108.16 Golden triangles founded on Kepler's triangle

Introduction

As is known, Kepler's triangle is a particular right-angled triangle ΔABC ($AC < AB < BC$) with $AC = 1$, $AB = \sqrt{\phi}$, $BC = \phi$, whose side-lengths are in geometric progression, with a common ratio $\sqrt{\phi}$. This triangle has some interesting properties ([1, 2, 3]) and links with other geometric figures. The purpose of this short Note is to show how it relates to the golden triangle, that is, an isosceles triangle in which the ratio between oblique side and base is equal to the golden ratio.

1. Let ΔABC be Kepler's triangle given in Figure 1a, and consider the isosceles triangle $\Delta BCC'$ obtained by the image ABC' symmetric to ABC with respect to side AB. Let H and H' be the points at which the circles centred in C, C', and of radius $BC = BC'$, respectively intersect line CC', so that $HC' = CH' = BC = \phi$.

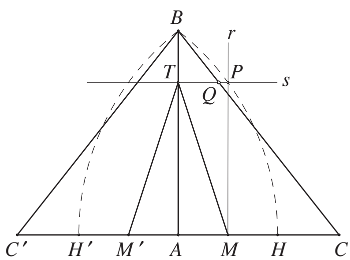

FIGURE 1a: Kepler's and BCC' triangles

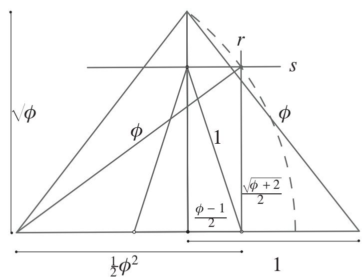

FIGURE 1b

Consider the midpoints M, M' of AH and AH' respectively, and let r be the line through M and perpendicular to CC'. Denote by P the common point of r and arc $BC'H$. Consider line s through P and perpendicular to r, so that T is the intersection of s and AB. In this way, the Golden triangle $\Delta MTM'$ is obtained, with $\frac{MT}{MM'} = \phi$. The Pythagorean theorem can be used to demonstrate this property.

As illustrated in Figure 1b, the circle centred in C' and of radius $C'B = \phi$ intersects CC' in H, implying immediately that $C'H = \phi$ and $AH = C'H - C'A = \phi - 1$.

Furthermore, the following equalities also hold:

$$AM = \frac{\phi - 1}{2}, \; C'M = C'A + AM = \frac{\phi^2}{2},$$

$$AT = MP = \sqrt{C'P^2 - C'M^2} = \sqrt{\phi^2 - \left(\frac{\phi^2}{2}\right)^2} = \frac{\sqrt{\phi + 2}}{2}.$$

As a consequence,

$$MT = \sqrt{AM^2 + AT^2} = \sqrt{\left(\frac{\phi - 1}{2}\right)^2 + \left(\frac{\sqrt{\phi + 2}}{2}\right)^2} = 1.$$

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In other words, $MT = AC$, and therefore

$$\frac{MT}{MM'} = \frac{1}{\phi - 1} = \phi,$$

which proves that $\Delta MTM'$ is a Golden triangle. Triangle $\Delta MTM'$, thus constructed, can be called the Golden triangle *associated* with Kepler's triangle ΔABC.

Since the right triangle ΔTBQ, in Figure 1a, is similar to Kepler's triangle ΔABC, the same construction can be repeated, leading to another Golden triangle, and this can be iterated *ad infinitum*.

2. The demonstrated equality $MT = AC$ allows us to construct the Golden triangle associated with Kepler's triangle, as just shown, bearing in mind that segment $AH = \phi - 1$ can be obtained as $BC - AC$.

Conversely, Kepler's triangle associated with an initial Golden triangle can also be constructed. In fact, as depicted in Figure 2, given the Golden triangle $\Delta MTM'$, having base MM', consider segments $AC = AC' = MT$. Let H be such that $AM = HM$. The circle of centre C' and radius $C'H$ intersects line AT at B. As a result, ΔABC is Kepler's triangle associated with the Golden triangle $\Delta MTM'$.

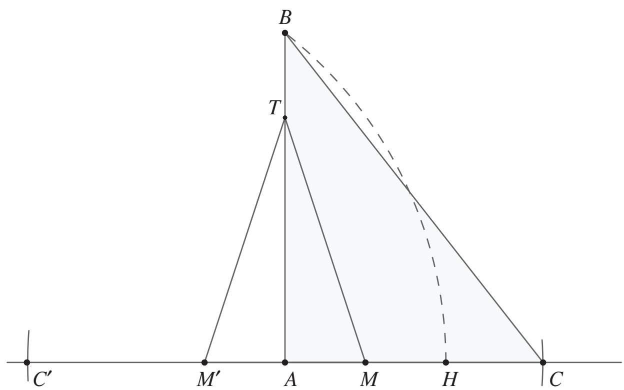

FIGURE 2: Kepler's triangle associated with the Golden triangle

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108.17 On a generalisation of the Lemoine axis

First let us briefly recall the definition of an *Apollonius* circle. Let A, B, P be three distinct points in the Euclidean plane. Then the ratio of the distances from P to A and B respectively is $\frac{|PA|}{|PB|}$. Then the locus of all points X in the plane with the property $\frac{|XA|}{|XB|} = \frac{|PA|}{|PB|}$ is a special *circle* if $\frac{|PA|}{|PB|} \neq 1$ (and, if $\frac{|PA|}{|PB|} = 1$, the perpendicular bisector of AB which can be viewed as a degenerated Apollonius circle with radius ∞). The diameter of this Apollonius circle is QR, where Q and R denote the intersection points of the angle bisector of $\angle APB$ with the straight line AB (Figure 1).

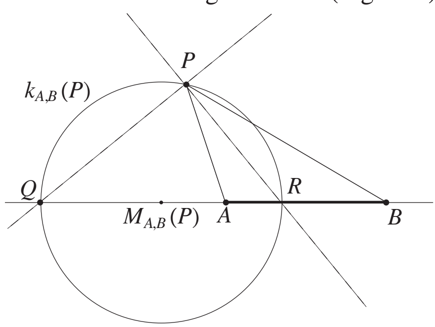

FIGURE 1: Construction of the Apollonius circle

In the following paragraphs we will denote this Apollonius circle by $k_{A,B}(P)$ and its centre by $M_{A,B}(P)$.

There is a well-known property of Apollonius circles, leading to the notion of the *Lemoine axis* (e.g. [1, pp. 294f]): Let ΔABC be a triangle and $k_{A,B}(C)$, $k_{B,C}(A)$ and $k_{C,A}(B)$ the three Apollonius circles through the vertices. Then the centres $M_{A,B}(C)$, $M_{B,C}(A)$ and $M_{C,A}(B)$ of the three circles are collinear (see Figure 2). The corresponding straight line L through these centres is called the *Lemoine axis* and the two intersection points of these circles are called the *isodynamic points* of the triangle ΔABC ($X(15)$ and $X(16)$ in Kimberling's *Encyclopedia of Triangle Centres* [2]).