11

Harmonic Measure and Elliptic Measures

For the harmonic measure in the plane, see [221]. Toro's survey [424] discusses relations between geometric measure theory and harmonic measure both in the plane and in higher dimensions.

11.1 Harmonic Measure

Let $\Omega \neq \mathbb{R}^n$ be an open connected subset of $\mathbb{R}^n, n \geq 2$. We assume that $\partial\Omega$ is not too small; it is not a polar set. This is true if $\mathcal{H}^{n-1}(\partial\Omega) > 0$. By classical potential theory, see, for example, [28], for any continuous function f on $\partial\Omega$ one can solve the Dirichlet problem to find a harmonic function u_f in Ω with boundary values f, in a generalized sense if $\partial\Omega$ is not sufficiently nice. Fixing $p \in \Omega$ for a while, the Riesz representation theorem and the maximum principle can be used to show that there is a probability measure $\omega_\Omega^p \in \mathcal{M}(\partial\Omega)$ such that

$$u_f(p) = \int f \, d\omega_\Omega^p. \tag{11.1}$$

Then $u(p) = \int u \, d\omega_\Omega^p$ if u is continuous in $\overline{\Omega}$ and harmonic in Ω. The measure ω_Ω^p is called the *harmonic measure* of Ω at p. It depends on p, but by the Harnack inequality for any two points $p, q \in \Omega$, the measures ω_Ω^p and ω_Ω^q are comparable.

According to Kakutani's probabilistic characterization, $\omega_\Omega^p(A)$ is the probability that the Brownian traveller starting from p hits A before hitting any other part of the boundary. This helps to visualize the fact that in the case of complicated boundaries ω_Ω^p lives on parts of the boundary which are more easily accessible from Ω. A more precise statement in the plane is a result of Wolff [442] saying that harmonic measure lives on a set of sigma-finite $\mathcal{H}^1$ measure. The corresponding statement in $\mathbb{R}^3$ is false by his example in [443]. However,

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Bourgain [79] proved that in $\mathbb{R}^n$ it lives on a set of dimension at most $n - \varepsilon(n)$, where $\varepsilon(n)$ is a small positive constant, whose best value is unknown.

From the point of view of this survey, the main question is: what are the relations between harmonic measure and the geometry of the boundary? More precisely, for $E \subset \partial\Omega$ does rectifiability of E imply something on the harmonic measure on E, and conversely, do some properties of the harmonic measure lead to rectifiability? The first general result was proved already in 1916 by the Riesz brothers: if $\Omega \subset \mathbb{R}^2$ is simply connected and $\mathcal{H}^1(\partial\Omega) < \infty$, then ω_Ω^p and $\mathcal{H}^1 \llcorner \partial\Omega$ are mutually absolutely continuous. Here the whole boundary is rectifiable, but Bishop and Jones extended this in [66] by showing that if $\Omega \subset \mathbb{R}^2$ is simply connected and $E \subset \partial\Omega \cap \Gamma$, where Γ is a rectifiable curve, then $\omega_\Omega^p(E) = 0$ if and only if $\mathcal{H}^1(E) = 0$. They used Jones's travelling salesman Theorem 3.16 for this.

In the plane, complex analytic tools are very effective, but in higher dimensions quite different methods are required and the situation is in many ways different. First, the analogue of the Riesz brothers theorem fails. Ziemer constructed in [447] a domain in $\mathbb{R}^3$ whose boundary is 2-rectifiable – it even has an ordinary tangent plane at each point – and it has finite $\mathcal{H}^2$ measure, but $\mathcal{H}^2$ is not absolutely continuous with respect to harmonic measure. To the other direction, Wu showed in [444] that there exists a domain $\Omega \subset \mathbb{R}^3$ and $E \subset \partial\Omega \cap V$, where V is a plane, with positive harmonic measure and $\mathcal{H}^2(E) = 0$, even $\dim E = 1$.

Starting from Dahlberg [130] in 1977 and Lipschitz boundaries and followed by David, Jerison, Kenig and Semmes and more general boundaries in the 1980s, a great number of people have produced results in the spirit that various geometric properties of the boundary imply absolute continuity of harmonic measure, often with quantitative estimates such as being an A_∞ weight. Uniform rectifiability of the boundary alone is not sufficient to get such results by an example of Bishop and Jones in [66]. But starting with some natural geometric conditions of the boundary, uniform rectifiability often comes into play leading to, or even characterizing, quantitative properties of the harmonic measure. Commonly used conditions are corkscrew and Harnack chain conditions. Roughly speaking, the former says that every ball centred in the boundary contains a ball of comparable size inside the domain and the latter that any two points in the domain can be joined with a chain of balls whose size is comparable to the distance to the boundary. The corkscrew condition is automatically satisfied if $\Omega = \mathbb{R}^n \setminus E$, where E is AD-m-regular, for any $0 < m < n$.

The proof of the following theorem was completed by Azzam, Hofmann, Martell, Mourgoglou and Tolsa in [36]. The authors explain by several examples that the result is in many ways optimal. It is a culmination of a long process involving many other people and articles, see [36, 235] for the history, references and other related results. In particular, that A_∞ implies uniform rectifiability was proved earlier by Hofmann and Martell, see [235], where this is obtained even for the non-linear p-harmonic equation.

Theorem 11.1 *Let Ω satisfy the corkscrew condition and have AD-$(n-1)$-regular boundary. Then $\partial\Omega$ is uniformly rectifiable and satisfies the weak local John condition if and only if ω_Ω^p is locally in weak A_∞.*

The weak local John condition is a quantitative connectivity condition saying, roughly, that each point of Ω can be connected to a large subset of the boundary by rectifiable curves in Ω staying away from the boundary. That ω_Ω^p is locally in weak A_∞ means that there is $s > 0$ such that for every $x \in \partial\Omega$ and $0 < r < d(\partial\Omega)/4$,

$$\omega_\Omega^p(A) \lesssim \left(\frac{\mathcal{H}^{n-1}(A)}{\mathcal{H}^{n-1}(\partial\Omega \cap B(x,r))}\right)^s \omega_\Omega^p(\partial\Omega \cap B(x,2r))$$

for all $p \in \Omega \setminus B(x,4r)$ and for all Borel sets $A \subset \partial\Omega \cap B(x,r)$.

The local weak A_∞ condition is known to be equivalent to the quantitative solvability of the L^p Dirichlet problem.

Let us now look at a qualitative rectifiability criterion. After many partial results by several people, Azzam, Hofmann, Martell, Mayboroda, Mourgoglou, Tolsa and Volberg proved in [35]:

Theorem 11.2 *Let $p \in \Omega$ and $E \subset \partial\Omega$ with $\mathcal{H}^{n-1}(E) < \infty$.*

(1) *If $\omega_\Omega^p \llcorner E \ll \mathcal{H}^{n-1} \llcorner E$, then $\omega_\Omega^p \llcorner E$ is $(n-1)$-rectifiable.*
(2) *If $\mathcal{H}^{n-1} \llcorner E \ll \omega_\Omega^p \llcorner E$, then E is $(n-1)$-rectifiable.*

By the Radon–Nikodym theorem, it is easy to see that these statements are equivalent. For example, if (1) holds and $\mathcal{H}^{n-1} \llcorner E \ll \omega_\Omega^p$, then $\mathcal{H}^{n-1} \llcorner E = g\omega_\Omega^p$ for some non-negative function g on E for which $g(x) > 0$ for $\mathcal{H}^{n-1}$ almost all $x \in E$. Then $\omega_\Omega^p \ll \mathcal{H}^{n-1} \llcorner \{x \in E : g(x) > 0\}$, and it follows from (1) that E is $(n-1)$-rectifiable.

The key to the proof of Theorem 11.2 is the relation between the Green function and the Riesz transform and Theorem 10.4 of Nazarov, Tolsa and Volberg. In classical potential theory, the *Green function* $G_\Omega : \Omega \times \Omega \setminus \{(p,x) : p = x\} \to \mathbb{R}$ is a basic tool to study harmonic measure. It is harmonic in both variables separately. For a fixed $p \in \Omega$, $G(p,\cdot)$ has zero boundary values. At $x = p$, it has the same singularity as the fundamental solution Γ of the Laplacian, which is a constant multiple of $|x|^{2-n}$, if $n \geq 3$, and of $\log|x|$, if $n = 2$. More precisely, $\Gamma(p-x) - G_\Omega(p,x)$ is a harmonic function of x in Ω.

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For nice domains, the harmonic measure is absolutely continuous with respect to the surface measure, and its density is given by the normal derivative of the Green function. By (11.1), the Green function can be written as

$$G_\Omega(p,x) = \Gamma(p-x) - \int \Gamma(p-x)\, d\omega_\Omega^p x,\ \ p, x \in \Omega, x \neq p.$$

Since $\nabla\Gamma = cR_{n-1}$, we have

$$\nabla_x G_\Omega(p,x) = cR_{n-1}(p-x) - c\int R_{n-1}(p-x)\, d\omega_\Omega^p x,\ \ p, x \in \Omega, x \neq p. \quad (11.2)$$

As in (1) of Theorem 11.2 suppose that $\omega_\Omega^p \llcorner E \ll \mathcal{H}^{n-1} \llcorner E$, so that we have $\omega_\Omega^p \llcorner E = g\mathcal{H}^{n-1} \llcorner E$ for some non-negative g. Given $M > 0$, it is enough to prove that $E_M := \{x \in E \colon 1/M < g(x) < M\}$ is $(n-1)$-rectifiable. One could then hope that, similarly to the case of nice boundaries, the left-hand side of (11.2) would have enough boundedness to give boundedness for the Riesz transform when $x \in E_M$. In this very general case this is not clear at all, but the authors of [35] managed to show something like this. A bit more precisely, they again used generalized dyadic cubes, now from [143] since there is no doubling, and they showed using (11.2) that there are enough cubes Q to cover a large part of E_M such that the truncated Riesz transform $R^{n-1}_{\omega_\Omega^p, r(Q)}(x)$ with a suitable $r(Q)$ is bounded for $x \in Q$, with a quantitative bound. This allows us to apply a $T(b)$-theorem of Nazarov, Treil and Volberg to get the $L^2(\omega_\Omega^p)$-boundedness of the Riesz transform on a subset of E_M with positive measure. From this an application of Theorem 10.4 yields rectifiability.

There are also results on two-phase problems involving rectifiability. The following was proved by Azzam, Mourgoglou, Tolsa and Volberg in [39], and the paper [38] contains a quantitative version:

Theorem 11.3 *Let Ω_1 and Ω_2 be disjoint open connected subsets of $\mathbb{R}^n$ and $E \subset \partial\Omega_1 \cap \partial\Omega_2$ a Borel set such that $\omega_{\Omega_1}^{p_1}$ and $\omega_{\Omega_2}^{p_2}, p_i \in \Omega_i$, are mutually absolutely continuous on E. Then E contains an $(n-1)$-rectifiable subset F such that $\omega_{\Omega_i}^{p_i}(E \setminus F) = 0$ and $\omega_{\Omega_1}^{p_1}$ and $\omega_{\Omega_2}^{p_2}$ are mutually absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner F$.*

The proof uses an interesting blow-up argument involving tangent measures and the Alt–Caffarelli–Friedman monotonicity formula for pairs of subharmonic functions applied to the Green functions. This method was introduced by Kenig, Preiss and Toro in [275], where a partial result and deep information about harmonic measures was obtained. The proof also relies on Theorem 11.2, and so on the Nazarov–Tolsa–Volberg Riesz transform Theorems 10.2 and 10.4, and on the result of Girela–Sarrión and Tolsa in [226].

11.2 Elliptic Measures in Codimension 1

The previous section dealt with the Laplace equation $\Delta u = 0$. It is natural to expect that the results would have analogues for more general elliptic equations, and this indeed is the case. Assuming that Ω is a uniform domain, Hofmann, Martell, Mayboroda, Toro and Zhao [239] characterized the A_∞ property with uniform rectifiability for elliptic measures corresponding to equations with optimal conditions for the coefficients. Prat, Puliatti and Tolsa proved in [381] an analogue of Theorem 11.2 for such elliptic measures with Hölder continuous coefficients using their singular integral results mentioned in Section 10.2. Then the same arguments as for the Laplace equation work.

Cao, Hidalgo-Palencia and Martell [86] investigated corona decompositions associated with quite general elliptic measures. They showed, among other things, that these are equivalent to square function estimates as in Theorem 9.12, also to a weaker form of them, which are equivalent to uniform rectifiability. The boundary of the domain is assumed to be AD-regular and to satisfy the cork-screw condition.

11.3 Elliptic Measures in Codimension Bigger Than One

If $E \subset \mathbb{R}^n$ is a closed set with $\mathcal{H}^{n-2}(E) < \infty$, then it is polar. In particular, if $E \subset \partial\Omega$, then $\omega_\Omega^p(E) = 0$ and the properties of the harmonic measure are in no way related to the geometric properties of E. The same is true for the elliptic equations as above. But considering suitable degenerate elliptic equations, a rich theory can be developed. This was realized and done by David, Feneuil and Mayboroda and their co-authors in many papers, only some of which are listed in the references.

Let $E \subset \mathbb{R}^n$ be AD-m-regular for some integer $0 < m < n-1$. The standard ellipticity conditions (10.5) and (10.6) for a measurable $n \times n$ matrix-valued function A are now replaced by

$$|\xi|^2 \lesssim d(E,x)^{n-m-1}A(x)\xi \cdot \xi \text{ for all } \xi, x \in \mathbb{R}^n, \tag{11.3}$$

$$d(E,x)^{n-m-1}A(x)\xi \cdot \zeta \lesssim |\xi||\zeta| \text{ for all } \xi, \zeta, x \in \mathbb{R}^n. \tag{11.4}$$

The authors developed in [142] a comprehensive theory for the degenerate elliptic operators $L = -\operatorname{div}(A\nabla)$ analogous to the classical theory, including – among others – solutions of the Dirichlet problem with continuous boundary data, the corresponding elliptic measure ω_L^p and its basic properties, and the

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existence and properties of the Green function and its relations to the elliptic measure. As remarked before, in the classical theory one often needs some conditions for the boundary to prevent it from being too massive. Here they are not needed. The AD-m-regularity with $m < n-1$ automatically gives the corkscrew and Harnack chain conditions.

To get absolute continuity of the elliptic measure on Lipschitz graphs or more general sets, one needs stronger assumptions on A, also in the classical theory. I now restrict to a particular operator, which is the authors' replacement of the Laplacian. Again there is a weight like $d(E,x)^{n-m-1}$, but this seems to be too rough and it is replaced by a regularized distance $D_{\alpha,\mu}$, where $\alpha > 0$ is a parameter and $\mu = \mathcal{H}^m \llcorner E$ or some other AD-m-regular measure with support E:

$$D_{\alpha,\mu}(x) = \left(\int |x-y|^{-m-\alpha}\, d\mu y\right)^{-1/\alpha}.$$

AD-regularity easily implies that $D_{\alpha,\mu}(x) \sim d(E,x)$.

The elliptic operator attached to μ is given by

$$L_{\alpha,\mu} = -\operatorname{div}(D_{\alpha,\mu}^{m+1-n}\nabla). \tag{11.5}$$

Denote now simply by ω^p the elliptic measure related to $L_{\alpha,\mu}$ and some base point $p \in \mathbb{R}^n \setminus E$. In this setting David and Mayboroda proved in [144]:

Theorem 11.4 *If E is uniformly m-rectifiable, $m \leq n-2$, then $\omega^p \ll \mu$. Moreover $\omega^p \in A_\infty(\mu)$, which means that for every $\varepsilon > 0$ there is $\delta > 0$ such that if $x \in E$, $r > 0$ and $F \subset E \cap B(x,r)$ is a Borel set, then*

$$\frac{\omega^p(F)}{\omega^p(E\cap B(x,r))} < \delta \implies \frac{\mu(F)}{\mu(E\cap B(x,r))} < \varepsilon, \tag{11.6}$$

where $p \in \mathbb{R}^n \setminus E$ is such that $d(E,p) \sim |p-x| \sim r$.

In [206], Feneuil gave a different simpler proof.

The converse of Theorem 11.4 fails completely for $\alpha = n-m-2$, then $m < n-2$. In this case, $\omega \in A_\infty(\mu)$ for any AD-m-regular measure μ, and m need not even be an integer. We shall say a bit more about this soon. The above authors believe that this value of α is a unique exception and the converse should be true for the other values.

In addition to extending from codimension one to general dimensions, the work of David, Feneuil, Mayboroda and their co-authors contains several interesting new results and phenomena also for codimension one. I present here two new characterizations of uniform rectifiability. Now m can be any integer with $0 < m < n$.

In [140], David, Engelstein and Mayboroda characterized uniform rectifiability in terms of the distance $D_{\alpha,\mu}$ in the spirit of Theorems 9.11 and 9.12.

Theorem 11.5 *Let $E \subset \mathbb{R}^n$ be AD-m-regular. Then E is uniformly rectifiable if and only if*

$$d(x,E)^{m+2-n}|\nabla(|\nabla D_{\alpha,\mu}|^2)(x)|^2 dx \tag{11.7}$$

is a Carleson measure on $\mathbb{R}^n \setminus E$.

The 'only if' direction uses Tolsa's α's, recall Theorem 5.10; (11.7) vanishes if μ is a flat measure and the uniform rectifiability leads to approximation of μ with flat measures. For the converse direction, the starting idea is that if (11.7) vanishes then $\nabla D_{\alpha,\mu}$ is constant, from which it follows, but with a lot of work, that μ is m-flat. They also characterized the rectifiability of E in terms of the non-tangential limits of $\nabla|D_{\alpha,\mu}|$.

Another characterization is in terms of the Green functions. Let $G_{\alpha,\mu}$ be the Green function corresponding to $L_{\alpha,\mu}$ with the pole at ∞. If E is an m-plane, then $G_{\alpha,\mu}(x) = cd(E,x)$, and the elliptic measure is a constant multiple of $\mathcal{H}^m \llcorner E$. Uniform rectifiability means that E is well approximated by m-planes, and this turns out to be equivalent to $G_{\alpha,\mu}$ being well approximated by distances to m-planes:

Theorem 11.6 *Let $E \subset \mathbb{R}^n$ be an unbounded AD-m-regular set. When $m = n-1$, assume also the corkscrew and Harnack chain conditions. Then E is uniformly rectifiable if and only if for every $\varepsilon > 0$ and $M > 1$ the set $E \times (0,\infty) \setminus \mathcal{G}(\varepsilon, M)$ is a Carleson set, where $\mathcal{G}(\varepsilon, M)$ is the set of $(x,r) \in E \times (0,\infty)$ such that there are an affine m-plane V and $c > 0$ for which*

$$|d(y,V) - cG_{\alpha,\mu}(y)| \leq \varepsilon r \textit{ for } y \in (\mathbb{R}^n \setminus E) \cap B(x, Mr).$$

This is a special case of the results proved by David and Mayboroda in [145]. Again the proof of the 'only if' direction uses Tolsa's α's. For the converse direction we have $G_{\alpha,\mu}(y) = 0$ when $y \in E$, which immediately gives some approximation of E by planes. So it is a good starting point, but it only gives a weak geometric lemma, and many more arguments are needed to get the bilateral approximation, recall Theorem 5.7.

The result in [145] is more general in that the authors considered much more general degenerate elliptic operators. They also have similar results with the regularized distance for $m = n-1$, where $d(y,V) - cG_{\alpha,\mu}(y)$ is replaced by $D_{\beta,\mu}(y) - cG_{a,\mu}(y), \beta > 0$. Then they can start with any $n-2 < m < n$ and the Carleson estimates force m to be an integer.

For $m < n-2$ and $\alpha = n-m-2$, the smallness of $|D_{\alpha,\mu} - cG_{a,\mu}|$ or the absolute

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continuity of the elliptic measure do not imply any kind of rectifiability, as was already stated after Theorem 11.4. The reason is that then by direct computation $L_{\alpha,\mu} D_{\alpha,\mu} = \frac{1}{\alpha} \Delta R_{\alpha,\mu}$, where, when $\alpha = n - m - 2$, $R_{\alpha,\mu}(x) = \int |x-y|^{2-n}\, d\mu y$ is harmonic, so $L_{\alpha,\mu} D_{\alpha,\mu} = 0$, from which it follows by the uniqueness of the Green function that $G_{\alpha,\mu}$ is a constant multiple of $D_{\alpha,\mu}$. This leads to $\omega_{L_{\alpha,\mu}} \sim \mu$. Further, if E is m-rectifiable, using basic properties of rectifiable sets, it follows that $\omega_{L_{\alpha,\mu}} = c\mathcal{H}^m \lfloor E$. The details can be found in [140].