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CURVATURE OF LEVEL CURVES OF HARMONIC FUNCTIONS

BY

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ABSTRACT. If H is an arbitrary harmonic function defined on an open set $\Omega \subset \mathbb{C}$, then the curvature of the level curves of H can be *strictly* maximal or *strictly* minimal at a point of Ω. However, if Ω is a doubly connected domain bounded by analytic convex Jordan curves, and if H is harmonic measure of Ω with respect to the outer boundary of Ω, then the minimal curvature of the level curves of H is attained on the boundary of Ω.

§1. **Introduction.** To our knowledge, all earlier theorems regarding the curvature of level sets of harmonic functions pertain exclusively to Green's functions on simply connected domains. For instance, [1] contains the well-known result that level curves of Green's functions on simply connected convex domains in the plane are convex Jordan curves. More difficult versions of these results (in higher dimensions) appear in [2], [3], [4].

In the present paper we prove a general extremum principle for the curvature of level curves of harmonic functions (Theorem 2.1) as well as a particular extremum principle for curvature in the case of harmonic measure on a *doubly connected* plane domain (Theorem 3.1). By the latter theorem the level curves of harmonic measure (with respect to the outer boundary) of an annular domain bounded by convex Jordan curves are, themselves, convex.

These two theorems suggest that broader extremum principles for the curvature may be valid. We formulate two conjectures to this effect. Then we provide a counter example to the stronger conjecture by constructing a particular harmonic function whose level curves attain strictly maximal curvature at an interior point of its domain. The weaker conjecture, which concerns harmonic measure, remains open.

§2. **The curvature function.** The following definition is convenient for the study of the curvature of level curves.

DEFINITION 2.1 Let H be harmonic in the open set $\Omega \subset \mathbb{C}$. Suppose the

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derived analytic function $g \equiv H_x - iH_y$ never vanishes in Ω and set

$$\kappa(H, z) \equiv |g(z)| \operatorname{Re}\left(\frac{1}{g}\right)'(z), \quad \text{all} \quad z \in \Omega.$$

We refer to $\kappa(H, \cdot)$ as the *curvature function* for H. □

To justify the terminology, let I be an open interval of real numbers and suppose $s \to z(s)$ is a differentiable function on I taking values in Ω and satisfying

$$\dot{z}(s) = \frac{\overline{ig(z(s))}}{|g(z(s))|}, \quad \text{all} \quad s \in I. \tag{2.1}$$

Then $s \to z(s)$ is a unit speed parametric arc and $H(z(s))$ is constant on the interval I. For $s \in I$ set $x(s) = \operatorname{Re} z(s)$, $y(s) = \operatorname{Im} z(s)$, and

$$Q(s) = |g(z(s))| \, (\dot{x}(s)\ddot{y}(s) - \ddot{x}(s)\dot{y}(s)) = \operatorname{Im}(\overline{\dot{z}(s)}\ddot{z}(s) \, |g(z(s))|).$$

Since

$$\operatorname{Im}\left(\overline{\dot{z}(s)}\dot{z}(s) \frac{d}{ds} |g(z(s))|\right) = 0 \quad \text{for all} \quad s \in I,$$

the product rule leads to

$$Q(s) = \operatorname{Im}\left[\overline{\dot{z}(s)} \frac{d}{ds} (\dot{z}(s) \, |g(\dot{z}(s))|)\right], \quad \text{all} \quad s \in I.$$

In this expression for $Q(s)$ replace $\dot{z}(s) \, |g(z(s))|$ by $\overline{ig(z(s))}$, then employ the chain rule, and then substitute for $\dot{z}(s)$ according to (2.1). We obtain

$$Q(s) = Re(g'(z(s))(\dot{z}(s))^2) = |g(z(s))|^2 \operatorname{Re}\left(\frac{1}{g}\right)'(z(s)), \quad \text{all} \quad s \in I.$$

Therefore

$$\dot{x}(s)\ddot{y}(s) - \ddot{x}(s)\dot{y}(s) = \kappa(H, z(s)), \quad \text{all} \quad s \in I.$$

Since $|\dot{z}(s)| \equiv 1$ the left-hand side above is the curvature at $z(s)$ of the parametric level arc of H defined by (2.1). So $\kappa(H, z)$ is the curvature at z of any parametric level arc of H passing through z.

There are simple extremum principles for the curvature function.

THEOREM 2.1. *Let Ω be a connected open subset of $\mathbb{C}$, let $z \in \Omega$, and let H be harmonic and have no critical points in Ω.*

(1) *Suppose $\kappa(H, w) \geq 0$, all $w \in \Omega$. Then*

$$\kappa(H, z) \geq \liminf_{w \to \partial\Omega} \kappa(H, w)$$

and equality holds if and only if $\kappa(H, \cdot)$ is a constant function.

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(2) *Suppose* $\kappa(H, w) \le 0$, *all* $w \in \Omega$. *Then*

$$\kappa(H, z) \le \limsup_{w \to \partial\Omega} \kappa(H, w)$$

and equality holds if and only if $\kappa(H, \cdot)$ *is a constant function.*

(3) *If* $\kappa(H, \cdot)$ *is a constant function then* $\kappa(H, w) = 0$, *all* $w \in \Omega$ *and there exist three real numbers* A, B, C, *and one complex number* $a \notin \Omega$ *such that* $H(x+iy) = Ax + By + C$, *all* $x + iy \in \Omega$ *or* $H(w) = A\arg(w-a) + B$, *all* $w \in \Omega$. □

Proof. Set $g = H_x - iH_y$. By hypothesis g has no zeros in Ω.

(1) The hypothesis of statement (1) implies $\text{Re}(1/g)'(w) \ge 0$ for all $w \in \Omega$. Hence there are two cases: (a) $\text{Re}(1/g)'(w) = 0$, all $w \in \Omega$; (b) $\text{Re}(1/g)'(w) > 0$, all $w \in \Omega$. In case (a) $\kappa(H, w) = 0$, all $w \in \Omega$. In case (b) Jensen's inequality and the mean value theorem for harmonic functions imply that the average of $\log \text{Re}(1/g)'$, over any sufficiently small circle centered at a point $w \in \Omega$, is less than $\log \text{Re}(1/g)'(w)$. Therefore, in case (b) $\log \text{Re}(1/g)'$ is super harmonic in Ω and $\log \kappa(H, \cdot)$ is super harmonic in Ω. By the minimum principle for super harmonic functions

$$\log \kappa(H, z) \ge \liminf_{w \to \partial\Omega} \log \kappa(H, w)$$

and $\log \kappa(H, \cdot)$ is a constant function if and only if equality holds. The conclusions of (1) are now immediate.

(2) Apply part (1) to $\kappa(-H, \cdot) = -\kappa(H, \cdot)$.

(3) Suppose $\kappa(H, w) = \text{constant}$, all $w \in \Omega$. Then either $\text{Re}(1/g)' = 0$ (if constant $= 0$) or $|g|^{-1} = (\text{constant})^{-1} \text{Re}(1/g)'$ is harmonic in Ω. In the second instance the mean value theorem implies (for appropriate $r > 0$)

$$\frac{1}{|g(z)|} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|g(z + re^{i\theta})|} \ge \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{g(z + re^{i\theta})} \right| = \frac{1}{|g(z)|}.$$

We conclude that $g(w)$ is identically constant in Ω. So in both cases constant $=$ 0 and $\text{Re}(1/g)' = 0$. Thus

$$g(w) = [i\ (\text{real constant } w + \text{complex constant}]^{-1}$$

and it is evident that H is of the expressed form. □

§3. Theorem 3.1 is a minimum principle for curvature of level curves of harmonic measure with respect to the outer boundary of an annulus bounded by convex Jordan curves. Thus, for completeness, we first define these terms precisely.

By definition, a *proper Jordan curve* Γ is a pair (Γ^*, p) for which the following statements are valid:

(i) p is an analytic function defined in a neighborhood of the real line with non-vanishing derivative.

(ii) $\Gamma^* = \{p(t): t \text{ real}\}$ is a Jordan curve.

(iii) $|dp/dt(t)| \equiv 1$, all $t \in (-\infty, \infty)$.

(iv). There is a number $e > 0$ such that if t is real, and $p(t) = x(t) + iy(t)$, and $0 < \varepsilon < e$, then $p(t) + \varepsilon(-\dot{y}(t) + i\dot{x}(t))$ lies in the interior of Γ^*.

If $z = p(t) \in \Gamma^*$, we define the *curvature of* Γ *at* z by

$$c(\Gamma, z) = \ddot{y}(t)\dot{x}(t) - \ddot{x}(t)\dot{y}(t).$$

The choice of the parameter t corresponding to z is irrelevant; it can be shown that p is *periodic* with some period $T > 0$ and that p maps $[0, T)$ one-to-one onto Γ^*.

If Γ_0 and Γ_1 denote proper Jordan curves with Γ_0^* contained in the *interior* of Γ_i^*, then $A = A(\Gamma_0, \Gamma_1)$ shall denote the region bounded by Γ_0^* and Γ_1^*. Since Γ_0 and Γ_1 are proper Jordan curves there is a region Ω and a function H such that $\bar{A} \subset \Omega$, H is harmonic in Ω, H has no critical points in Ω, H is identically zero on Γ_0^*, and H is identically one on Γ_1^*. There is *at most one* such function H in *any fixed* region Ω containing $\bar{A}$ and any candidate will be called *harmonic measure for the annulus* A *with respect to* Γ_1. Then, if $0 \le r \le 1$, we set $\Gamma_r^* = \{z \in \bar{A} : H(z) = r\}$ and assert that there is a function p_r such that $\Gamma_r = (\Gamma_r^*, p_r)$ is a proper Jordan curve. Moreover, Γ_r^* lies in the interior of Γ_s^* and $r \le H(z) \le s$ if $0 \le r < s \le 1$ and $z \in A(\Gamma_r, \Gamma_s)$. These statements are standard (but non-trivial). Finally, we introduce the following notation for the *minimum* and *maximum* curvatures of $\Gamma_0 \cup \Gamma_1$:

$$k(\Gamma_0, \Gamma_1) = \min\{c(\Gamma_r, z) : (r = 0 \text{ or } r = 1) \text{ and } z \in \Gamma_r^*\}$$

$$K(\Gamma_0, \Gamma_1) = \max\{c(\Gamma_r, z) : (r = 0 \text{ or } r = 1) \text{ and } z \in \Gamma_r^*\}$$

THEOREM 3.1. *Let* $\Gamma_0 = (\Gamma_0^*, p_0)$ *and* $\Gamma_1 = (\Gamma_1^*, p_1)$ *denote proper Jordan curves with* Γ_0^* *lying in the interior of* Γ_1^*, *let* H *denote harmonic measure for the annulus* $A = A(\Gamma_0, \Gamma_1)$ *with respect to* Γ_1, *and let* $\Gamma_r = (\Gamma_r^*, p_r)$, $0 < r < 1$, *denote proper Jordan curves for which* $\Gamma_r^* = \{z \in A : H(z) = r\}$. *Then*

(1) $c(\Gamma_r, z) = \kappa(H, z)$, *all* $r \in [0, 1]$ *and all* $z \in \Gamma_r^*$;

(2) *If* $\kappa(\Gamma_0, \Gamma_1) \ge 0$ *and* $0 < r < 1$ *and* $z \in \Gamma_r^*$ *we have*

$$\kappa(H, z) = c(\Gamma_r, z) > \kappa(\Gamma_0, \Gamma_1)$$

and Γ_r^* *is strictly convex.* □

Proof. (1) Fix $r \in [0, 1]$ and set $p(t) \equiv p_r(t) \equiv x(t) + iy(t)$, all $t \in (-\infty, \infty)$. We claim that p is a solution of the differential equation (2.1) with $g = H_x - iH_y$ (for the case $r = 0$ or 1 recall that H is harmonic with no critical points in a *neighborhood* of $\bar{A}$). Since $H(p(t)) \equiv r$ we have $H_x(p(t))\dot{x}(t) + H_y(p(t))\dot{y}(t) \equiv 0$. Thus

$$(*) \qquad \operatorname{Re} \dot{p}(t) g(p(t)) = 0.$$

By the discussion of terminology, $H(z) \le r$ if z lies in the interior of Γ_r^*;

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therefore a number $e(r)>0$ can be chosen so that $H(p(t)+\varepsilon(-\dot{y}(t)+i\dot{x}(t)))-H(p(t))\le 0$, all $\varepsilon\in(0, e(r)]$ and all $t\in(-\infty,\infty)$. Divide by ε and allow $\varepsilon\to 0$. We see

$$(**)\qquad \operatorname{Im}\dot{p}(t)g(p(t))\ge 0$$

Together (*) and (**) imply

$$\dot{p}(t)g(p(t))\equiv i\,|\dot{p}(t)|\,|g(p(t))|,$$

and since $|p(t)|\equiv 1$ and $|g(p(t))|\ne 0$, it follows that p satisfies (2.1). We conclude

$$\kappa(H, p(t))=\ddot{y}(t)\dot{x}(t)-\ddot{x}(t)\dot{y}(t)=c(\Gamma_r, p(t))$$

as required.

(2) By the hypothesis and part (1) $\kappa(H, z)=c(\Gamma_r, z)\ge 0$ if $r=0$ or 1 and $z\in\Gamma_r^*$. Thus $\operatorname{Re}(1/g)'\ge 0$ on ∂A. And this implies that both $\operatorname{Re}(1/g)'$ and $\kappa(H,\cdot)$ are non-negative at each point of A. Moreover

$$\liminf_{w\to\partial A}\kappa(H, w)=\min_{w\in\partial A}\kappa(H, w)=k(\Gamma_0,\Gamma_1)$$

by part (1) of the present theorem. We appeal to part (1) of Theorem 2.1 and conclude $c(\Gamma_r, z)=\kappa(H, z)>k(\Gamma_0,\Gamma_1)$, all $r\in(0, 1)$ and all $z\in\Gamma_r^*$.

It is well known that Jordan curves with non-vanishing curvature are strictly convex. □

A conjecture. To our knowledge there is no counterexample to any part of the following conjecture.

CONJECTURE 3.2. *Let Γ_0 and Γ_1 denote proper Jordan curves with Γ_0^* in the interior of Γ_1^*, let H denote harmonic measure for $A=A(\Gamma_0,\Gamma_1)$ with respect to Γ_1, and let Γ_r denote a proper Jordan curve with $\Gamma_r^*=\{z\in A: H(z)=r\}$. Then $k(\Gamma_0,\Gamma_1)\le c(\Gamma_r, z)=\kappa(H, z)\le K(\Gamma_0,\Gamma_1)$, all $r\in[0, 1]$ and all $z\in\Gamma_r^*$.* □

Conjecture 3.2 pertains to the theory of conformal mapping because H is essentially the logarithm of the modulus of a conformal map of A onto a circular annulus. Moreover, the upper and lower bounds in the conjectured inequalities are geometrical quantities associated with A, while $\kappa(H,\cdot)$ has a simple analytic expression in terms of the mapping function mentioned above.

§4. **Counterexample to a stronger conjecture.** The purpose of this section is to provide a counterexample to the following conjecture, the truth of which would immediately imply the truth of Conjecture 3.2.

CONJECTURE 4.1. *Let H be harmonic with no critical points in a region Ω. Then*

$$\liminf_{w\to\partial\Omega}\kappa(H, w)\le\kappa(H, z)\le\limsup_{w\to\partial\Omega}\kappa(H, w),\quad \textit{all}\quad z\in\Omega.\ \square$$

Because Conjecture 4.1 is false the geometrical hypotheses of Conjecture 3.2 are seen to be essential. We require Lemma 4.2 in the construction.

LEMMA 4.2. *Set* $h(w)=\exp[-(w-1)+\frac{1}{4}(w-1)^2]$. *Then there are positive numbers r and* δ *and there is an analytic function g defined for* $|z|<r$ *such that the following statements are valid.*

(1) $|h(w)|\,\mathrm{Re}\, w<h(1)$ *if* $0<|w-1|<\delta$

(2) $g(w)\neq 0$ *if* $|w|<r$.

(3) $(1/g)'(0)=1$ *and* $g(0)=1$.

(4) $0<|(1/g)'(z)-1|<\delta$ *if* $0<|z|<r$.

(5) $h((1/g)'(z))=g(z)$ *if* $|z|<r$. □

Proof. (1) The remainder after two terms in Taylor's expansion of $\log x$ about $x=1$ is $0(|x-1|^3)$. Thus we may choose $\delta>0$ so that

$$\log x-(x-1)+\tfrac{1}{2}(x-1)^2<\tfrac{1}{4}(x-1)^2 \quad \text{if} \quad 0<|x-1|<\delta.$$

This implies $\log x-(x-1)+\frac{1}{4}[(x-1)^2-y^2]<0$ if $0<(x-1)^2+y^2<\delta^2$; or equivalently, with $w=x+iy$,

$$|\exp[-(w-1)+\tfrac{1}{4}(w-1)^2]|\,\mathrm{Re}\, w<1 \quad \text{if} \quad 0<|w-1|<\delta.$$

This is statement (1).

(2) → (5) The function h is one-to-one in a neighborhood N_0 of the complex number 1; also $h(1)=1$. Denote by h^{-1} the inverse function of $h|_{N_0}$, denote the domain of h^{-1} (a neighborhood of 1) by N_1, and note that $h^{-1}(1)=1$. Since $h^{-1}(1)\neq 0$, there is a unique analytic solution g of the initial value problem

$$(*) \qquad g'(z)=-g(z)^2h^{-1}(g(z)), \qquad g(0)=1,$$

defined in a neighborhood N_2 of zero (so that $g(N_2)\subset N_1$).

The following properties of g are successively evident in light of (*) and the fact that h^{-1} is one-to-one in N_1 with $h^{-1}(1)=1$.

(a) There is a neighborhood N_3 of 0 such that $N_3\subset N_2$, $g(z)\neq 0$ for z in N_3, $g'(z)\neq 0$ for z in N_3, and g is one-to-one in N_3.

(b) $(1/g)'(z)=h^{-1}(g(z))$, all $z\in N_3$.

(c) $(1/g)'$ is one-to-one in N_3 and $(1/g)'(0)=1$.

(d) With δ as in statement (1), there is a neighborhood N_4 of 0 such that $N_4\subset N_3$, $(1/g)'(z)\in N_0$ if $z\in N_4$, and

$$0<\left|\left(\frac{1}{g}\right)'(z)-1\right|<\delta, \quad \text{all} \quad z\in N_4-\{0\}.$$

Now choose $r>0$ so that $\{|z|<r\}\subset N_4$ and choose g as above, but restricted to $\{|z|<r\}$. Then (2) follows from (a), (3) follows from (c) and (*), and (4)

follows from (d). Finally, by (d), $(1/g)'(z) \in N_0$ if $|z|<r$. Hence, by (b),

$$h\left(\left(\frac{1}{g}\right)'(z)\right) = h(h^{-1}(g(z))) = g(z) \quad \text{if} \quad |z|<r.$$

The proof is complete. □

We may now exhibit a curvature function which assumes a *strict* global maximum, or a strict global minimum, and thus serves as a counterexample to Conjecture 4.1. Strictness is the main point of the construction; it is easy to find examples with *weak* global extrema.

EXAMPLE 4.3. With g and r as in Lemma 4.2 set

$$H(z) = \operatorname{Re} \int_0^z g(t)\, dt \quad \text{if} \quad |z|<r.$$

Then $\kappa(H, z) < \kappa(H, 0)$ if $0<|z|<r$ and $\kappa(-H, z) > \kappa(-H, 0)$ if $0<|z|<r$. □

Proof. By (2) of Lemma 4.2, $\kappa(H, z) = |g(z)| \operatorname{Re}(1/g)'(z)$ is defined in $|z|<r$. By (5) of Lemma 4.2,

$$\kappa(H, z) < h(1) \quad \text{if} \quad 0<|z|<r.$$

Then, by (4) and (1),

Finally (3) implies

$$h(1) = 1 = |g(0)| \operatorname{Re}\left(\frac{1}{g}\right)'(0) = \kappa(H, 0).$$

Thus $\kappa(H, z) < \kappa(H, 0)$ if $0<|z|<r$. The second statement follows since $\kappa(-H, \cdot) = -\kappa(H, \cdot)$. □

REFERENCES

1. L. V. Ahlfors, *Conformal Invariants*. McGraw-Hill, New York, 1974 (pp. 5, 6).

2. R. M. Gabriel, *A result concerning convex level surfaces of 3-dimensional harmonic functions*. J. London Math. Soc. 32 (1957), 286–294. (MR 19-848).

3. R. M. Gabriel, *Further results concerning the level surfaces of the Green's function for a 3-dimensional convex domain. I.* J. London Math. Soc. 32 (1957), 295–302. (MR 19-848).

4. R. M. Gabriel, *Further results concerning the level surfaces of the Green's function for a 3-dimensional convex domain. II.* J. London Math. Soc. 32 (1957), 303–306. (MR 19-848).

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