Canad. Math. Bull.Vol. **34** (4), 1991 pp. 520–524

INDEPENDENCE FOR SETS OF TOPOLOGICAL SPHERES

LEWIS PAKULA AND SOL SCHWARTZMAN

ABSTRACT. Consider a collection of topological spheres in Euclidean space whose intersections are essentially topological spheres. We find a bound for the number of components of the complement of their union and discuss conditions for the bound to be achieved. This is used to give a necessary condition for independence of these sets. A related conjecture of Grünbaum on compact convex sets is discussed.

1. **Introduction.** The notion of independence for a class of sets was introduced by Marczewski [3] in connection with some problems in measure theory. It is related to a variety of problems in combinatorial geometry; see the discussions and references in [1], [2], and [5]. For a collection of sets $\mathcal{C} = \{S_1, S_2, \dots S_k\}$ let

$$m(\mathcal{C}) := \operatorname{card}\{T \neq \emptyset : T = T_1 \cap \cdots \cap T_k, \text{ where each } T_i \text{ is either } S_i \text{ or } S_i^c\}$$

and we say that $\mathcal{C}$ is *independent* iff $m(\mathcal{C}) = 2^k$. It is well known that if $S_1, \dots, S_k$ are Euclidean balls in $\mathbf{R}^n$ then they cannot be independent if $k > n+1$; in fact, for such $\mathcal{C} = \{S_1, S_2, \dots S_k\}$ an upper bound for $m(\mathcal{C})$ is

$$(1) \qquad M_{n,k} := 2\sum_{\ell=0}^{m} \binom{k-1}{\ell}, \qquad m = \min(k-1, n)$$

REMARK 1. Note that $M_{n,k} = 2^k$ when $k \leq n+1$ and $M_{n,k} < 2^k$ if $k > n+1$.

The most elementary setting for these ideas is the familiar Venn diagram in the plane: circular regions can be used to illustrate set-theoretic statements for 3 sets, but not 4.

The result on failure of independence for Euclidean balls can be obtained using linear algebra [5]. The bound for $m(\mathcal{C})$ was obtained in [6] via induction using stereographic projection and our purpose here is to establish a topological version of this result, using some basic ideas in algebraic topology.

For $n \geq 0$ we call a topological space A a *topological n-sphere* if it is homeomorphic to $\mathbf{S}^n = \{\mathbf{x} \in \mathbf{R}^{n+1} : \|\mathbf{x}\| = 1\}$. If A is a topological n-sphere in $\mathbf{R}^{n+1}$ we will call the bounded component of $\mathbf{R}^{n+1} \setminus A$ its *inside* and the unbounded component(s) its *outside*. If $\mathcal{C}$ is a collection $\{A_1, \dots, A_k\}$ of topological $(n-1)$-spheres in $\mathbf{R}^n$ we call a set of the form $T_1 \cap \cdots \cap T_k$, where T_i is either the inside or the outside of A_i, a *Venn cell* of $\mathcal{C}$. We then define $m(\mathcal{C})$ to be the number of nonempty Venn cells of $\mathcal{C}$ and call $\mathcal{C}$ *independent* if $m(\mathcal{C}) = 2^k$. We will say that an indexed collection of topological spaces $\mathcal{C}$ has *spherical intersections* if the intersection of the sets in any subcollection of $\mathcal{C}$ is either empty, a single point, or a topological sphere. We will consider the empty set a sphere of dimension -1. Then

Received by the editors November 15, 1989.
AMS subject classification: 52A37, 05B99.

https://doi.org/10.4153/CMB-1991-082-1 Published online by Cambridge University Press

THEOREM 1. *Suppose* $C = \{A_1, \ldots, A_k\}$ *is a collection of topological* $(n-1)$-*spheres in* $\mathbf{R}^n$ *having spherical intersections. Then* $m(C) \leq M_{n,k}$. *If* C *is independent then (i)* $k \leq n+1$, *(ii) the intersection of any distinct* r *of the* A_i*'s is a topological* $(n-r)$-*sphere, and (iii) each bounded Venn cell has the homology of a point, and the unbounded Venn cell has the homology of an* $(n-1)$-*sphere; in particular, the Venn cells are connected.*

2. **Main results.** Theorem 1 will follow from Theorems 2 and 3.

Suppose $A_1, \ldots, A_k$ are subsets of a topological n-sphere. We will say that $C = \{A_1, \ldots, A_k\}$ has *proper* spherical intersections if, for $r \leq \min(k, n+1)$ and distinct indices $i_1, \ldots, i_r$, we have that $A_{i_1} \cap \cdots \cap A_{i_r}$ is a topological $(n-r)$-sphere.

Suppose $K = \cup_1^k A_\ell \subset \mathbf{S}^n$. $\tilde{H}^*$ will denote *reduced* Čech-Alexander cohomology with integer coefficients, and we define $b^q(K) = \operatorname{rank} \tilde{H}^q(K)$. (See [7], Chapter 6, and [4] Chapter 6.) We will need the following special case of the Alexander duality theorem:

$$\tilde{H}_q(\mathbf{S}^n \setminus K) \approx \tilde{H}^{n-q-1}(K) \tag{1}$$

where $\tilde{H}_*$ denotes reduced homology. This follows e.g. from [4] Theorem 6.6. By [7], Corollary 4.4.8, the number of nonempty components of $\mathbf{S}^n \setminus K$ is $b^{n-1}(K) + 1$.

THEOREM 2. *Suppose* $C = \{A_1, \ldots, A_k\}$ *is an indexed collection of subsets of* $\mathbf{S}^n$, $n \geq 1$, *having spherical intersections. Then*

(i) $M_{n,k}$ *is an upper bound for the number of components of* $\mathbf{S}^n \setminus \cup_{\ell=1}^k A_\ell$;

(ii) the upper bound in (i) is achieved if and only if C *has proper spherical intersections; and*

(iii) if C *has proper spherical interesections then* $\tilde{H}^q(\cup_1^k A_\ell) = 0$ *for* $q < n-1$.

PROOF. By our remarks above, (i) is equivalent to $b^{n-1}(\cup_1^k A_\ell) + 1 \leq M_{n,k}$. We proceed by induction. Since $M_{1,k} = 2k$, (i), (ii) and (iii) are easily seen to hold for $k \geq 1$ when $n = 1$. Since $M_{n,1} = 2$ for $k \geq 1$ it we also check that (i), (ii) and (iii) hold for $n \geq 1$ when $k = 1$. Suppose we have established (i), (ii) and (iii) for (n', k') when $1 \leq n' < n$ or $n' = n$ and $k' < k$. Now let $X = A_1$ and $Y = A_2 \cup \ldots \cup A_k$. We can assume that $X \not\approx \mathbf{S}^n$ since otherwise $\mathbf{S}^n \setminus A_1 = \emptyset$. By the argument leading to Theorem 6.1.13 in [7], the Mayer-Vietoris sequence

$$\longrightarrow \tilde{H}^{q-1}(X) \oplus \tilde{H}^{q-1}(Y) \longrightarrow \tilde{H}^{q-1}(X \cap Y) \longrightarrow \tilde{H}^q(X \cup Y) \longrightarrow \tilde{H}^q(X) \oplus \tilde{H}^q(Y) \longrightarrow$$

is exact. It follows that

$$b^q(X \cup Y) \leq b^{q-1}(X \cap Y) + b^q(X) + b^q(Y). \tag{2}$$

with equality holding if $\tilde{H}^{q-1}(X) = \tilde{H}^{q-1}(Y) = \tilde{H}^q(X \cap Y) = 0$. Note that by using Remark 1 together with $\binom{k-1}{\ell} = \binom{k-2}{\ell} + \binom{k-2}{\ell-1}$ one can see by induction that

$$M_{n,k} = M_{n,k-1} + M_{n-1,k-1} \tag{3}$$

https://doi.org/10.4153/CMB-1991-082-1 Published online by Cambridge University Press

Observe that by our hypotheses $X \cap Y$ is a union of $k-1$ topological spheres or points, with spherical intersections, which can be regarded as subsets of an $(n-1)$-sphere, and Y is a union of $k-1$ topological spheres or points in $\mathbf{S}^n$. Thus, by the induction hypothesis, we have

$$(4.1) \qquad b^{n-2}(X \cap Y) + 1 \leq M_{n-1,k-1};$$

$$(4.2) \qquad b^{n-1}(Y) + 1 \leq M_{n,k-1};$$

$$(4.3) \qquad b^{n-1}(X) \leq 1.$$

Thus, from (3) and (2) with $q = n-1$, we get

$$(5) \qquad b^{n-1}(X \cup Y) + 1 \leq M_{n-1,k-1} + M_{n,k-1} = M_{n,k}$$

which proves (i) for (n,k). Next we prove (iii) for (n,k). Suppose $q < n-1$. $\mathcal{C}$ having proper spherical intersections in $\mathbf{S}^n$ implies that $X = A_1 \approx \mathbf{S}^{n-1}$ and also that $\{A_2, \ldots, A_k\}$ has proper spherical intersections in $\mathbf{S}^n$ so $\tilde{H}^q(X) = \tilde{H}^q(Y) = 0$ by inductive hypothesis. Now $X \cap Y = \cup_{i=2}^k (A_1 \cap A_i)$ and $\{A_1 \cap A_2, \ldots, A_1 \cap A_k\}$ have proper spherical intersections in $A_1 \approx \mathbf{S}^{n-1}$ so also, inductively, $\tilde{H}^{q-1}(X \cap Y) = 0$. Then by exactness of the Mayer-Vietoris sequence, $\tilde{H}^q(X \cup Y) = 0$, proving (iii). Now we prove (ii) for (n,k). In view of (3) and part (i), for the upper bound to be achieved, i.e. for $b^{n-1}(X \cup Y) + 1 = M_{n,k}$ to hold, it is necessary and sufficient to have equality in both (2) (with $q = n-1$) and (4.1)–(4.3). Suppose first that the upper bound is achieved. If $r \leq k-1$, we can assume that the r sets under consideration are among the $k-1$ sets $A_2, \ldots, A_k$; since we must have equality in (4.2) we see by induction that any r of these must intersect in a topological sphere of the appropriate dimension. The only remaining case is $r = k$ (so we must have $k \leq n+1$ by assumption). By equality in (4.3) we have $X = A_1 \approx \mathbf{S}^{n-1}$ and then $\cap_{i=1}^k A_i = \cap_{i=2}^k (A_1 \cap A_i)$ is an intersection of $k-1$ topological spheres in $A_1 \approx \mathbf{S}^{n-1}$. Using the induction hyphotheses and (4.1) with equality we get $\cap_{i=1}^k A_i \approx \mathbf{S}^{(n-1)-(k-1)} = \mathbf{S}^{n-k}$. Conversely, suppose $\mathcal{C}$ has proper spherical intersections. It then follows from (iii) that $b^{n-2}(X) = b^{n-2}(Y) = b^{n-1}(X \cap Y) = 0$. Thus we have equality in (2) with $q = n-1$. Moreover, since $\{A_2, \ldots, A_k\}$ has proper spherical intersections, we have equality in (4.2) by induction. Similarly, we conclude that equality holds in (4.1) and (4.3) and hence in (5). This finishes the proof.

THEOREM 3. *Suppose $\mathcal{C} = \{A_1, \ldots, A_k\}$ is a collection of subsets of $\mathbf{S}^n$ with proper spherical intersections. Then each component of $\mathbf{S}^n \setminus \cup A_\ell$ has the homology of a point.*

PROOF. We need only show that $\tilde{H}_q(\mathbf{S}^n \setminus K) = 0$ for $q \geq 1$ where $K = \cup_{\ell=1}^k A_\ell$. But this follows immediately from (1) and part (iii) of Theorem 2.

PROOF OF THEOREM 1. We can regard the A_i's as compact subsets of $\mathbf{R}^n \cup \{\infty\} \approx \mathbf{S}^n$ Each nonempty Venn cell is a union of components of $\mathbf{R}^n \setminus K$ with $K = \cup_{\ell=1}^k A_\ell$ so $m(\mathcal{C}) \leq M_{n,k}$. By Theorem 2 and Remark 1, $m(\mathcal{C}) = 2^k$ implies that $k \leq n+1$ and that furthermore, each nonempty Venn cell consists of a single component of $\mathbf{R}^n \setminus K$, and so is connected. The bounded components of $\mathbf{R}^n \setminus K$ have the homology of point by

https://doi.org/10.4153/CMB-1991-082-1 Published online by Cambridge University Press

Theorem 3. The result is clear for $n = 1$ so assume $n > 1$. The unbounded component U of $\mathbf{R}^n \setminus K$ corresponds to the the component of $\mathbf{S}^n \setminus K$ which contains ∞. It follows from a simple excision argument that U has the homology of $\mathbf{S}^{n-1}$.

EXAMPLES. If C is a compact convex subset of $\mathbf{R}^n$ with nonempty interior then the boundary of C is a topological sphere. We call a set *homothetic* to C if it is a translate of kC for some $k > 0$. Following [2], let $h(C)$ be the maximal number of sets in an independent collection consisting of sets homothetic to C. Thus $h(C) = n + 1$ when C is the Euclidean sphere $\mathbf{S}^{n-1} \subset \mathbf{R}^n$.

Suppose C is a regular tetrahedron in $\mathbf{R}^3$. If we are given a finite collection of sets homothetic to C we can find, if needed, arbitrarily small translations of these sets which will make their boundaries have spherical intersections and will not increase the number of components of the complement of the union of their boundaries. It follows from our results that $h(C) = 4$. In the same way we can show that the maximal number of independent *translates* of a fixed cube is 4. However, it is possible to find 3 homothetic cubes whose boundaries intersect in 4 points (we thank R. Sine for this observation) so the hypothesis of Theorem 1 do not hold and in fact it is possible to find 5 independent cubes homothetic to a fixed one. Figure 1 shows the intersection of the surfaces of 4 such cubes on the surface of a fifth.

Grünbaum [2] conjectured that $h(C) = n + 1$ holds for any compact convex C; the last example shows that this is false (although it holds when $n = 2$, [2]). In fact, it is possible to construct a C in $\mathbf{R}^3$ such that $h(C) = \infty$. This follows from the 2.13 and Proposition 3.19 in [1].

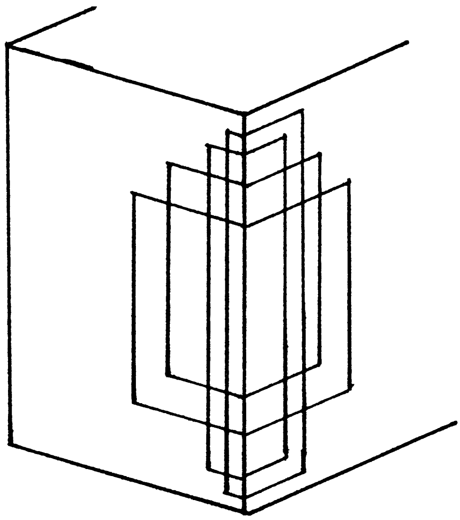

FIGURE 1

References

1. P. Assouad, *Densité et dimension*, Ann. Inst. Fourier (Grenoble) (3)**33**(1983), 233–282.

2. B. Grünbaum, *Venn diagrams and independent families of sets*, Mathematics Magazine **48**(1975), 12–22.

3. E. Marczewski, *Indépendance d'ensembles et prolongements de mesures*, Colloq. Math. **1**(1947), pp. 122–132.

4. W. S. Massey, *Singular Homology Theory*. Springer-Verlag, 1980.

5. L. Pakula, *A note on Venn diagrams*, American Math. Monthly (1)**96**(1989), 38–39.

6. A. Rényi, C. Rényi and J. Surányi, *Sur l'indépendance des domaines simples dans l'espace Euclidien à n dimensions*, Colloq. Math. **2**(1951), 130–135.

7. E. Spanier, *Algebraic Topology*. McGraw-Hill, Inc., 1966.

Department of Mathematics
University of Rhode Island
Kingston RI
USA 02881