COMMON FIXED POINTS OF COMMUTING MONOTONE MAPPINGS

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We are concerned here with the existence of fixed or common fixed points of commuting monotone self-mappings of a partially ordered set into itself. Let X be a partially ordered set. A self-mapping f of X into itself is called an *isotone mapping* if $x \geqslant y$ implies $f(x) \geqslant f(y)$. Similarly, a self-mapping f of X into itself is called an *antitone mapping* if $x \geqslant y$ implies $f(x) \leqslant f(y)$. An element $x_0 \in X$ is called *well-ordered complete* if every well-ordered subset with x_0 as its first element has a supremum. An element $x_0 \in X$ is called *chain-complete* if every non-empty chain $C \subseteq X$ such that $x \geqslant x_0$ for all $x \in C$, has a supremum. X is called a *well-ordered-complete semi-lattice* if every non-empty well-ordered subset has a supremum. X is called a *complete semi-lattice* if every non-empty subset of X has a supremum. X is called a *well-ordered-complete lattice* if X has a greatest element e and a least element 0, and every non-empty well-ordered subset of X has a supremum and an infimum. We denote, for simplicity,

$$P_f = \{x:x \in X, x \leqslant f(x)\}, \qquad Q_f = \{x:x \in X, f(x) \leqslant x\},$$

and $\Phi_f = P_f \cap Q_f$, the fixed point set of f. For notations and terminologies not explained here, we refer to Birkhoff **(2)**. In the following, we need the following strong form of Zorn's lemma due to Bourbaki **(3)**:

LEMMA (Bourbaki). *Let S be a partially ordered set. If every well-ordered set in S has an upper bound, then S has a maximal element.*

THEOREM 1. *Let X be a partially ordered set and F be a non-empty family of commuting isotone mappings of X into itself. If there exists a well-ordered complete element*

$$x_0 \in \bigcap_{f \in F} P_f, \qquad \textit{then} \bigcap_{f \in F} \Phi_f \neq \emptyset.$$

Moreover, there exists a maximal common fixed point in the set

$$N_{x_0} = \{x:x \in X, x \geqslant x_0\}.$$

Proof. Consider the set $S = \bigcap_{f \in F} P_f \cap N$. Since $x_0 \in S$, S is non-empty. Let $W \subseteq S$ be any well-ordered subset. By hypothesis, $c = \sup W$ exists. Note that $f(c) \geqslant f(x) \geqslant x$ for all $x \in W$; hence $f(c) \geqslant c$, showing that $c \in S$. From the above lemma, there exists a maximal element $a \in S$. For any two mappings $f, g \in F$, we have $f(a) \geqslant a$ and $g(a) \geqslant a$. On the other hand,

Received March 4, 1966. Part of this work was presented to the Canadian Mathematical Congress, June 10, 1965.

$g(f(a)) = f(g(a)) \geqslant f(a)$ implies $f(a) \in S$. Since a is maximal, $f(a) = a$ for all $f \in F$. Obviously $a \in \bigcap_{f \in F} \Phi_f$ is maximal in N_{x_0}.

COROLLARY 1. *Let X be a well-ordered-complete semi-lattice and F be a non-empty commutative family of isotone mappings of X into itself. If $\bigcap_{f \in F} P_f \neq \emptyset$, then there exists an element $a \in \bigcap_{f \in F} \Phi_f$ which is maximal in X.*

Remark 1. Theorem 1 reduces to a recent result of DeMarr (**4**, Theorem 1) if the well-ordered completeness of x_0 in the hypothesis is replaced by the stronger assumption that x_0 be chain-complete.

Remark 2. If F consists of one single mapping, Corollary 1 extends a result of Abian and Brown (**1**, Theorems 2 and 3). It should be noted that their results are proved without using any form of the Axiom of Choice. However, their result does not imply the existence of maximal fixed points.

Remark 3. In view of the above two remarks, results of Pelczar (**7**; **8**) and Wolk (**11**), and hence the classical theorem of Tarski (**9**), become special cases of Corollary 1; for an up-to-date account of related results, we refer to van der Walt (**10**).

THEOREM 2. *Let X be a complete semi-lattice, and F be a non-empty family of commuting isotone mappings of X into itself. If $\bigcap_{f \in F} P_f \neq \emptyset$, then there exists a greatest common fixed point.*

Proof. By Theorem 1, $\bigcap_{f \in F} \Phi_f \neq \emptyset$ and hence $a = \sup \bigcap_{f \in F} \Phi_f$ exists. For any $f \in F$, clearly $f(a) \geqslant a$. Now by Theorem 1 again there is an element $b \in \bigcap_{f \in F} \Phi_f$ such that $b \geqslant f(a) \geqslant a$; consequently $f(a) = a$ for all $f \in F$, proving the assertion.

Remark 4. Theorem 2 also extends the above-mentioned result of DeMarr (**4**, Theorem 1), since for any subset $S \subseteq X$, $\sup S \in S$ is certainly a maximal element of S.

THEOREM 3. *Let X be a partially ordered set and let f, g be two commutative isotone mappings of X into itself. Let $h = f \circ g$. If $\Phi_h \neq \emptyset$ and $\sup \Phi_h \in \Phi_h$, then $\Phi_f \cap \Phi_g \neq \emptyset$.*

Proof. Let $a = \sup \Phi_h$. Since f and g commute with h, $f(a)$ and $g(a)$ both belong to Φ_h. Thus, $f(a) \leqslant a$ and $g(a) \leqslant a$. Since f, g are isotone, we have

$$a = h(a) = g(f(a)) \leqslant g(a) \quad \text{and} \quad a = h(a) = f(g(a)) \leqslant f(a);$$

hence $g(a) = f(a) = a$.

As an immediate consequence of Theorems 2 and 3, we obtain the following corollary, which also generalizes results mentioned in Remark 2.

COROLLARY 2. *Let X be a complete semi-lattice and f be an isotone mapping of X into itself. If $P_{f^n} \neq \emptyset$ for some positive integer n, then $\Phi_f \neq \emptyset$.*

Remark 5. It is clear that the above results may be formulated dually. In fact, part of the results in (**4**) are stated dually as compared to assertions presented here. Since the procedure of such a formulation is obvious (**8**), the details will be omitted.

Remark 6. We also note that the notion of chain-completeness is called chain-compactness in (**4**).

THEOREM 4. *Let X be a partially ordered set and F be a non-empty family of commuting antitone mappings of X into itself. If* $\sup P_f \in P_f$ *and* $\inf f(P_f) \leqslant \sup P_f$ *for some $f \in F$, then there exists a unique common fixed point of F.*

Proof. Let $a = \sup P_f$, and $b = \inf f(P_f)$. Since f is antitone, $f(a) \leqslant f(x)$ for all $x \in P_f$, and hence $f(a) \leqslant b$. By hypothesis $a \leqslant f(a)$, and $b \leqslant a$. Thus $b = a$, and $f(a) = a$. Now suppose that $c \in \Phi_f$. Since $c \leqslant a$, we have $a = f(a) \leqslant f(a) = c$. Finally, for any $g \in F$ we have $g(a) \in \Phi_f$. Since f has exactly one fixed point, namely a, $g(a) = a$ for all $g \in F$.

Remark 7. Theorem 4 reduces to a result of Pelczar (**8**, Theorem 4) if F consists of one single mapping.

THEOREM 5. *Let X be a relatively complemented well-ordered-complete lattice and let f be a mapping of X into itself satisfying:*

(i) $f(\bigcup_{x \in A} x) = \bigcap_{x \in A} f(x)$ *for every well-ordered subset A of X,*

(ii) $f(x) \cap x \neq 0$ *if* $x \neq 0$.

Then f has a unique fixed point.

Proof. By (i), f is in particular an antitone mapping. Denote the ordinals by Greek letters, ξ, η, ζ, σ, τ, μ, ν, and ϕ. Let $x_0 = f(e)$. For each ordinal ξ, we define inductively $x_\xi = \bigcup_{\eta < \xi} f^2(x_\eta)$. (This definition is justified since X is well-ordered complete.) It is easy to see that $x_\xi \leqslant x_\zeta$ if $\xi \leqslant \zeta$. Hence there must exist an ordinal ϕ, such that $x_\phi \geqslant x_\xi$ for each ordinal ξ, from which it readily follows that $f^2(x_\phi) = x_\phi$. We also claim that $f(x_\sigma) \geqslant x_\tau$ for any pair of ordinals σ, τ. To prove this, we proceed by induction and assume that $f(x_\mu) \leqslant x_\nu$ for every μ and all $\nu \leqslant \tau$. Observe that for each pair of ordinals σ, τ, we have

$$f(x_\sigma) = f\Big(\bigcup_{\mu<\sigma} f^2(x_\mu)\Big) = \bigcap_{\mu<\sigma} f^3(x_\mu) \geqslant \bigcup_{\nu<\tau} f^2(x_\nu) = x_\tau,$$

which proves the assertion. Denote x_ϕ by y. We have $f^2(y) = y$ and $y \leqslant f(y)$. From (ii), it follows that $f(e) \neq 0$, and hence $y \neq 0$. Since X is relatively complemented, there exists $z \in X$ such that $z \cup y = f(y)$ and $z \cap y = 0$. By (i), we have $y = f^2(y) = f(y \cup z) = f(y) \cap f(z)$ and hence

$$f(z) \cap z = y \cap z = 0.$$

We may now conclude from (ii) that $z = 0$ and $f(y) = y$. To complete the proof, choose $b \in X$ such that $f(b) = b$. From $b \leqslant e$, we can again prove by transfinite induction that $b \geqslant y$, from which we obtain $b = f(b) \leqslant f(y) = y$.

By a similar argument to that given in Theorem 4, we may extend the above result as follows.

COROLLARY 3. *Let X be a relatively complemented well-ordered-complete lattice and let F be a non-empty family of commutative mappings of X into itself. If there exists $f \in F$ satisfying the assumptions* (i) *and* (ii) *of Theorem* 5, *then* $\bigcap_{f \in F} \Phi_f \neq \emptyset$.

Remark 8. Restricting Theorem 5 to the case that X is a Boolean algebra and f is a mapping satisfying the stronger hypothesis that condition (i) holds for every subset $A \subseteq X$, we obtain a solution to a recent problem proposed by Forcade **(5)**.

Remark 9. On account of these results, some applications of Tarski's fixed point theorem may readily be generalized; cf. **(4; 6; 8)**.

Acknowledgment. The author wishes to thank R. Bercov for commenting on the manuscript.

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