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ON CERTAIN QUOTIENTS OF GROTHENDIECK GROUPS

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ABSTRACT. A general categorical construction is described which has as special cases the construction of the Brauer group of a field and the construction of the Witt ring of a field of characteristic $\neq 2$.

The striking parallel between the construction of the Brauer group and the construction of the Witt ring—division rings playing the same special role in the one case as anisotropic forms in the other—was noticed from the start (e.g. by Witt himself in the Introduction to [3]). We give here a general categorical construction which has these two as particular cases.

Let $(\mathscr{C}, *)$ be a category with product in the sense of [1, Ch. VII, §1]. We assume that subcategories of $\mathscr{C}$ always contain, with any object, all objects isomorphic to it in $\mathscr{C}$. This convention involves no loss of generality, and simplifies the verbiage somewhat. For example, since the isomorphism classes of $\mathscr{C}$ form a set, we can speak of $\mathrm{Ob}\,\mathscr{C}' \cap \mathrm{Ob}\,\mathscr{C}''$ for subcategories $\mathscr{C}'$ and $\mathscr{C}''$. Throughout, $\simeq$ denotes isomorphism in $\mathscr{C}$.

Let $\mathscr{W}$ be a full cofinal subcategory of $\mathscr{C}$, closed under $*$. (*Cofinal* means given $X \in \mathrm{Ob}\,\mathscr{C}$ there exists $X' \in \mathrm{Ob}\,\mathscr{C}$ with $X * X' \in \mathrm{Ob}\,\mathscr{W}$.) Let $[\mathscr{W}]$ denote the subgroup of the Grothendieck group $K_0(\mathscr{C}, *)$ generated by $\mathrm{Ob}\,\mathscr{W}$, and define $K(\mathscr{C}/W) = K_0(\mathscr{C}, *)/[\mathscr{W}]$.

Let $\mathscr{S}$ be a full subcategory of $\mathscr{C}$. We say that $(\mathscr{C}, *, \mathscr{W}, \mathscr{S})$ is a *π-category* if the following condition is satisfied:

(π): Given $X \in \mathrm{Ob}\,\mathscr{C}$ there exist $w(X) \in \mathrm{Ob}\,\mathscr{W}$ and $s(X)$ in $\mathrm{Ob}\,\mathscr{S}$, the latter unique up to isomorphism, such that $X \simeq w(X) * s(X)$.

EXAMPLE 1. Let K be a field and let $\mathscr{C}$ be the category of finite dimensional central simple K-algebras, with product $* = \otimes_K$. Let $\mathscr{W}$ (resp. $\mathscr{S}$) be the full subcategory whose objects are the algebras isomorphic to endomorphism rings of K-vector spaces (resp. the central division algebras). $\mathscr{W}$ is cofinal in $\mathscr{C}$, since if A is central simple and A^0 is its opposite algebra, $A \otimes_K A^0$ is isomorphic to a full matrix ring over K. Wedderburn's theorem shows that condition (π) is satisfied, and $K(\mathscr{C}/\mathscr{W})$ is the Brauer group of K.

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EXAMPLE 2. Let K be a field of characteristic $\neq 2$ and let $\mathscr{C}$ be the category of nondegenerate quadratic forms over K (with isometries as morphisms), with product $*$=orthogonal sum. Let $\mathscr{W}$ (resp. $\mathscr{S}$) be the full subcategory whose objects are the orthogonal sums of hyperbolic planes (resp. the anisotropic forms). $\mathscr{W}$ is cofinal since q nondegenerate implies $q*(-q)$ isometric to a sum of hyperbolic planes, and condition (π) follows from Witt's theorem. $K(\mathscr{C}/\mathscr{W})$ is the Witt ring of K (or rather its underlying additive group).

In neither example is $\mathscr{S}$ closed under $*$.

We call the objects of $\mathscr{W}$ (resp. $\mathscr{S}$) the *weak* (resp. *strong*) objects of $\mathscr{C}$. Condition (π) says that each object X of $\mathscr{C}$ has a weak part $w(X)$ and a strong part $s(X)$, the latter at least well-defined up to isomorphism. The terminology is suggested by:

THEOREM 3. *Let* $(\mathscr{C}, *, \mathscr{W}, \mathscr{S})$ *be a* π-*category. Then the elements of* $K(\mathscr{C}/\mathscr{W})$ *are in one-to-one correspondence with the isomorphism classes of strong objects. If* $x, y \in K(\mathscr{C}/\mathscr{W})$ *are represented by strong objects* $X, Y \in \text{Ob}\,\mathscr{C}$ *respectively then* $x+y$ *is represented by* $s(X*Y)$.

(The theorem makes precise the assertion—plausible enough in light of the decomposition given by condition (π)—that when we kill the weak part of $\mathscr{C}$ in the Grothendieck group, the strong part survives intact.)

Proof. For $X, Y \cdots$ in $\text{Ob}\,\mathscr{C}$ let $x, y \cdots$ denote the corresponding element of $K_0(\mathscr{C}, *)$ and let $z \to \bar{z}$ denote the canonical projection $K_0(\mathscr{C}, *) \to K(\mathscr{C}/\mathscr{W})$. Every element of $K_0(\mathscr{C}, *)$ is of the form $x-w$ with $X \in \text{Ob}\,\mathscr{C}$ and $W \in \text{Ob}\,\mathscr{W}$ by [1, Proposition (1.3.a)]. Hence every element of $K(\mathscr{C}/\mathscr{W})$ is represented by a strong object of C: $\overline{x-w}=\bar{x}-\bar{w}=\bar{x}=\bar{s}$ where $S=s(X)$. If two strong objects X and Y give the same element of $K(\mathscr{C}/\mathscr{W})$ we have $x-y \in [\mathscr{W}]$, i.e., $x+w=y+w'$ in $K_0(\mathscr{C}, *)$ for some $W, W' \in \text{Ob}\,\mathscr{W}$, or equivalently $X*W'' \simeq Y*W'''$ for some $W'', W''' \in \text{Ob}\,\mathscr{W}$ (by [1, Proposition (1.3.b)], and the closure of $\mathscr{W}$ under $*$). The uniqueness requirement in condition (π) then implies $X \simeq Y$. The statement about the sum is trivial: $\bar{x}+\bar{y}=\overline{x+y}=\bar{s}$ where $S=s(X*Y)$.

It is easy, if slightly clumsy, to jazz up the definition so as to include the multiplicative structure in Example 2, as follows. Let $(\mathscr{C}, *)$ be a category with product and $\mathscr{D}$ a full subcategory, closed under $*$, which is itself a category with product under an operation $\circ$. We say that $(\mathscr{D}, \circ)$ is a multiplication for $(\mathscr{C}, *)$ if $\circ$ distributes over $*$, i.e., if $X\circ(Y*Z)$ and $(X\circ Y)*(X\circ Z)$ are isomorphic functors $\mathscr{D}\times\mathscr{D}\times\mathscr{D}\to\mathscr{D}$. We say the multiplication has a unit if there is an $I \in \text{Ob}\,\mathscr{D}$ (necessarily unique up to natural isomorphism) such that $X \to I\circ X$ is isomorphic to the identity functor on $\mathscr{D}$.

PROPOSITION 4. *Let* $(\mathscr{D}, \circ)$ *be a multiplication for* $(\mathscr{C}, *)$. *Then* $\circ$ *gives* $K_0(\mathscr{D}, *)$ *the structure of a commutative ring, and the unit for* $\circ$ (*if there is one*) *represents a unit element for this ring.*

Proof. Write $x, y \cdots$ for the elements of $K_0(\mathcal{D}, *)$ corresponding to objects $X, Y \cdots$ of $\mathcal{D}$, as before. Then every element of $K_0(\mathcal{D}, *)$ is of the form $x-w$ with $X, W \in \text{Ob}\,\mathcal{D}$. Define the product $(x-w)(y-z)$ of two elements to be $x \circ y - x \circ z - w \circ y + w \circ z$ where $x \circ y$ means the element of $K_0(\mathcal{D}, *)$ corresponding to $X \circ Y$. Everything follows if we show this is well-defined for elements x, y. Thus assume that $X, X' \in \text{Ob}\,\mathcal{D}$ represent the same element x of $K_0(\mathcal{D}, *)$ and that Y, Y' represent the same element y. Then $X * W \simeq X' * W$ and $Y * Z \simeq Y' * Z$ for some W and Z in $\text{Ob}\,\mathcal{D}$, and consequently $X * S \simeq X' * S$ and $Y * S \simeq Y' * S$ (where $S = W * Z$). Hence $(X \circ Y) * (S \circ (S * X * Y)) \simeq (X * S) \circ (Y * S) \simeq (X' * S) \circ (Y' * S) \simeq (X' \circ Y') * (S \circ (S * X' * Y'))$. Now letting $T = S \circ (X * S * Y * S)$ we have $(X \circ Y) * T \simeq (X' \circ Y') * T$, which shows that $X \circ Y$ and $X' \circ Y'$ represent the same element, as required.

The empty subcategory $\mathcal{D}$ gives any category with product a trivial (i.e. never-defined) multiplication. At the other extreme if $\mathcal{D} = \mathcal{C}$ we say $\circ$ is a *full* multiplication for $(\mathcal{C}, *)$. We need one further definition. If $(\mathcal{C}, *, \mathcal{W}, \mathcal{S})$ is a π-category and $(\mathcal{D}, \circ)$ is a multiplication for $(\mathcal{C}, *)$, we call the whole works $(\mathcal{C}, *, \mathcal{D}, \circ, \mathcal{W}, \mathcal{S})$ a *multiplicative π-category* provided: if $X \in \text{Ob}\,\mathcal{D}$ and $Y \in \text{Ob}\,\mathcal{D} \cap \text{Ob}\,\mathcal{W}$ then $X \circ Y \in \text{Ob}\,\mathcal{W}$.

COROLLARY 5. *Let $(\mathcal{C}, *, \mathcal{D}, \circ, \mathcal{W}, \mathcal{S})$ be a multiplicative π-category. Then we have the conclusions of Theorem* 3. *If the multiplication is full, it gives $K(\mathcal{C}/\mathcal{W})$ the structure of a commutative ring; if $x, y \in K(\mathcal{C}/\mathcal{W})$ are represented by strong objects X, Y respectively then xy is represented by $s(X \circ Y)$; the unit for $\circ$, if there is one, represents a unit element for this ring.*

Proof. To get the conclusions of Theorem 3 we simply forget the multiplication. The definition is rigged to make the rest of the proof equally trivial: multiplicativity makes $[\mathcal{W}]$ an ideal in $K_0(\mathcal{C}, *)$.

In applying Corollary 5 to Example 1 we impose the empty multiplication and get (again) the Brauer group. In the case of Example 2, tensor product of quadratic forms is a full multiplication, and Corollary 5 yields the Witt ring of K (not just its additive group). The strong form $\langle 1 \rangle$ (i.e. $q(x) = x^2$) represents a unit. Note, however, that nondegeneracy of $\langle 1 \rangle$ requires the invertibility of 2 in K; the Witt ring defined for any commutative ring by the construction described here will in general be a ring without unit. (See [2, Ch. V, §2] for a "natural remedy".) The Brauer group can of course also be defined for any commutative ring as a quoteint of a Grothendieck group as above (again see [2]). In both cases, however, in passing from fields to arbitrary commutative rings one loses the strong objects: the categories are not π-categories.

Exercise. Ignoring the pessimistic tone of the preceding sentence, find natural examples of π-categories other than the two the notion was invented to accommodate.

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