CONSTANT HOLOMORPHIC CURVATURE

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We shall present in this paper a certain theorem concerning complex manifolds provided with an Hermitian metric satisfying the Kaehler restriction. The variables $z_1, z_2, \ldots, z_n$ denote local complex coordinates in the manifold and $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n$ their conjugates. The subscripts $a, b, c, \ldots$ run from 1 to n and by $\bar{a}$ we mean $a + n \pmod{2n}$, e.g., $\bar{\bar{a}} = a$.

An Hermitian line element is given by

$$ds^2 = g_{a\bar{b}} dz_a d\bar{z}_b \tag{1}$$

(summation over a, b from 1 to n); the coefficients $g_{a\bar{b}}$ shall satisfy the conditions

$$g_{ab} = g_{\bar{a}\bar{b}} = 0, \quad g_{a\bar{b}} = g_{\bar{b}a} = \bar{g}_{\bar{a}b}. \tag{2}$$

In matric notation (2) has the following form:

$$G = \begin{pmatrix} g_{1,n+1}, & \ldots, & g_{1,2n} \\ g_{n,n+1}, & \ldots, & g_{n,2n} \end{pmatrix}.$$

If we let $\bar{G}'$ be the transposed conjugate of G, then by (2), $\bar{G}' = G$ and $\mathfrak{G}$, the matrix of the fundamental tensor, is as follows:

$$\mathfrak{G} = \begin{pmatrix} O & G \\ \bar{G} & O \end{pmatrix},$$

so we see that

$$\mathfrak{G}' = \begin{pmatrix} O & \bar{G}' \\ G' & O \end{pmatrix} = \begin{pmatrix} O & G \\ \bar{G} & O \end{pmatrix} = \mathfrak{G}.$$

The line element (1) is said to satisfy the Kaehler condition if

$$\frac{\partial g_{a\bar{b}}}{\partial z_c} = \frac{\partial g_{c\bar{b}}}{\partial z_a}. \tag{3}$$

The relations (2) and (3) thus imply

$$\frac{\partial g_{\bar{a}b}}{\partial \bar{z}_c} = \frac{\partial g_{\bar{c}b}}{\partial \bar{z}_a}. \tag{4}$$

We may form an affine connection and a Riemann-Christoffel tensor from the fundamental tensor by the usual formulae. The only components different from zero of the Riemann-Christoffel tensor are

Received August 22, 1951.

https://doi.org/10.4153/CJM-1953-007-1 Published online by Cambridge University Press

$$R_{a\bar{b}c\bar{d}},\ R_{\bar{a}\bar{b}\bar{c}\bar{d}},\ R_{\bar{a}bc\bar{d}},\ R_{a\bar{b}\bar{c}d}. \tag{5}$$

A two-dimensional element at a point is called an holomorphic section if it is tangent to a complex curve through that point. If the *sectional curvature* at a point is the same for all holomorphic sections at that point then the Riemann-Christoffel tensor can be written in the form

$$R_{a\bar{b}c\bar{d}} = -\tfrac{1}{2}\mathrm{b}(g_{a\bar{b}}\,g_{c\bar{d}} \pm g_{a\bar{d}}\,g_{c\bar{b}}), \tag{6}$$

where b is the curvature on holomorphic sections; moreover b is a constant on the manifold if at each point the sectional curvature is the same for all holomorphic sections at that point [**1**, p. 184]. For a more thorough-going exposition of this subject matter, one is referred to the papers of Bochner [**1**; **2**].

Bochner has shown that relation (6) implies the existence of an analytic coordinate system in which the line element has the form

$$ds^2 = \frac{2\sum_a |dz_a|^2 + \mathrm{b}\{\sum_a |z_a|^2 \sum_b |dz_b|^2 - |\sum_a \bar{z}_a dz_a|^2\}}{(1 + \tfrac{1}{2}\mathrm{b}\sum_a |z_a|^2)^2} \tag{7}$$

The line element (7) is, for $\mathrm{b} > 0$, the Fubini-Study line element for complex projective space. If $\mathrm{b} = -2$ then (7) is the invarient line element of the unit cell $|z_1|^2 + \ldots + |z_n|^2 < 1$ under the group of all linear fractional transformations into itself. Let us call the complex projective space P^* and the unit cell E^* when they have these line elements.

We can now state our theorem.

THEOREM I. *If a complex manifold S has an Hermitian metric satisfying the Kaehler restriction and if it has constant holomorphic curvature and if the space is complete in this metric then its universal covering space is analytically isometrically equivalent with E^* for* $\mathrm{b} = -2$ *and with P^* for* $\mathrm{b} > 0$.

Let us assume from the start that S is simply connected. From Bochner's result we know that there is a local coordinate neighbourhood of each point of S such that the line element can be put into the form (7). Since S is complete there is a $\delta > 0$ such that each coordinate neighbourhood can be chosen with radius $> 2\delta$. Let $K(x, \delta)$ denote the cell of radius δ and centre x in S and let $L(y, \delta)$, denote the cell of radius δ and centre y in E^* or P^* according to whether $\mathrm{b} = -2$ or $\mathrm{b} > 0$. Then $K(x_0, \delta)$ can be mapped analytically isometrically onto $L(y_0, \delta)$ where x_0 is any point of S and y_0 is any point of E^*(or P^* as the case may be). Let x_1 be a point of $K(x_0, \delta)$ and let ϕ be the analytic isometry of $K(x_0, \delta)$ onto $L(y_0, \delta)$ considered, and let $y_1 = \phi(x_1)$. We now wish to show that ϕ can be extended to an analytic isometry of $K(x_0, \delta) \cup K(x_1, \delta)$ onto $L(y_0, \delta) \cup L(y_1, \delta)$.

So far we have discussed only the mapping of a $K(x, \delta)$ onto an $L(y, \delta)$, but we could also discuss the analytic isometry of a $K(x, 2\delta)$ onto an $L(y, 2\delta)$ since one of these is contained in a suitable coordinate neighbourhood (in which the line element has the form (7)) about each point. We now observe that there exists a mapping of $K(x_0, 2\delta)$ onto $L(y_0, 2\delta)$ which agrees with ϕ on $K(x_0, \delta)$, for an isometry of $K(x_0, \delta)$ is characterized by the directions at y_0 into which the directions at x_0 are sent. Since ϕ is analytic it sends holomorphic directions into holomorphic directions. Now the unimodular unitary group acts transitively on the set of all holomorphic directions and we have the unimodular unitary group acting on both $K(x_0, 2\delta)$ and $L(y_0, 2\delta)$ (leaving invariant the metrics in each). Thus an analytic isometry ψ of $K(x_0, 2\delta)$ onto $L(y_0, 2\delta)$ can be made into one which agrees with ϕ on $K(x_0, \delta)$ by first performing ψ and then performing an unimodular transformation of $L(y_0, 2\delta)$ into itself. Therefore ϕ can be extended from $K(x_0, \delta)$ to $K(x_0, 2\delta)$ so in particular to $K(x_0, \delta) \cup K(x_1, \delta)$. Now let x'_0 be any point of S then we can continue (analytically and isometrically) the mapping ϕ to x'_0. To do this choose a path C with end points x_0 and x'_0; take points $x_1, \ldots, x_s$ on C so that $x_1 \in K(x_0, \delta)$, $x_2 \in K(x_1, \delta), \ldots, x'_0 \in K\ (x_s, \delta)$, then by the method described above ϕ can be extended to each cell in turn and so to x'_0. In this way we extend ϕ to all of S. But ϕ is single valued since S is simply connected. Now S satisfies all the conditions of being a covering space over part of E^* (or P^*), however since E^* (as well as P^*) is also simply connected ϕ is actually an homeomorphism of S into E^* (or P^*). Also ϕ is an analytic isometry by the way it was defined, and it must be onto since S is complete; thus we have the theorem.

Since we know that the complex projective space cannot cover, we have the following corollary:

COROLLARY. *If* b > 0 *then S must already be simply connected, hence* $S = P^*$.

On the other hand, E^* may well occur as the covering space of a non-simply connected S, even of a compact S. We now have

THEOREM II. *If S is a compact complex manifold with Hermite-Kaehler metric which has constant negative holomorphic curvature then the group of analytic homeomorphisms of S is finite.*

The proof of Theorem II follows quickly from Theorem I and [**3**, Theorem VII]. For S has E^* as universal covering space and E^* is a bounded domain in E^{2n} hence *a fortiori* a Picard domain. (In Theorem I we assumed that b $= -2$, but this was only so that E^* would be the *unit cell;* this is clearly inessential and $b < 0$ is all that matters.)

We now state: if S satisfies the conditions of Theorem II it uniformizes an algebraic variety. For let $\tilde{S}$ be a fundamental domain for S in E^*, then $\tilde{S}$ generates a discontinuous group Γ in E^* such that $E^* \pmod{\Gamma} = S$. Given such a group, where $\tilde{S}$ is compact modulo Γ, Siegel has proved there exist n analytically independent automorphic functions in E^* relative to Γ [**4**, pp. 132–136]. To obtain an algebraic variety which S uniformizes we note that any $n + 1$ functions on E^*, automorphic relative to Γ, satisfy an algebraic relation [**4**, pp. 137–145]. Making this polynomial homogeneous we obtain a projective model of our algebraic variety.

Finally one notes that for S compact and $b > 0$ or $b < 0$ we have the existence of n analytically independent functions meromorphic on S. But for $b = 0$ we have no information as to the existence of functions. Indeed in this case S is a multi-torus which may have any number from $0, 1, \ldots, n$ analytically independent non-constant meromorphic functions. This phenomenon arises because of the need for *period relations* in the theory of Abelian functions; which in turn arises from the fact that in some multi-tori a local sub-variety is not part of a sub-variety in the large, not because the local sub-variety cannot be extended (it can be continued indefinitely), but because it winds infinitely often around the multi-torus and never meets itself (much as the familiar everywhere-dense integral curve winds on the two-dimensional torus). In fact a multi-torus (in n complex variables, $n > 1$) may be constructed which has no proper sub-varieties other than points. (For an example see [**4**, pp. 104—106].)

In concluding I wish to thank Professor S. Bochner for suggesting Theorem I. It is also explicitly stated in [**2**, p. 21].

References

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