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NORM INEQUALITIES FOR ULTRASPHERICAL AND HANKEL CONJUGATE FUNCTIONS

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1. Introduction. The notion of conjugate functions associated with ultraspherical expansions and their continuous analogues, the Hankel transforms, was introduced by Muckenhoupt and Stein [**14**], to which we refer the reader for general background and an excellent discussion of the motivation underlying these notions. The operation of passing from a given function to its conjugate is in many ways analogous to the passage from a function to its Hilbert transform, indeed, Muckenhoupt and Stein proved, among other things, that these operations acting on appropriate weighted Lebesgue spaces, $L^p(\mu)$, satisfy inequalities of M. Riesz type analogous to those satisfied by the Hilbert transform on the usual Lebesgue spaces, $L^p(-\infty, \infty)$. The purpose of this paper is to show that even in the broader context of rearrangement invariant spaces, of which the Lebesgue spaces are but one example, the conjugate function operators associated with ultraspherical expansions and Hankel transforms satisfy inequalities analogous to those satisfied by the classical trigonometric conjugate function operator (periodic Hilbert transform) and the Hilbert transform, respectively. We give necessary and sufficient conditions for these operations to map a given rearrangement invariant space continuously into another, in particular, we obtain new results for these operations as they act on the Lorentz and Orlicz spaces.

We refer to [**3**] for definitions and elementary properties of rearrangement invariant spaces, and although our notation is not identical to that of [**3**], the reader should have no difficulty making the translation. The Lorentz spaces $\Lambda(\phi, p)$ and the Orlicz spaces $L_{M\Phi}$ are all examples of rearrangement invariant spaces, see [**1**] for the definitions. Since they are used frequently in what follows, we note here for easy reference, two facts concerning rearrangement invariant norms σ:

$$(1.1)\quad f^{**}(t) \leqq g^{**}(t)\,(t > 0) \Rightarrow \sigma(f^*) \leqq \sigma(g^*)$$

where as usual f^{**} denotes the integral mean of f^*, the non-increasing rearrangement of f, and if m denotes Lebesgue measure,

$$(1.2)\quad m\{t : |f(t)| > y\} \leqq cm\,\{t : |g(t)| > y\} \quad (y > 0) \Rightarrow \sigma(|f|) \leqq c\sigma(|g|)$$

provided $c \geqq 1$.

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As usual, we adopt the convention that letters A, B, c denote absolute constants but which are not necessarily the same at each occurrence.

2. Ultraspherical conjugate functions. For $\lambda > 0$, let $dm_\lambda(\theta) = \sin^{2\lambda}\theta d\theta$ and denote by $L^p(m_\lambda)$ the space of measurable functions f defined on $(0, \pi)$ for which the (m_λ-rearrangement invariant) norm

$$||f||_{p,\lambda} = \left(\int_0^\pi |f(\theta)|^p \sin^{2\lambda}\theta d\theta\right)^{1/p} \quad (1 \leqq p < \infty)$$

is finite. The ultraspherical polynomials of type λ are denoted by $P_n^\lambda(t)$ and are defined by the generating relation

$$\sum_{n=0}^\infty w^n P_n^\lambda(t) = (1 - 2tw + w^2)^{-\lambda}.$$

For each fixed $\lambda > 0$, the set $\{P_n^\lambda (cos\ \theta)\}_0^\infty$ is orthogonal with respect to the measure $m_\lambda(\theta)$ on the interval $(0, \pi)$ and is in fact complete in $L^2(m_\lambda)$.

Now, if $f \in L^1(m_\lambda)$ has the ultraspherical expansion

$$f(\theta) \sim \sum_{n=0}^\infty a_n P_n^\lambda(\cos\theta),$$

the *conjugate harmonic function* associated with f, denoted $\tilde{f}(r, \theta)$, was defined by Muckenhoupt and Stein [**14**] and is given by the expansion

$$\tilde{f}(r, \theta) \sim 2\lambda \sum_{n=1}^\infty \frac{a_n r^n \sin\theta}{n + 2\lambda} P_{n-1}^{\lambda+1}(\cos\theta) \quad (r < 1).$$

One of their main results is that the *conjugate function* $\tilde{f}(\theta)$, given by

$$\tilde{f}(\theta) = \lim_{r\to 1^-} \tilde{f}(r, \theta),$$

exists a.e., and that the operation $T : f \to \tilde{f}$ is of weak-type $(1, 1)$ and of (strong) type (p, p) for all p, $1 < p < \infty$, that is

$$m_\lambda\{\theta : |Tf(\theta)| > y\} \leqq (A/y)||f||_{1,\lambda} \quad (y > 0)$$
$$||Tf||_{p,\lambda} \leqq A_p||f||_{p,\lambda} \quad (1 < p < \infty)$$

where A, A_p denote absolute constants, independent of f.

In this section, we shall extend these results to the situation of rearrangement invariant spaces by obtaining necessary and sufficient conditions in order that inequalities of the form

$$(2.1) \quad ||Tf||_{L^{\rho_2}} \leqq A||f||_{L^{\rho_1}}$$

should hold for some constant A. Indeed, if we define P_a, $P_a' : \mathscr{M}(0, a) \to \mathscr{M}(0, a)$, (abbreviated: $P = P_\infty$, $P' = P_\infty'$), by

$$(P_a f)(t) = \frac{1}{t}\int_0^t f(s)ds, \quad (P_a' f)(t) = \int_t^a f(s)\frac{ds}{s}$$

whenever the required integrals exist for almost all t, $0 < t < a$, we have:

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THEOREM 2.1. *Let ρ_1 and ρ_2 be m_λ-rearrangement invariant function norms on $(0, \pi)$ which are generated by σ_1 and σ_2 respectively, and let $a = m_\lambda(0, \pi)$, Then, in order that there exist an absolute constant A such that*

$$(2.2) \quad \rho_2(|Tf|) \leqq A\rho_1(|f|)$$

for all $f \in L^{\rho_1}(0, \pi)$, it is both necessary and sufficient that there exist an absolute constant B such that

$$(2.3) \quad \sigma_2(|(P_a + P_a')f|) \leqq B\sigma_1(|f|)$$

for all $f \in L^{\sigma_1}(0, a)$.

Remark 2.1. The constant B which appears in (2.3) depends in general on the value of a, however, it is easily verified that if (2.3) holds for some a, $0 < a < \infty$, then it also holds (with a suitable modification of B) for every other choice of a, $0 < a < \infty$.

Proof of Theorem 2.1. According to [**14**, p. 42], both T and its adjoint T^* are of type (2, 2) and of weak-type (1, 1), and hence the appendix of [**4**, p. 299] shows that there exists c, such that

$$(Tf)^{**}(t) \leqq c \int_0^\infty \frac{f^{**}(s)}{(s^2 + t^2)^{\frac{1}{2}}} ds.$$

But then it follows by an elementary inequality and Fubini's theorem, that

$$\frac{1}{t} \int_0^t (Tf)^*(s)ds \leqq c\frac{1}{t} \int_0^t ((P_a + P_a')f^*)(s)ds$$

so that, if (2.3) holds, by (1.1) we have

$$\rho_2(|Tf|) = \sigma_2((Tf)^*) \leqq c\sigma_2((P_a + P_a')f^*) \leqq cB\sigma_1(f^*) = cB\rho_1(|f|),$$

so (2.2) holds.

Conversely, according to [**14**, Lemma 4, p. 35], $\tilde{f}(r, \theta)$ is given by

$$\tilde{f}(r, \theta) = \int_0^\pi Q(r, \theta, \phi)f(\phi) \sin^{2\lambda}\phi d\phi$$

where

$$Q(r, \theta, \phi) = c_\lambda r^\lambda(\sin\theta \sin\phi)^{-\lambda} \frac{\sin(\theta - \phi)}{1 - 2r\cos(\theta - \phi) + r^2} + O\left[(\sin\theta)^{-2\lambda-1}\left(1 + \log^+\left(\frac{\sin\theta\sin\phi}{1 - \cos(\theta - \phi)}\right)\right)\right]$$

provided θ and ϕ are related by $\theta/2 \leqq \phi \leqq 2\theta$. Hence, if $0 < \theta < \pi/6, 0 < \phi <$

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$\pi/6$, we have

$$Q\left(r, \frac{\pi}{2} + \theta, \frac{\pi}{2} - \phi\right) = c_\lambda r^\lambda (\cos\theta\cos\phi)^{-\lambda} \frac{\sin(\theta+\phi)}{1 - 2r\cos(\theta+\phi) + r^2}$$
$$+ O\left[(\cos\theta)^{-2\lambda-1}\left(1 + \log^+\left(\frac{\cos\theta\cos\phi}{1-\cos(\theta+\phi)}\right)\right)\right]$$

and therefore, there exists a constant b, $0 < b \leqq \pi/6$, and a positive constant c such that $0 < \theta, \phi < b$ implies

$$Q\left(1, \frac{\pi}{2} + \theta, \frac{\pi}{2} - \phi\right) \geqq \frac{c}{\phi + \theta}.$$

Now, let $f \geqq 0$ with support in $(0, b)$ and define $g(\phi)$ by

$$g(\phi) = \begin{cases} f(\pi/2 - \phi), \text{ if } \pi/2 - b < \phi < \pi/2. \\ 0, \text{ otherwise.} \end{cases}$$

Then

$$(2.4) \quad m\{\phi : g^*(\phi) > y\} = m_\lambda\{\phi \in (0, \pi) : g(\phi) > y\} \leqq m\{\phi : f(\phi) > y\}$$

and if $0 < \theta < b$, simple manipulations show that

$$\tilde{g}(\pi/2 + \theta) \geqq c((P_b + P_b')f)(\theta).$$

Hence, if $h(\theta)$ is defined on $(0, \infty)$ by

$$h(\theta) = \begin{cases} \tilde{g}(\pi/2 + \theta), \text{ if } 0 < \theta < b \\ 0, \text{ otherwise} \end{cases}$$

we get

$$(2.5) \quad \begin{aligned} m\{\theta : h(\theta) > y\} &\leqq \cos^{-2\lambda} b \, m_\lambda\{\theta \in (\pi/2, \pi/2 + b) : \tilde{g}(\theta) > y\} \\ &\leqq \cos^{-2\lambda} b \, m\{\theta : (\tilde{g})^*(\theta) > y\} \end{aligned}$$

and therefore, if (2.2) holds, applying (1.2) to each of (2.4) and (2.5), we have

$$\sigma_2((P_b + P_b')f) \leqq cA\rho_1(|g|) \leqq B\sigma_1(|f|)$$

and (2.3) follows in view of Remark 2.1. The theorem is proved.

Remark 2.2. Since the estimates for the adjoint of T, T^*, are essentially the same, (2.3) is also necessary and sufficient for (2.2) to hold with T^* in place of T.

The analogue of Theorem 2.1 for the classical trigonometric conjugate function operator C, given by the principal value integral

$$(Cf)(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot\left(\frac{\theta - t}{2}\right) dt, \quad \theta \in (0, 2\pi),$$

was proved by Kerman [**8**]. Applying his result to our Theorem 2.1, we obtain

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THEOREM 2.2. *Let ρ_1 and ρ_2 be m_λ-rearrangement invariant function norms on* $(0, \pi)$ *which are generated by σ_1 and σ_2 respectively. Then, there exists an absolute constant A such that*

$$\rho_2(|Tf|) \leqq A\rho_1(|f|)$$

for all $f \in L^{\rho_1}(0, \pi)$ if and only if there is an absolute constant B such that

$$(2.7) \quad \sigma_2(|Cf|) \leqq B\sigma_1(|f|)$$

for all $f \in L^{\sigma_1}(0, 2\pi)$.

For fixed, ν, $\nu \geqq 1$, let the Young's functions Φ and Φ_1 be given by

$$\Phi(u) = u(\log^+ u)^\nu, \quad \Phi_1(u) = u(\log (2 + u))^{\nu-1}.$$

Then according to [**16**, Exercise 6, p. 296 and Theorem 10.14, p. 174] we have

$$||Cf||_{L_{M\Phi_1}} \leqq B||f||_{L_{M\Phi}}$$

and hence, by Theorem 2.2,

$$||Tf||_{L_{M\Phi_1}} \leqq A||f||_{L_{M\Phi}}.$$

In particular, the special case $\nu = 1$ yields $L_{M\Phi_1} = L^1$, $L_{M\Phi} = L\ log^+ L$ and since we have $||f||_{L \log^+ L} \leqq 1 + M_\Phi(|f|)$ (see [**12**, Lemma 1, p. 45)] we obtain an analogue of Theorem 2.8 of [**16**]:

COROLLARY 2.1. *There exist absolute constants A, B such that*

$$\int_0^\pi |Tf(\theta)| \sin^{2\lambda} \theta d\theta \leqq A \int_0^\pi |f(\theta)| \log^+ |f(\theta)| \sin^{2\lambda} \theta d\theta + B.$$

Now, if $L^\rho(0, \pi)$ is an Orlicz space, the result of Ryan [**15**], together with Theorem 2.2 yields

COROLLARY 2.2. *If L^ρ is an Orlicz space, then there exists an absolute constant A such that*

$$||Tf||_{L^\rho} \leqq A||f||_{L^\rho}$$

if and only if L^ρ is reflexive.

Conditions under which (2.3) (equivalently (2.7)) hold have been studied extensively by Kerman [**8**] and result in:

COROLLARY 2.3. *A necessary condition for* (2.2) *to hold is that $L^{\rho_1} \subseteq L^{\rho_2}$ and a sufficient condition is that*

$$\int_0^\infty \min (1, 1/s)h(s, L^{\rho_1}, L^{\rho_2})ds < \infty.$$

COROLLARY 2.4. *In order that* (2.2) *hold with $\rho_1 = \rho_2 = \rho$ it is both necessary and sufficient that the upper and lower indices α, β of L^ρ satisfy* $0 < \beta \leqq \alpha < 1$.

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COROLLARY 2.5. *In order that* (2.2) *hold with* $L^{\rho_1} = L^{\rho_2} = \Lambda(\phi, p)$, $p > 1$, *it is both necessary and sufficient that* $\Lambda(\phi, p)$ *be uniformly convex.*

Conditions under which an inequality of the form

$$A\rho(|f|) \leqq \rho(|Tf|) \tag{2.8}$$

should hold for every $f \in L^\rho$ with

$$\int_0^\pi f(\phi) \sin^{2\lambda} \phi d\phi = 0$$

can be given. Indeed, examining the proof given in [**14**, p. 43-44] for the L^p case, and in view of Remark 2.2, an inequality of the form (2.8) will hold provided:

(i) $\rho(|Tf|) \leqq B\rho(|f|)$ for all $f \in L^\rho$

and

(ii) polynomials of the form $\sum_0^N a_n P_n^\lambda(\cos\theta)$ are dense in L^ρ.

Now it follows from the closed graph theorem and [**3**, (6)] that for any m_λ-rearrangement invariant norm ρ, there exists a constant c such that

$$\rho(|f|) \leqq c\|f\|_{\infty,\lambda} \quad (f \in L^\infty(m_\lambda)),$$

and since polynomials of the form $\sum_0^N a_n P_n^\lambda(\cos\theta)$ are dense in the space of continuous functions on $(0, \pi)$ with the $L^\infty(m_\lambda)$ norm, it follows from [**12**, Theorem 7] and Lusin's theorem that (ii) may be replaced by (ii)$'$:

(ii)$'$ L^ρ is separable.

Now, according to [**12**, Theorem 4 and Corollary 1, p. 35], it follows that (ii)$'$ holds for any reflexive space L^ρ, and we therefore obtain an immediate strengthening of our Corollary 2.2:

COROLLARY 2.2.$'$ *Let* L^σ *be an Orlicz space. Then there exists a constant* B *such that*

$$\rho(|Tf|) \leqq B\rho(|f|), \quad f \in L^\rho$$

if and only if L^ρ *is reflexive, and in that case there is a constant* A *such that*

$$A\rho(|f|) \leqq \rho(|Tf|) \leqq B\rho(|f|)$$

for every $f \in L^\rho$ *with*

$$\int_0^\pi f(\phi) \sin^{2\lambda} \phi d\phi = 0.$$

Further, since for any Banach space X, X is uniformly convex implies X is reflexive (due to Milman), our Corollary 2.5 is similarly strengthened.

We note, however, that our Corollary 2.4 can not be so strengthened. For example, the space M_α which is the associate of $\Lambda_\alpha = \Lambda(\alpha t^{\alpha-1}, 1)$, $0 < \alpha < 1$, has both upper and lower indices equal to $1 - \alpha$, so (i) holds, but M_α is never separable.

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3. Hankel conjugate functions. In this section we describe briefly how results analogous to those presented in the previous section for the ultraspherical expansions may be obtained for the expansions associated with Hankel transforms.

For $\lambda > 0$, let $d\mu_\lambda(z) = z^{2\lambda}dz$ and denote by $L^p(\mu_\lambda)$ the Lebesgue space of μ_λ-measurable functions for which the norm

$$||f||_{p,\lambda} = \left(\int_0^\infty |f(z)|^p z^{2\lambda}dz\right)^{1/p} \quad (1 \leqq p < \infty)$$

is finite. As usual, $J_\alpha(t)$ denotes the Bessel function of order α. Corresponding to an $f \in L^p(\mu_\lambda)$, Muckenhoupt and Stein [**14**, p. 84] define the *conjugate harmonic function*, $\tilde{f}(x, y)$ by

$$\tilde{f}(x, y) = \int_0^\infty Q(x, y, z)f(z)z^{2\lambda}dz \quad (x > 0, y > 0)$$

where

$$Q(x, y, z) = -(yz)^{-\lambda+1/2}\int_0^\infty e^{-xt}J_{\lambda+1/2}(yt)J_{\lambda-1/2}(zt)tdt.$$

Their main result is that for $f \in L^p(\mu_\lambda)$, $\lim_{x\to 0+}\tilde{f}(x, y) = \tilde{f}(y)$ exists a.e. and the operation $\mathscr{H} : f \to \tilde{f}$ is of (strong) type (p, p), $1 < p < \infty$, and of weak type $(1, 1)$. We shall extend these results to the situation of μ_λ-rearrangement invariant spaces by giving necessary and sufficient conditions in order that inequalities of the form

$$\rho_2(|\mathscr{H}f|) \leqq A\rho_1(|f|), \quad f \in L^{\rho_1}(0, \infty)$$

should hold. Recall the definition of P and P' given in § 2. We have:

THEOREM 3.1. *Let ρ_1 and ρ_2 be μ_λ-rearrangement invariant function norms on $(0, \infty)$ which are generated by σ_1 and σ_2 respectively. Then, in order that there exist an absolute constant A such that*

$$(3.1) \quad \rho_2(|\mathscr{H}f|) \leqq A\rho_1(|f|), \quad f \in L^{\rho_1}(0, \infty)$$

it is both necessary and sufficient that there exist an absolute constant B such that

$$(3.2) \quad \sigma_2(|(P + P')f|) \leqq B\sigma_1(|f|), \quad f \in L^{\sigma_1}(0, \infty).$$

The proof of this theorem requires the following easy lemma.

LEMMA. *Let $f \in \mathscr{M}(0, \infty)$. Then for any s, $s > 0$, and $(E_sf)(t) = f(st)$,*
(i) $m\{t : |f(t)| > y\} = s\, m\{t : |(E_sf)(t)| > y\} \quad (y \geqq 0)$
(ii) $(E_sf)^*(t) = (E_{s^{2\lambda+1}}f^*)(t) \quad (t > 0)$
(iii) *if $\tilde{f}$ exists,* $(E_sf)^\sim(y) = (E_s\tilde{f})(y) \quad (y > 0)$.

Proof of Theorem 3.1. The proof of the sufficiency follows the same line as the sufficiency part of Theorem 2.1 and is therefore omitted. To prove the

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necessity, note first that according to [**14**, p. 87], if y, z satisfy $y/2 \leqq z \leqq 2y$, then

$$Q(x, y, z) = c_\lambda (yz)^{-\lambda} \frac{y - z}{x^2 + (y - z)^2} + O\left[y^{-2\lambda-1}\left(1 + \log^+\left(\frac{yz}{y - z}\right)\right)\right].$$

Therefore, if n is an integer, $n \geqq 9$, and $0 < y, z < \sqrt{n}$, we have

$$\begin{aligned} Q(0, n + y, n - z) &= c_\lambda \frac{[(n + y)(n - z)]^{-\lambda}}{y + z} + O\Bigg[(n + y)^{-2\lambda-1} \\ &\qquad \times \left(1 + \log^+\left(\frac{(n + y)(n - z)}{y + z}\right)\right)\Bigg] \\ &= \frac{[(n + y)(n - z)]^{-\lambda}}{y + z}\Bigg(c_\lambda + O\Bigg[\frac{y + z}{n} \\ &\qquad \times \left(1 + \log^+\left(\frac{2n^2}{y + z}\right)\right)\Bigg]\Bigg) \end{aligned}$$

and hence, there is a positive constant c and an integer N such that $n \geqq N$ implies

$$(3.3) \quad Q(0, n + y, n - z) \geqq c \frac{[(n + y)(n - z)]^{-\lambda}}{y + z} \quad (0 < y, z < \sqrt{n}).$$

Now let $f \in L^{\sigma_1}(0, \infty)$ with $f \geqq 0$ and define $g(z) = f(n - z)$ if $n - \sqrt{n} < z < n$ and $g(z) = 0$ otherwise. Then for $n \geqq N$, $0 < y < \sqrt{n}$, we have, by (3.3),

$$(3.4) \quad \tilde{g}(n + y) \geqq c[(P + P')f\chi_{(0,\sqrt{n})}](y).$$

Hence, if we define $h(y) = \tilde{g}(n + y)$ if $0 < y < \sqrt{n}$ and $h(y) = 0$ otherwise, from (3.4) we have

$$(3.5) \quad \sigma_2([(P + P')f\chi_{(0,\sqrt{n})}]\chi_{(0,\sqrt{n})}) \leqq c\sigma_2(h).$$

Now, noting the support of h we easily obtain

$$(3.6) \quad m\{z : h(z) > y\} \leqq n^{-2\lambda} m\{z : (\tilde{g})^*(z) > y\}$$

and if we put $s^{2\lambda+1} = n^{2\lambda}$ in the lemma, (3.6) becomes

$$\begin{aligned} m\{z : h(z) > y\} &\leqq m\{z : [E_{s^{2\lambda+1}}(\tilde{g})^*](z) > y\} \\ &= m\{z : [(E_s g)^\sim]^*(z) > y\} \end{aligned}$$

so that, by (1.2),

$$(3.7) \quad \sigma_2(h) \leqq \sigma_2([(E_s g)^\sim]^*) = \rho_2((E_s g)^\sim).$$

Again, noting the support of g, we have

$$n^{-2\lambda} m\{z : g^*(z) > y\} \leqq m\{z : f(z)\chi_{(0,\sqrt{n})}(z) > y\}$$

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and therefore if (3.1) holds, (3.7) and the lemma together with (1.2) yield

$$(3.8)\quad \sigma_2(h) \leqq A\sigma_1((E_s g)^*) = A\sigma_1(E_{s^{2\lambda+1}} g^*) = A\sigma_1(E_{n^{2\lambda}} g^*) \leqq A\sigma_1(f\chi_{(0,\sqrt{n})}).$$

Finally, taking (3.5) and (3.8) together we get

$$\sigma_2([(P + P')f\chi_{(0,\sqrt{n})}]\chi_{(0,\sqrt{n})}) \leqq cA\sigma_1(f\chi_{(0,\sqrt{n})}) \leqq cA\sigma_1(f)$$

which, together with the Fatou property of σ_2, yields (3.2). The proof is complete.

Boyd [**1**, Theorem 2.1, p. 602] has proved the analogue of Theorem 3.1 for the classical Hilbert transform H, defined by the principal value integral

$$(Hf)(x) = \int_{-\infty}^{\infty} \frac{f(t)}{x-t}\,dt, \quad x \in (-\infty, \infty).$$

His result together with Theorem 3.1 proves:

THEOREM 3.2. *Let ρ_1 and ρ_2 be μ_λ-rearrangement invariant function norms on* $(0, \infty)$ *generated by σ_1 and σ_2 respectively. Then, in order that there exist an absolute constant A such that*

$$\rho_2(|\mathscr{H}f|) \leqq A\rho_1(|f|), \quad f \in L^{\rho_1}(0, \infty)$$

it is both necessary and sufficient that there exist an absolute constant B such that

$$\sigma_2(|Hf|) \leqq B\sigma_1(|f|), \quad f \in L^{\sigma_1}(0, \infty).$$

Now the results of [**1**] and Theorem 3.2 show that the obvious analogues of Corollaries 2.2–2.5 given in § 2 for the operator T also hold for $\mathscr{H}$.

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