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NAMIOKA SPACES AND TOPOLOGICAL GAMES

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We introduce a class of $\beta - v$-unfavourable spaces, which contains some known classes of β-unfavourable spaces for topological games of Choquet type. It is proved that every $\beta - v$-unfavourable space X is a Namioka space, that is for any compact space Y and any separately continuous function $f : X \times Y \to \mathbb{R}$ there exists a dense in X G_δ-set $A \subseteq X$ such that f is jointly continuous at each point of $A \times Y$.

1. INTRODUCTION

Investigations of the joint continuity point set of separately continuous functions were started by Baire in [2] and were continued in papers of many mathematicians. A Namioka's result [7] on a massivity of the joint continuity point set of separately continuous function on the product of two topological spaces, one of which satisfies compactness type conditions, gave a new impulse to a further investigation of this topic.

A topological space X is called *a strongly countably complete space* if there exists a sequence $(\mathcal{U}_n)_{n=1}^{\infty}$ of open coverings of X such that $\bigcap\limits_{n=1}^{\infty} F_n \neq \emptyset$ for every centred sequence $(F_n)_{n=1}^{\infty}$ of closed in X sets F_n with $F_n \subseteq U_n$ for every $n \in \mathbb{N}$ and some $U_n \in \mathcal{U}_n$.

THEOREM 1.1. (Namioka.) *Let X be a strongly countably complete space, Y be a compact space and $f : X \times Y \to \mathbb{R}$ be a separately continuous function. Then there exists a dense in X G_δ-set $A \subseteq X$ such that f is jointly continuous at each point of $A \times Y$.*

The following notions were introduced in [9].

A mapping $f : X \times Y \to \mathbb{R}$ has *the Namioka property* if there exists a dense in X G_δ-set $A \subseteq X$ such that $A \times Y \subseteq C(f)$, where $C(f)$ means the joint continuity point set of f.

A Baire space X is called *a Namioka space* if for any compact space Y, every separately continuous function $f : X \times Y \to \mathbb{R}$ has the Namioka property.

It was shown in [4] that a topological games technique can be useful in a study of Namioka spaces.

Let $\mathcal{P}$ be a system of subsets of a topological space X. Define a $G_{\mathcal{P}}$-game on X in which two players α and β participate. A nonempty open in X set U_0 is the first move

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of β and a nonempty open in X set $V_1 \subseteq U_0$ and set $P_1 \in \mathcal{P}$ are the first move of α. Further β chooses a nonempty open in X set $U_1 \subseteq V_1$ and α chooses a nonempty open in X set $V_2 \subseteq U_1$ and a set $P_2 \in \mathcal{P}$ and so on. The player α wins if

$$\left(\bigcap_{n=1}^{\infty} V_n\right) \cap \left(\overline{\bigcup_{n=1}^{\infty} P_n}\right) \neq \emptyset.$$

Otherwise β wins.

A topological space X is called *α-favourable in the $G_{\mathcal{P}}$-game* if α has a winning strategy in this game. A topological space X is called *β-unfavourable in the $G_{\mathcal{P}}$-game* if β has no winning strategy in this game. Clearly, any α-favourable topological space X is a β-unfavourable space.

In the case of $\mathcal{P} = \{X\}$ the game $G_{\mathcal{P}}$ is the classical Choquet game and X is β-unfavourable in this game if and only if X is a Baire space (see [**9**]). If $\mathcal{P}$ is the system of all finite (or one-point) subsets of X then $G_{\mathcal{P}}$-game is called a *σ-game.*

Note that Christensen in [**4**] generalising the Namioka theorem considered an s-game which is a modification of the σ-game. So, for the s-game α and β play in the same way as in the σ-game, and α wins if each subsequence $(x_{n_k})_{k=1}^{\infty}$ of sequence $(x_n)_{n=1}^{\infty}$ has a cluster point in the set $\bigcap\limits_{n=1}^{\infty} V_n$, where $P_n = \{x_n\}$. It was proved in [**4**] that any $\alpha - s$-favourable space is a Namioka space.

Saint-Raymond showed in [**9**] that for usage the topological games method in these investigations it is enough to require a weaker condition of β-unfavourability instead of the α-favourability. He proved that any $\beta - \sigma$-unfavourable space is a Namioka space and generalised the Christensen result.

A further development of this technique leads to a consideration of another topological games which based on wider systems $\mathcal{P}$ of subsets of a topological space X.

Let T be a topological space and $\mathcal{K}(T)$ be a collection of all compact subsets of T. Then T is said to be *$\mathcal{K}$-countably-determined* if there exist a subset S of the topological space $\mathbb{N}^{\mathbb{N}}$ and a mapping $\varphi : S \to \mathcal{K}(T)$ such that for every open in T set $U \subseteq T$ the set $\{s \in S : \varphi(s) \subseteq U\}$ is open in S and $T = \bigcup\limits_{s \in S} \varphi(s)$; and it is called *$\mathcal{K}$-analytical* if there exists such a mapping φ for the set $S = \mathbb{N}^{\mathbb{N}}$.

A set A in a topological space X is called *bounded* if for any continuous function $f : X \to \mathbb{R}$ the set $f(A) = \{f(a) : a \in A\}$ is bounded.

The following theorem gives further generalisations of Saint-Raymond result.

Theorem 1.2. *Any β-unfavourable in $G_{\mathcal{P}}$-game topological space X is a Namioka space if:*

(i) *$\mathcal{P}$ is the system of all compact subsets of X (see Talagrand [**11**]);*

(ii) *$\mathcal{P}$ is the system of all $\mathcal{K}$-analytical subsets of X (see Debs [**5**]);*

(iii) *$\mathcal{P}$ is the system of all bounded subsets of X (see Maslyuchenko [**6**]);*

(iv) *$\mathcal{P}$ is a system of all $\mathcal{K}$-countable-determined subsets of X (see Rybakov* [8]*).*

It is easy to see that (iv)$\Rightarrow$(ii) $\Rightarrow$(i) and (iii)$\Rightarrow$(i).

In this paper, using a technique which is related to the dependence of functions on products upon some quantity of coordinates, we prove a result which generalises (iii) and (iv).

2. Dependence of functions upon some quantity of coordinates and Namioka property

Proposition 2.1. *Let X be a topological space, $A \subseteq X$ be a dense in X set, $Y \subseteq \mathbb{R}^T$ be a topological space, $f : X \times Y \to \mathbb{R}$ be a continuous in the first variable function, $\varepsilon \geqslant 0$ and $S \subseteq T$ be such that $|f(a,y') - f(a,y'')| \leqslant \varepsilon$ for every $a \in A$ and every $y', y'' \in Y$ with $y'|_S = y''|_S$. Then $|f(x,y') - f(x,y'')| \leqslant \varepsilon$ for every $x \in X$ and every $y', y'' \in Y$ with $y'|_S = y''|_S$.*

Proof: Suppose that $y', y'' \in Y$ with $y'|_S = y''|_S$. Put

$$h' : X \to \mathbb{R},\ h'(x) = f(x,y'), \quad h'' : X \to \mathbb{R},\ h''(x) = f(x,y'').$$

Since h' and h'' are continuous, the set

$$G = \left\{x \in X : |h'(x) - h''(x)| > \varepsilon\right\}$$

is open. But $G \cap A = \emptyset$ and $\overline{A} = X$. Thus, $G = \emptyset$ and $f(x,y') = f(x,y'')$ for each $x \in X$. ∎

Proposition 2.2. *Let $Y \subseteq \mathbb{R}^T$ be a compact space, $(Z, |\cdot - \cdot|_Z)$ be a metric space, $f : Y \to Z$ be a continuous mapping, $\varepsilon \geqslant 0$ and $S \subseteq T$ be such that $|f(y') - f(y'')|_Z \leqslant \varepsilon$ for every $y', y'' \in Y$ with $y'|_S = y''|_S$. Then for every $\varepsilon' > \varepsilon$ there exist a finite set $S_0 \subseteq S$ and a real $\delta > 0$ such that $|f(y') - f(y'')|_Z \leqslant \varepsilon'$ for every $y', y'' \in Y$ with $|y'(s) - y''(s)| < \delta$ for each $s \in S_0$.*

Proof: Fix some $\varepsilon' > \varepsilon$. Suppose that the proposition is false for this ε'. Put $A = \{(R,n) : R \subseteq S$ is finite and $n \in \mathbb{N}\}$. Consider on A the following order: $(R',n') \leqslant (R'',n'')$ if $R' \subseteq R''$ and $n' \leqslant n''$. By the assumption, for every $a = (R,n) \in A$ there exist $y'_a, y''_a \in Y$ such that $|f(y'_a) - f(y''_a)|_Z > \varepsilon'$ and $|y'_a(s) - y''_a(s)| < 1/n$ for each $s \in R$. Since Y^2 is a compact, the net $(y'_a, y''_a)_{a\in A}$ has a subnet $(z'_b, z''_b)_{b\in B}$ which converges in Y^2 to some point (y', y''). The continuity of f implies $|f(y') - f(y'')|_Z \geqslant \varepsilon' > \varepsilon$. For every $a_0 = (\{s_0\}, n_0) \in A$ there exists $b_0 \in B$ such that $a \geqslant a_0$ for every $b \geqslant b_0$ where $a \in A$ is such that $(z'_b, z''_b) = (y'_a, y''_a)$. Therefore $|y'(s_0) - y''(s_0)| \leqslant 1/n_0$ for every $s_0 \in S$ and $n_0 \in \mathbb{N}$. Thus $y'(s) = y''(s)$ for every $s \in S$ and $|f(y') - f(y'')|_Z \leqslant \varepsilon$, but it is impossible. ∎

Corollary 2.3. *Let $X \subseteq \mathbb{R}^T$ be a compact space, $\varphi : X \to \mathbb{R}^S$ be a continuous mapping and $S_0 \subseteq S$ with $|S_0| \leqslant \aleph_0$. Then there exists a set $T_0 \subseteq T$ with $|T_0| \leqslant \aleph_0$ such that $\varphi(x')\big|_{S_0} = \varphi(x'')\big|_{S_0}$ for every $x', x'' \in X$ with $x'|_{S_0} = x''|_{T_0}$.*

Proof: Consider a continuous mapping $f : X \to \mathbb{R}^{S_0}$, $f(x) = \varphi(x)|_{S_0}$. Note that $\mathbb{R}^{S_0}$ is metrisable. Fix a metric d which generates the topology on $\mathbb{R}^{S_0}$. Using Proposition 2.2 for $\varepsilon = 0$ we obtain that for every $n \in \mathbb{N}$ there exists a finite set $T_n \subseteq T$ such that $d\big(f(x'), f(x'')\big) \leqslant 1/n$ for any $x', x'' \in X$ with $x'|_{T_n} = x''|_{T_n}$. The set $T_0 = \bigcup_{n=1}^{\infty} T_n$ is to be found. ∎

Let X be a topological space, (Y, d) be a metric space, $f : X \to Y$ and $A \subseteq X$ be a nonempty set. The real $\omega_f(A) = \sup_{x',x'' \in A} d\big(f(x'), f(x'')\big)$ is called *the oscillation of f on A*, and the real $\omega_f(x_0) = \inf_{U \in \mathcal{U}} \omega_f(U)$, where $\mathcal{U}$ is the system of all neighbourhoods of $x_0 \in X$ in X, is called *the oscillation of f at x_0*.

The following result illustrates relations between the Namioka property and the dependence of mappings upon some quantity of coordinates.

Theorem 2.4. *Let X be a Baire space, $Y \subseteq \mathbb{R}^T$ be a compact space and $f : X \times Y \to \mathbb{R}$ be a separately continuous function. Then the following conditions are equivalent:*

(i) *f has the Namioka property;*

(ii) *for every open in X nonempty set U and every $\varepsilon > 0$ there exist an open in X nonempty set $U_0 \subseteq U$ and a set $S_0 \subseteq T$ with $|S_0| \leqslant \aleph_0$ such that $\big|f(x, y') - f(x, y'')\big| \leqslant \varepsilon$ for every $x \in U_0$ and every $y', y'' \in Y$ with $y'|_{S_0} = y''|_{S_0}$;*

(iii) *for every open in X nonempty set U and every $\varepsilon > 0$ there exist an open in X nonempty set $U_0 \subseteq U$, a finite set $S_0 \subseteq T$ and $\delta > 0$ such that $\big|f(x, y') - f(x, y'')\big| \leqslant \varepsilon$ for every $x \in U_0$ and every $y', y'' \in Y$ with $\big|y'(s) - y''(s)\big| < \delta$, if $s \in S_0$.*

Proof: (i)$\Rightarrow$ (ii). Let f has the Namioka property and U be an open in X nonempty set. Then there exists an open in X nonempty set $U_0 \subseteq U$ such that $\big|f(x', y) - f(x'', y)\big| \leqslant \varepsilon/2$ for every $x', x'' \in U_0$ and every $y \in Y$. Pick an arbitrary point $x_0 \in U_0$. The continuous function $g : Y \to \mathbb{R}$, $g(y) = f(x_0, y)$ depends upon countable quantity of coordinates, that is there exists a set $S_0 \subseteq T$ with $|S_0| \leqslant \aleph_0$ such that $g(y') = g(y'')$ for every $y', y'' \in Y$ with $y'|_{S_0} = y''|_{S_0}$. For every $x \in U_0$ we have

$$\begin{aligned}\big|f(x,y') - f(x,y'')\big| &\leqslant \big|f(x,y') - f(x_0,y')\big| + \big|f(x_0,y') - f(x_0,y'')\big| + \big|f(x_0,y'') - f(x,y'')\big| \\ &\leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

(ii)$\Rightarrow$(iii) . Fix an open in X nonempty set U and $\varepsilon > 0$. By (ii) there exist an open in X nonempty set $U' \subseteq U$ and a set $S' \subseteq S$ with $|S'| \leqslant \aleph_0$ such that $\big|f(x, y')$

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$- f(x,y'')| \leqslant \varepsilon/2$ for every $x \in U'$ and every $y', y'' \in Y$ with $y'|_{S'} = y''|_{S'}$. Let $S' = \{s_1, s_2, \dots, s_n, \dots\}$. For every $n \in \mathbb{N}$ denote by F_n the set of all points $x \in X$ such that $|f(x,y') - f(x,y'')| \leqslant \varepsilon$ for every $y', y'' \in Y$ with $|y'(s_k) - y''(s_k)| < 1/n$ by $k = 1, 2, \dots, n$. The continuity of f in the variable x implies that all sets F_n are closed. By Proposition 2.2 the continuity of f in the variable y implies $U' \subseteq \bigcup_{n=1}^{\infty} F_n$. Since X is a Baire space, there exist an integer $n_0 \in \mathbb{N}$ and an open in X nonempty set $U_0 \subseteq U'$ such that $U_0 \subseteq \operatorname{int}(F_{n_0})$. It remains to put $\delta = 1/n_0$.

(iii)$\Rightarrow$(i). Suppose that for an open in X nonempty set U and any real $\varepsilon > 0$ there exist an open in X nonempty set $U_0 \subseteq U$ and a finite set $S_0 \subseteq T$ and a $\delta > 0$ such that $|f(x,y') - f(x,y'')| \leqslant \varepsilon$ for every $x \in U_0$ and every $y', y'' \in Y$ with $|y'(s) - y''(s)| < \delta$, if $s \in S_0$. Then for every $\varepsilon > 0$ the open set

$$G_\varepsilon = \left\{x \in X : (\forall y \in Y)\big(\omega_f(x,y) < \varepsilon\big)\right\}$$

is dense in X. Put $A = \bigcap_{n=1}^{\infty} G_{1/n}$. Clearly that f is continuous at each point of $A \times Y$. Thus, f has the Namioka property. ∎

Proposition 2.5. *Let $X \subseteq \mathbb{R}^S$ be a compact space, $f : X \to \mathbb{R}$ be a continuous function, $T \subseteq S$ be a set such that $f(x_1) = f(x_2)$ for every $x_1, x_2 \in X$ with $x_1|_T = x_2|_T$, $\varphi : X \to \mathbb{R}^T$, $\varphi(x) = x|_T$ and $Y = \varphi(X)$. Then the function $g : Y \to \mathbb{R}$, $g(x|_T) = f(x)$ is continuous.*

Proof: Let $y_0 \in Y$, $x_0 \in X$ be such that $\varphi(x_0) = y_0$ and $\varepsilon > 0$. The set

$$F = \{x \in X : |f(x) - f(x_0)| \geqslant \varepsilon\}$$

is closed in X, thus F is compact. Therefore the set $\varphi(F)$ is compact subset of Y, besides $y_0 \notin \varphi(F)$. Thus the set $V = Y \setminus \varphi(F)$ is a neighbourhood of y_0. For every $y \in V$ we have $|g(y) - g(y_0)| < \varepsilon$. Hence g is continuous at y_0. ∎

3. Valdivia compacts and the Namioka property

In this section we establish a result which we shall use in the proof of a generalisation of Theorem 1.2.

Recall that a compact space Y is called *a Valdivia compact* if Y is homeomorphic to a compact $Z \subseteq \mathbb{R}^S$ such that a set $B = \{z \in Z : |\operatorname{supp} z| \leqslant \aleph_0\}$ is dense in Z, where $\operatorname{supp} f$ means *the support* $\{x \in X : f(x) \neq 0\}$ of a function $f : X \to \mathbb{R}$.

Theorem 3.1. *Let X be a Baire space, $Y \subseteq \mathbb{R}^T$ be a Valdivia compact, $\varepsilon > 0$ and $f : X \times Y \to \mathbb{R}$ be a continuous in the firsts variable function such that $\omega_{f^x}(y) < \varepsilon$ for every $x \in X$ and every $y \in Y$, where $f^x : Y \to \mathbb{R}$, $f^x(y) = f(x,y)$. Then there exist an open in X nonempty set U_0 and a set $T_0 \subseteq T$ with $|T_0| \leqslant \aleph_0$ such that $|f(x,y') - f(x,y'')| \leqslant 3\varepsilon$ for every $x \in U_0$ and every $y', y'' \in Y$ with $y'|_{T_0} = y''|_{T_0}$.*

PROOF: Note that Y is homeomorphic to a compact $Z \subseteq \mathbb{R}^S$ such that the set $B = \{z \in Z : |\operatorname{supp} z| \leqslant \aleph_0\}$ is dense in Z. Let $\varphi : Y \to Z$ be a homeomorphism and $g : X \times Z \to \mathbb{R}$, $g(x,z) = f(x, \varphi^{-1}(z))$. Clearly, g is continuous in the first variable and $\omega_{g^x}(z) < \varepsilon$ for every $x \in X$ and every $z \in Z$, where $g^x : Z \to \mathbb{R}$, $g^x(z) = g(x,z)$.

For each $x \in X$ pick a finite covering $\mathcal{W}_x$ of Z by open in Z basic sets such that $\omega_{g^x}(W) < \varepsilon$ for every $W \in \mathcal{W}_x$. For each $W \in \mathcal{W}_x$ choose a finite set $R(W) \subseteq S$ such that for every $z', z'' \in Z$ the conditions $z' \in W$ and $z''(s) = z'(s)$ for any $s \in R(W)$ imply that $z'' \in W$. For a finite set $S_x = \bigcup\limits_{W \in \mathcal{W}_x} R(W)$ we have $|g(x,z') - g(x,z'')| < \varepsilon$ for every $z', z'' \in Z$ with $z'|_{S_x} = z''|_{S_x}$. For each $n \in \mathbb{N}$ we put $X_n = \{x \in X : |S_x| \leqslant n\}$. Since X is a Baire space, there exist an open in X nonempty set $\widetilde{U}$ and an integer $n_0 \in \mathbb{N}$ such that $\widetilde{U} \subseteq \overline{X}_{n_0}$.

Show that there exist an open in X nonempty set $U_0 \subseteq \widetilde{U}$ and a set $S_0 \subseteq S$ with $|S_0| \leqslant \aleph_0$ such that $|g(x,b') - g(x,b'')| \leqslant \varepsilon$ for every $x \in U_0$ and every $b', b'' \in B$ with $b'|_{S_0} = b''|_{S_0}$.

Assume that it is false. Pick a set $S_1 \subseteq S$ with $|S_1| \leqslant \aleph_0$ put $U_1 = \widetilde{U}$. By the assumption, there exist $x_1 \in U_1$ and $b_1, c_1 \in B$ such that $|g(x_1,b_1) - g(x_1,c_1)| > \varepsilon$ and $b_1|_{S_1} = c_1|_{S_1}$. Using the continuity of g in the first variable, we find an open in X nonempty set $U_2 \subseteq U_1$ such that $|g(x,b_1) - g(x,c_1)| > \varepsilon$ for every $x \in U_2$. Put $S_2 = S_1 \cup (\operatorname{supp} b_1) \cup (\operatorname{supp} c_1)$. By the assumption, there exist $x_2 \in U_2$ and $b_2, c_2 \in B$ such that $|g(x_2,b_2) - g(x_2,c_2)| > \varepsilon$ and $b_2|_{S_2} = c_2|_{S_2}$. Doing like that step by step n_0 times, we obtain a decreasing sequence $(U_n)_1^{n_0+2}$ of open in X nonempty sets U_n, an increasing sequence $(S_n)_{n=1}^{n_0+2}$ of at most countable sets $S_n \subseteq S$ and sequences $(b_n)_{n=1}^{n_0+1}$ and $(c_n)_{n=1}^{n_0+1}$ of points $b_n, c_n \in B$ such that the following conditions hold:

(a) $U_{n+1} \subseteq U_n$;

(b) $S_{n+1} = S_n \cup (\operatorname{supp} b_n) \cup (\operatorname{supp} c_n)$;

(c) $b_n|_{S_n} = c_n|_{S_n}$;

(d) $|g(x,b_n) - g(x,c_n)| > \varepsilon$ for each $x \in U_{n+1}$

for every $n = 1, 2, \dots, n_0 + 1$.

Since $U_{n_0+2} \subseteq U_1 = \widetilde{U} \subseteq \overline{X}_{n_0}$, $U_{n_0+2} \cap X_{n_0} \neq \emptyset$. Pick a point $x_0 \in U_{n_0+2} \cap X_{n_0}$ and fix $n \in \{1, 2, \dots, n_0+1\}$. Then $|g(x_0,b_n) - g(x_0,c_n)| > \varepsilon$ by Condition (d). The definition of S_{x_0} implies $b_n|_{S_{x_0}} \neq c_n|_{S_{x_0}}$. Besides, $b_n|_{S_n} = c_n|_{S_n}$ by Condition (c) and Condition (b) implies that $b_n|_{S \setminus S_{n+1}} = c_n|_{S \setminus S_{n+1}} \equiv 0$. Therefore $b_n|_{S \setminus (S_{n+1} \setminus S_n)} = c_n|_{S \setminus (S_{n+1} \setminus S_n)}$. Thus $S_{x_0} \not\subseteq S \setminus (S_{n+1} \setminus S_n)$, that is $S_{x_0} \cap (S_{n+1} \setminus S_n) \neq \emptyset$ for every $n = 1, 2, \dots, n_0 + 1$. But this contradicts $|S_{x_0}| \leqslant n_0$.

Now we show that $|g(x,z') - g(x,z'')| \leqslant 3\varepsilon$ for every $x \in U_0$ $z', z'' \in Z$ with $z'|_{S_0} = z''|_{S_0}$.

Fix $x \in U_0$ and $z', z'' \in Z$ with $z'|_{S_0} = z''|_{S_0}$. Since the countably compact set B is dense in Z, $\omega_{g^x}(z') < \varepsilon$, $\omega_{g^x}(z'') < \varepsilon$ and $|S_0| \leqslant \aleph_0$, there exist $b', b'' \in B$ such that

$b'|_{S_0} = z'|_{S_0}$, $b''|_{S_0} = z''|_{S_0}$, $|g(x,z') - g(x,b')| < \varepsilon$ and $|g(x,z'') - g(x,b'')| < \varepsilon$. Therefore $b'|_{S_0} = b''|_{S_0}$ and $|g(x,b') - g(x,b'')| < \varepsilon$. Thus

$$\begin{aligned}|g(x,z') - g(x,z'')| &\leqslant |g(x,z') - g(x,b')| + |g(x,b') - g(x,b'')| + |g(x,b'') - g(x,z'')| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.\end{aligned}$$

Applying Corollary 2.3 to the compact space Y, the mapping φ and the set S_0, we find an at most countable set $T_0 \subseteq T$ such that $\varphi(y')|_{S_0} = \varphi(y'')|_{S_0}$ for every $y', y'' \in Y$ with $y'|_{T_0} = y''|_{T_0}$. Then

$$|f(x,y') - f(x,y'')| = \Big|g\big(x,\varphi(y')\big) - g\big(x,\varphi(y'')\big)\Big| \leqslant 3\varepsilon$$

for every $x \in U_0$. ∎

In particular, Theorem 2.4 and Theorem 3.1 imply the following result which was obtained in [**3**, Corollary 1.2].

COROLLARY 3.2. *Any separately continuous function on the product of a Baire space and a Valdivia compact has the Namioka property, that is any Valdivia compact is a co-Namioka space.*

4. NAMIOKA SPACES AND $\beta - v$-UNFAVOURABLE SPACES

In this section we prove a generalisation of Theorem 1.2.

Let $\mathcal{P}$ be a system of subsets of topological space X with the following conditions

- (v_1) $\mathcal{P}$ is closed with respect to finite unions;
- (v_2) for every set $E \in \mathcal{P}$ and every compact $Y \subseteq C_p(X)$ a compact $\varphi(Y)$ is a Valdivia compact, where $\varphi : C_p(X) \to C_p(E)$, $\varphi(y) = y|_E$.

A topological space X is called *$\beta - v$-unfavourable* if the player β has no winning strategy in the $G_{\mathcal{P}}$-game for some system $\mathcal{P}$ with (v_1) and (v_2).

Recall that a compact space Y is called *an Eberlein compact* if Y is homeomorphic to a compact subset of $C_p(X)$ for some compact X. A compact space Y is called *a Corson compact* if Y is homeomorphic to a compact $Z \subseteq \mathbb{R}^S$ such that $|\operatorname{supp} z| \leqslant \aleph_0$ for every $z \in Z$. It is known that any Eberlein compact is a Corson compact and clearly that any Corson compact is a Valdivia compact.

PROPOSITION 4.1. *Let X be a topological space and $\mathcal{K}$ be a system of all nonempty sets $E \subseteq X$ such that for every compact $Y \subseteq C_p(X)$ a compact $\varphi(Y)$ is a Corson compact, where $\varphi : C_p(X) \to C_p(E)$, $\varphi(y) = y|_E$. Then $\mathcal{K}$ has (v_1) and (v_2).*

PROOF: Let $E_1, E_2 \in \mathcal{K}$, $E = E_1 \cup E_2$, then

$$\begin{aligned}\varphi_1 &: C_p(X) \to C_p(E_1), \quad \varphi_1(y) = y|_{E_1},\\ \varphi_2 &: C_p(X) \to C_p(E_2), \quad \varphi_2(y) = y|_{E_2},\\ \varphi &: C_p(X) \to C_p(E), \quad \varphi(y) = y|_E\end{aligned}$$

and $Y \subseteq C_p(X)$ be a compact. Note that a mapping $\psi : \varphi(Y) \to \varphi_1(Y) \times \varphi_2(Y)$, $\psi(y) = (y|_{E_1}, y|_{E_2})$, is a homeomorphic embedding. Therefore the compact $\varphi(Y)$ is a Corson compact. Thus $\mathcal{K}$ has (v_1).

The property (v_2) of system $\mathcal{K}$ is obvious. ∎

Proposition 4.2. *Any β-unfavourable in $G_{\mathcal{P}}$-game topological space X, where $\mathcal{P}$ is the system of all bounded subsets of X or $\mathcal{P}$ is a system of all $\mathcal{K}$-countably-determined subsets of X, is a $\beta - v$-unfavourable space.*

Proof: Let E be a bounded set in a topological space X and $Y \subseteq C_p(X)$ be a compact. Consider a continuous mapping $\psi : X \to C_p(Y)$. Clearly, the set $T = \psi(E)$ is bounded in $C_p(Y)$. Therefore by [**1**, Theorem III.4.1] the closure $\overline{T}$ of T in $C_p(Y)$ is a compact. Then a compact $Z = \psi_1(Y)$, where $\psi_1 : Y \to C_p(T)$, $\psi_1(y)(t) = t(y)$ for every $y \in Y$ and $t \in T$, is an Eberlein compact, because Z is homeomorphic to a compact subset of $C_p(\overline{T})$. Since compacts Z and $\varphi(Y)$, where $\varphi : C_p(X) \to C_p(E)$, $\varphi(y) = y|_E$, are homeomorphic, $\varphi(Y)$ is an Eberlein compact, in particular, $\varphi(Y)$ is a Corson compact.

It follows analogously from [**10**, Theorem 3.7] that for every $\mathcal{K}$-countably-determined set $E \subseteq X$ and every compact $Y \subseteq C_p(X)$ a compact $\varphi(Y)$, where $\varphi : C_p(X) \to C_p(E)$, $\varphi(y) = y|_E$, is a Corson compact.

Thus the systems $\mathcal{P}_1$ of all bounded subsets and $\mathcal{P}_2$ of all $\mathcal{K}$-countable-determined subsets of the topological space X are contained in the system $\mathcal{K}$, by Proposition 4.1. Therefore any β-unfavourable space in the $G_{\mathcal{P}_1}$-game or in the $G_{\mathcal{P}_2}$-game is a β-unfavourable in $G_{\mathcal{K}}$-game and it is $\beta - v$-unfavourable by Proposition 4.1. ∎

Theorem 4.3. *Any $\beta - v$-unfavourable space is a Namioka space.*

Proof: Let X be a $\beta - v$-unfavourable space. Then there exists a system $\mathcal{P}$ of subsets E of topological space X which satisfies (v_1) and (v_2) and such that X is β-unfavourable in the $G_{\mathcal{P}}$-game.

Assume that X is not a Namioka space. Then there exist a compact space Y and a separately continuous function $f : X \times Y \to \mathbb{R}$ which does not have the Namioka property. Consider a continuous mapping $\varphi : Y \to C_p(X)$, $\varphi(y)(x) = f(x, y)$. Put $Z = \varphi(Y)$. Clearly, Z is a compact subspace of $\mathbb{R}^X$ and a separately continuous mapping $g : X \times Z \to \mathbb{R}$, $g(x, z) = z(x)$, does not have the Namioka property. Note that X is a β-unfavourable space in the Choquet game, that is, X is a Baire space. Therefore by Theorem 2.4 there exist an open in X nonempty set U_0 and an $\varepsilon > 0$ such that for every open in X nonempty set $U \subseteq U_0$ and every at most countable set $A \subseteq X$ there exist $x \in U$ and $z', z'' \in Z$ such that $z'|_A = z''|_A$ and $|g(x, z') - g(x, z'')| > \varepsilon$.

Show that for every set $E \in \mathcal{P}$ the set $F(E) = \Big\{x \in U_0 : |g(x, z') - g(x, z'')| \leqslant \varepsilon/8$ for every $z', z'' \in Z$ with $z'|_E = z''|_E\Big\}$ is nowhere dense in U_0.

Suppose that it is false. Since U_0 is a Baire space with the topology induced by X and by Proposition 2.1 all sets $F(E)$ are closed in U_0, there exist a set $E_0 \in \mathcal{P}$ and an open in X nonempty set $V_0 \subseteq U_0$ such that $V_0 \subseteq F(E_0)$. Suppose $\psi : Z \to \mathbb{R}^{E_0}$, $\psi(z) = z|_{E_0}$ and $\widetilde{Z} = \psi(Z)$. Since the system $\mathcal{P}$ satisfies (v_2), $\widetilde{Z}$ is a Valdivia compact. For every $\widetilde{z} \in \widetilde{Z}$ choose a point $\tau(\widetilde{z}) \in Z$ such that $\psi(\tau(\widetilde{z})) = \widetilde{z}$. Consider a mapping $h : V_0 \times \widetilde{Z} \to \mathbb{R}$, $h(x, \widetilde{z}) = g(x, \tau(\widetilde{z}))$. Since g is continuous in the first variable, h is continuous in the first variable.

Fix $x_0 \in V_0$ and $\widetilde{z}_0 \in \widetilde{Z}$. The set

$$B = \left\{ z \in Z : \left| g(x_0, z) - g(x_0, \tau(\widetilde{z}_0)) \right| \geqslant \varepsilon/4 \right\}$$

is a compact subset of Z. Besides, $x_0 \in F(E_0)$ implies $\widetilde{z}_0 \notin \psi(B)$. Since ψ is continuous, the set $\psi(B)$ is a compact subset of $\widetilde{Z}$. Thus $\psi(B)$ is a closed subset of $\widetilde{Z}$. Therefore the set $\widetilde{W} = \widetilde{Z} \setminus \psi(B)$ is a neighbourhood of $\widetilde{z}_0$. Then $\tau(\widetilde{z}) \notin B$ for every $\widetilde{z} \in \widetilde{W}$, that is $\left| g(x_0, \tau(\widetilde{z})) - g(x_0, \tau(\widetilde{z}_0)) \right| = |h(x_0, \widetilde{z}) - h(x_0, \widetilde{z}_0)| < \varepsilon/4$ for every $\widetilde{z} \in \widetilde{W}$. Hence $\omega_{h^{x_0}}(z_0) \leqslant \varepsilon/4$, where $h^{x_0} : \widetilde{Z} \to \mathbb{R}$, $h^{x_0}(\widetilde{z}) = h(x_0, \widetilde{z})$.

Thus, h satisfies the conditions of Theorem 3.1. Therefore there exist an open in X nonempty set $\widetilde{U} \subseteq V_0$ and an at most countable set $A_0 \subseteq E_0$ such that $|h(x, \widetilde{z}') - h(x, \widetilde{z}'')| \leqslant 3\varepsilon/4$ for every $x \in \widetilde{U}$ and every $\widetilde{z}', \widetilde{z}'' \in \widetilde{Z}$ with $\widetilde{z}'|_{A_0} = \widetilde{z}''|_{A_0}$.

Pick arbitrary points $x \in \widetilde{U}$ and $z', z'' \in Z$ such that $z'|_{A_0} = z''|_{A_0}$. Put $\widetilde{z}' = \psi(z')$ and $\widetilde{z}'' = \psi(z'')$. Clearly, $\widetilde{z}'|_{A_0} = \widetilde{z}''|_{A_0}$. Therefore $|h(x, \widetilde{z}') - h(x, \widetilde{z}'')| \leqslant 3\varepsilon/4$. Since

$$z'|_{E_0} = \tau(\widetilde{z}')|_{E_0}, \; z''|_{E_0} = \tau(\widetilde{z}'')|_{E_0}$$

and

$$x \in \widetilde{U} \subseteq V_0 \subseteq F(E_0), \; \left| g(x, z') - g(x, \tau(\widetilde{z}')) \right| = |g(x, z') - h(x, \widetilde{z}')| \leqslant \varepsilon/8$$

and

$$\left| g(x, z'') - g(x, \tau(\widetilde{z}'')) \right| = |g(x, z'') - h(x, \widetilde{z}'')| \leqslant \varepsilon/8.$$

Then

$$\begin{aligned} |g(x, z') - g(x, z'')| &\leqslant |g(x, z') - h(x, \widetilde{z}')| + |h(x, \widetilde{z}') - h(x, \widetilde{z}'')| + |h(x, \widetilde{z}'') - g(x, z'')| \\ &\leqslant \frac{\varepsilon}{8} + \frac{3\varepsilon}{4} + \frac{\varepsilon}{8} = \varepsilon. \end{aligned}$$

But this contradicts the choice of U_0.

Thus the set $F(E)$ is nowhere dense in U_0 for every $E \in \mathcal{P}$.

Describe a strategy for the player β in the $G_{\mathcal{P}}$-game. The set U_0 is the first move of β. Let $(V_1, \widetilde{E_1})$ be the first move of α, where $V_1 \subseteq U_0$ is an open in X nonempty set and $\widetilde{E}_1 \in \mathcal{P}$. Then $U_1 = V_1 \setminus F(E_1)$ is the second move of β, where $E_1 = \widetilde{E}_1$. If $V_2 \subseteq U_1$ is an open in X nonempty set and $\widetilde{E}_2 \in \mathcal{P}$ then $U_2 = V_2 \setminus F(E_2)$ where $E_2 = E_1 \cup \widetilde{E}_2$. Continuing the procedure of choice by the obvious manner, we obtain decreasing sequences $(U_n)_{n=0}^{\infty}$ and $(V_n)_{n=1}^{\infty}$ of open in X nonempty sets U_n and V_n and an increasing sequence $(E_n)_{n=1}^{\infty}$ of sets $E_n \in \mathcal{P}$ such that $V_n \subseteq U_{n-1}$, $U_n = V_n \setminus F(E_n)$ and $\widetilde{E}_n \subseteq E_n$ for every $n \in \mathbb{N}$, where $\widetilde{E}_n \in \mathcal{P}$ is the corresponding part of the n-th move of α.

Put $E = \bigcup\limits_{n=1}^{\infty} E_n$. Clearly, $\bigcup\limits_{n=1}^{\infty} \widetilde{E}_n \subseteq E$. Pick a point $x_0 \in \overline{E}$. Note that $g(x_0, z') = g(x_0, z'')$ for every $z', z'' \in Z$ with $z'|_E = z''|_E$, that is the continuous function $g^{x_0} : Z \to \mathbb{R}$, $g^{x_0}(z) = g(x_0, z)$, is concentrated on E. Using Proposition 2.2, we obtain that there exists a finite set $A \subseteq E$ such that $\left|g(x_0, z') - g(x_0, z'')\right| < \varepsilon/8$ for every $z', z'' \in Z$ with $z'|_A = z''|_A$. Pick $n_0 \in \mathbb{N}$ such that $A \subseteq E_{n_0}$. Then $x_0 \in \overline{F(E_{n_0})}$ therefore $x_0 \notin U_{n_0}$. Thus $x_0 \notin \bigcap\limits_{n=0}^{\infty} U_n$ and $\overline{E} \bigcap \left(\bigcap\limits_{n=0}^{\infty} U_n \right) = \emptyset$. In particular, $\left(\overline{\bigcup\limits_{n=1}^{\infty} \widetilde{E}_n} \right) \bigcap \left(\bigcap\limits_{n=0}^{\infty} U_n \right) = \emptyset$. Hence the strategy described above is a winning strategy for β in the $G_{\mathcal{P}}$-game, but it is impossible.

Thus, our assumption is false and the theorem is proved. □

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