_n) in $\ell^\infty(X)$, we can find, for each n, another sequence of subspaces $(X_{n,m})_m$ in X, such*

https://doi.org/10.1017/S0004972700016798 Published online by Cambridge University Press

that $X_{n,m} \subset X_{n+1,m}$, $X_n \subset Y_n = (\oplus_m X_{n,m})_\infty$ *and* $(\oplus_n Y_n)_\infty$ *has the Dunford-Pettis property. Then,* $L^\infty(\mu, X)$ *has it.*

PROOF: Arguing, as in the proof above, one shows that $\ell^\infty(X)$ has the Dunford-Pettis property. Then, applying Theorem 1, we obtain that $L^\infty(\mu, X)$ has this property. ∎

REMARK. Theorem 1 shows that the measure μ does not play a significant role. This is in contrast to other vector-valued situations. Indeed, the behaviour of purely atomic and atomless measures with respect to the Dunford-Pettis property is quite different in $L^1(\mu, X)$ (see [**2**, Corollary 2.4(c)] and [**9**, Théorème 3]). Something similar happens with perfect and dispersed compacts in $C(K, X)$ (see [**1**, Theorem 2] and [**9**, Théorème 3]). It is worth mentioning that, in general, $\ell^\infty(X)$ and $L^\infty(\mu, X)$ are not isomorphic [**5**, Corollary 1].

We recall that a Banach space X is said to be an $\mathcal{L}^p$-space $(1 \leqslant p \leqslant \infty)$, in the sense of Lindenstrauss-Pełczyński, if there is $\lambda \geqslant 1$ such that, for every finite-dimensional subspace Y of X, there is another finite-dimensional subspace Z of X such that Y is contained in Z and $d(Z, \ell_k^p) \leqslant \lambda$, for some $k \in \mathbb{N}$.

COROLLARY 1. *Denote by* X *either any* $\mathcal{L}^1$*-space or any* $\mathcal{L}^\infty$*-space. Then,* $L^\infty(\mu, X)$ *has the Dunford-Pettis property.*

PROOF: Bearing in mind the definition of $\mathcal{L}^p$-space and Theorem 2, it is not difficult to see that we only need to show that $(\oplus_n \ell_{r_n}^p)_\infty$ $(r_n \in \mathbb{N};\ p = 1, \infty)$ has the Dunford-Pettis property. The case $p = 1$ follows from Bourgain's result [**3**, Theorem 1] and, for the case $p = \infty$, we note that $(\oplus_n \ell_{r_n}^\infty)_\infty \cong \ell^\infty$. ∎

COROLLARY 2. *Denote by* X *either the disc algebra* A*, or the space of bounded analytic functions on the disc* H^∞*. Then,* $L^\infty(\mu, X)$ *has the Dunford-Pettis property.*

PROOF: This follows directly from Theorem 1 and [**4**, proof of Theorem 2]. ∎

REFERENCES

[1] F. Bombal, 'On weakly compact operators on spaces of vector valued continuous functions', *Proc. Amer. Math. Soc.* **97** (1986), 93–96.

[2] F. Bombal, 'Distinguished subsets in vector sequence spaces', in *Progress in Functional Analysis*, (K.D. Bierstedt, J. Bonet, J. Horváth and M. Maestre, Editors) (North-Holland /Elsevier Science Publishers B.V., Amsterdam, New York, Oxford, 1992), pp. 293–306.

[3] J. Bourgain, 'On the Dunford-Pettis property', *Proc. Amer. Math. Soc.* **81** (1981), 265–272.

[4] M.D. Contreras and S.Díaz, '$C(K, A)$ and $C(K, H^\infty)$ have the Dunford-Pettis property', *Proc. Amer. Math. Soc.* (to appear).

[5] S. Díaz, 'Complemented copies of c_0 in $L^\infty(\mu, X)$', *Proc. Amer. Math. Soc.* **120** (1994), 1167–1172.

[6] J. Diestel, 'A survey of results related to the Dunford-Pettis property', in *Contemporary Math.*, **2**, Proc. of the Conf. on Integration, Topology and Geometry in Linear Spaces (Amer. Math. Soc., Providence, RI, 1980), pp. 15–60.

[7] J. Diestel and J.J. Uhl, Jr., *Vector measures*, Math. Surveys, **15** (Amer. Math. Soc., Providence, RI, 1977).

[8] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Lecture Notes in Mathematics, **338** (Springer-Verlag, Berlin, Heidelberg, New York, 1973).

[9] M. Talagrand, 'La propriété de Dunford-Pettis dans $C(K,E)$ et $L_1(E)$', *Israel J. Math.* **44** (1983), 317–321.

[10] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Studies in Advanced Mathematics, **25** (Cambridge University Press, Cambridge, 1991).

Departamento de Matemática Aplicada II
Universidad de Sevilla
E.S. Ingenieros Industriales
41012 Sevilla
Spain
e-mail: contreras@cica.es
madrigal@cica.es