Proceedings of the Edinburgh Mathematical Society (2015) **58**, 81–106
DOI:10.1017/S0013091513000679

REDUCTION OF HOMOGENEOUS RIEMANNIAN STRUCTURES

M. CASTRILLÓN LÓPEZ[1] AND I. LUJÁN[2]

[1] *ICMAT (CSIC-UAM-UC3M-UCM), Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain* (mcastri@mat.ucm.es)

[2] *Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain* (ilujan@mat.ucm.es)

(Received 17 March 2012)

Abstract The goal of this paper is the study of homogeneous Riemannian structure tensors within the framework of reduction under a group H of isometries. In a first result, H is a normal subgroup of the group of symmetries associated with the reducing tensor $\bar{S}$. The situation when H is any group acting freely is analyzed in a second result. The invariant classes of homogeneous tensors are also investigated when reduction is performed. It turns out that the geometry of the fibres is involved in the preservation of some of them. Some classical examples illustrate the theory. Finally, the reduction procedure is applied to fibrings of almost contact manifolds over almost Hermitian manifolds. If the structure is, moreover, Sasakian, the obtained reduced tensor is homogeneous Kähler.

Keywords: connection; contact and Hermitian structures; homogeneous structures; fibre bundle; reduction

2010 *Mathematics subject classification:* Primary 53C30
Secondary 53D35; 22F30

1. Introduction

Since their introduction [**2**], homogeneous structure tensors have proved to be a powerful tool in the study of homogeneous Riemannian manifolds. Their nature is twofold. On one hand, they belong to the tensor algebra. In particular, representation theory techniques classify them into eight different invariant classes with respect to a convenient action of the orthogonal group. On the other hand, homogeneous tensors satisfy a system of partial differential equations (Ambrose–Singer equations). Many works in the literature combine these aspects to provide geometric properties of the underlying Riemannian manifold. The first characterizations were given to hyperbolic space and naturally reductive spaces (see [**18**]). These techniques were subsequently generalized to Riemannian manifolds with special holonomy by many authors (see, for example, [**1**, **4**, **6**, **10**, **11**]). It is interesting to point out that there is no bijection between tensors and possible groups acting isometrically and transitively. The same tensor can be defined by two different groups and the same group can provide different tensors. In this context, it is remarkable how little is known about all homogeneous structures and tensors for even well-known spaces. There is still much work to do.

Manifolds endowed with symmetries are relevant in many situations. In particular, symmetries represent a classical tool in reduction schemes intimately related to different topics such as systems of differential equations, variational principles, symplectic or other geometric structures, etc. In particular, reduction is recurrently applied in homogeneous manifolds. The goal of this paper is the study of the behaviour of homogeneous tensors by reduction under subgroups of the group of isometries. In particular, this gives rise to new homogeneous tensors in the orbit space of the action. Additionally, the reduction process reveals and sheds light on some previously known properties of some homogeneous structures. Finally, the reduction technique opens up a reverse way to get new homogeneous tensors in the unreduced space from tensors in the orbit space.

The paper has the following structure. In § 2 we recall basic definitions on homogeneous structure tensors and their classification. Moreover, the model for reduction is a Riemannian principal bundle $\bar{M} \to M$, endowed with the compatible connection defined as the orthogonal complements to the fibres. This connection is ubiquitously used for reduction schemes in mechanics (see, for example, [**14, 15**]), where it is called the mechanical connection. Section 3 begins with reduction of homogeneous tensors $\bar{S}$ on $\bar{M}$ by the action under a normal subgroup H of the group of symmetries $\bar{G}$ associated with $\bar{S}$ (see Theorem 3.4). The space of all tensors $\bar{S}$ projecting to the same tensor S on $M = \bar{M}/H$ is also determined. The expression of the reduced tensors leads to a generalization of the reduction result (see Theorem 3.7) to the case where $\bar{S}$ is not explicitly associated with a precise group $\bar{G}$. For example, this is the case of non-simply connected or incomplete manifolds where the existence of homogenous tensors still provides interesting geometric properties. Without the presence of the group $\bar{G}$, the normality of the structure group H of the bundle $\bar{M} \to M$ needs to be replaced by a suitable differential condition on the mechanical connection. Finally, the behaviour of the classification of homogeneous tensors under the reduction process is analyzed. It is interesting to point out that the geometry of the orbits of the H-action is involved in some of the classes in this classification. Section 4 provides many examples of the main results of the paper. In particular, they explore the possible scenarios with respect to the classes when reduction is performed. Section 5 applies the reduction theorem to fibrings of almost contact manifolds over almost Hermitian manifolds (see [**16**]). It turns out that the differential condition on the mechanical connection is automatically satisfied for homogeneous almost contact or Sasakian tensors. Hence, they project to homogeneous almost Hermitian or Kähler tensors in a natural way. This is connected with other constructions found in the literature (see [**8**]).

2. Preliminaries

2.1. Homogeneous Riemannian structures

Let (M, g) be a connected Riemannian manifold of dimension n. Let ∇ be the Levi-Civita connection of g and let R be its curvature tensor with the convention

$$R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

A *homogeneous Riemannian structure* on (M,g) is a $(1,2)$-tensor field S satisfying the so-called *Ambrose–Singer equations*

$$\tilde{\nabla} g = 0, \qquad \tilde{\nabla} R = 0, \qquad \tilde{\nabla} S = 0, \tag{2.1}$$

where $\tilde{\nabla} = \nabla - S$ [**18**]. We also denote by S the associated $(0,3)$-tensor field obtained by lowering the contravariant index, $S_{XYZ} = g(S_X Y, Z)$.

We now suppose that (M,g) is homogeneous Riemannian. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ acting effectively and transitively on M by isometries. And let K be the isotropy group at a point $x \in M$ with Lie algebra $\mathfrak{k}$. A decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ is said to be a *reductive decomposition* of $\mathfrak{g}$ if $\mathrm{Ad}(K)(\mathfrak{m}) \subset \mathfrak{m}$. Let μ be the infinitesimal action of $\mathfrak{g}$ at the point x, that is,

$$\begin{aligned} \mu \colon \mathfrak{g} &\to T_x M \\ \xi &\mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varPhi_{\exp(t\xi)}(x), \end{aligned}$$

where $\varPhi_a$ denotes the action of an element $a \in G$. Then, for all $k \in K$, the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \mu\ } & T_x M \\ {\scriptstyle \mathrm{Ad}(k)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle (\varPhi_k)_*} \\ \mathfrak{g} & \xrightarrow{\ \mu\ } & T_x M \end{array} \tag{2.2}$$

The restriction of μ to $\mathfrak{m}$ gives an isomorphism $\mu \colon \mathfrak{m} \to T_x M$, and the canonical connection $\tilde{\nabla}$ (see [**12**]) with respect to the reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ is determined by its value at x:

$$(\tilde{\nabla}_X Y)_x = \mu([\mu^{-1}(X), \mu^{-1}(Y)]_{\mathfrak{m}}), \quad X, Y \in T_x M. \tag{2.3}$$

The tensor field $S = \nabla - \tilde{\nabla}$ is the homogeneous Riemannian structure associated with the reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$.

The Ambrose–Singer theorem states that a connected, simply connected and complete Riemannian manifold is homogeneous Riemannian if and only if it admits a homogeneous structure tensor. In the case where (M,g) is just a connected Riemannian manifold, the existence of a homogeneous structure tensor implies that (M,g) is locally homogeneous. Tricerri and Vanhecke [**18**] gave a classification of the homogeneous Riemannian structure tensors in eight invariant classes: the class $\{S = 0\}$ of symmetric structures, the total space denoted by $\mathcal{S}$, three irreducible classes under the action of the group $\mathrm{O}(n)$,

$$\begin{aligned} \mathcal{S}_1 &= \{S \in \mathcal{S} / S_{XYZ} = g(X,Y)\varphi(Z) - g(X,Z)\varphi(Y),\ \varphi \in \varGamma(T^*M)\}, \\ \mathcal{S}_2 &= \left\{S \in \mathcal{S} \middle/ \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0,\ c_{12}(S) = 0\right\}, \\ \mathcal{S}_3 &= \{S \in \mathcal{S} / S_{XYZ} + S_{YXZ} = 0\}, \end{aligned}$$

https://doi.org/10.1017/S0013091513000679 Published online by Cambridge University Press

and their direct sums,

$$\mathcal{S}_1 \oplus \mathcal{S}_2 = \left\{ S \in \mathcal{S} \Big/ \underset{XYZ}{\mathfrak{S}} S_{XYZ} = 0 \right\},$$
$$\mathcal{S}_1 \oplus \mathcal{S}_3 = \{ S \in \mathcal{S} / S_{XYZ} + S_{YXZ} = 2g(X,Y)\varphi(Z) - g(X,Z)\varphi(Y) - g(Y,Z)\varphi(X), \quad \varphi \in \Gamma(T^*M) \},$$
$$\mathcal{S}_2 \oplus \mathcal{S}_3 = \{ S \in \mathcal{S} / c_{12}(S) = 0 \},$$

where $c_{12}(S)_p(Z) = \sum_i S_{e_i e_i Z}$ for any orthonormal base $\{e_i\}_{i=1,\dots,n}$ of T_pM.

2.2. The reduced metric in a principal bundle

Let $\pi\colon \bar{M} \to M$ be an H-principal bundle, where $\bar{M}$ is a Riemannian manifold with metric $\bar{g}$, and H acts on $\bar{M}$ by isometries. Although it is not essential, the action of H is understood as left and, hence, π is a left principal bundle. Let $\bar{x} \in \bar{M}$ and let $V_{\bar{x}}\bar{M}$ denote the vertical subspace at $\bar{x}$. If we take the orthogonal complement $H_{\bar{x}}\bar{M} = (V_{\bar{x}}\bar{M})^{\perp}$ of $V_{\bar{x}}\bar{M}$ in $T_{\bar{x}}\bar{M}$ with respect to the metric $\bar{g}$, we have that

$$T_{\bar{x}}\bar{M} = V_{\bar{x}}\bar{M} \oplus H_{\bar{x}}\bar{M}. \tag{2.4}$$

Moreover, as H acts by isometries, the horizontal subspaces $H_{\bar{x}}\bar{M}$ are preserved by the action of H, and the decomposition (2.4) leads to the so-called *mechanical connection* in the principal bundle $\bar{M} \to M$. In this situation there is a unique Riemannian metric g in M such that the restriction $\pi_* : H_{\bar{x}}\bar{M} \to T_{\pi(\bar{x})}M$ is an isometry at every $\bar{x} \in \bar{M}$. Obviously, the metric g satisfies

$$g(X,Y) \circ \pi = \bar{g}(X^{\mathrm{H}}, Y^{\mathrm{H}}) \quad \forall X, Y \in \mathfrak{X}(M), \tag{2.5}$$

where X^{H} and Y^{H} denote the horizontal lift of X and Y, respectively, with respect to the mechanical connection. To complete the notation, in the following, for a vector $Z \in T_{\bar{x}}\bar{M}$, we denote by $Z^h \in H_{\bar{x}}\bar{M}$ the horizontal part of Z with respect to the mechanical connection. In particular,

$$Z^h = (\pi_*(Z))^{\mathrm{H}}. \tag{2.6}$$

Proposition 2.1. *In the situation above, if $\bar{\nabla}$ is the Levi-Civita connection for the metric $\bar{g}$, then the Levi-Civita connection ∇ for the reduced metric g is given by*

$$\nabla_X Y = \pi_*(\bar{\nabla}_{X^{\mathrm{H}}} Y^{\mathrm{H}}) \quad \forall X, Y \in \mathfrak{X}(M). \tag{2.7}$$

Proof. Since the structure group H acts by isometries, it also acts by affine transformations of $\bar{\nabla}$. Thus, the vector field $\bar{\nabla}_{X^{\mathrm{H}}} Y^{\mathrm{H}}$ is projectable and the operator $D_X Y = \pi_*(\bar{\nabla}_{X^{\mathrm{H}}} Y^{\mathrm{H}})$ is well defined. It is a direct computation to show that D fulfils the properties of a linear connection in M. For $X, Y, Z \in \mathfrak{X}(M)$, from (2.5) and (2.6) we have that

$$\begin{aligned} g(D_X Y, Z) \circ \pi + g(Y, D_X Z) \circ \pi &= \bar{g}((\bar{\nabla}_{X^{\mathrm{H}}} Y^{\mathrm{H}})^h, Z^{\mathrm{H}}) + \bar{g}(Y^{\mathrm{H}}, (\bar{\nabla}_{X^{\mathrm{H}}} Z^{\mathrm{H}})^h) \\ &= \bar{g}(\bar{\nabla}_{X^{\mathrm{H}}} Y^{\mathrm{H}}, Z^{\mathrm{H}}) + \bar{g}(Y^{\mathrm{H}}, \bar{\nabla}_{X^{\mathrm{H}}} Z^{\mathrm{H}}) \\ &= X^{\mathrm{H}}(\bar{g}(Y^{\mathrm{H}}, Z^{\mathrm{H}})). \end{aligned}$$

Hence, $g(D_XY,Z)+g(Y,D_XZ)=X(g(Y,Z))$ and the connection D is metric. Finally, as $[X,Y]^{\mathrm{H}}=[X^{\mathrm{H}},Y^{\mathrm{H}}]^h$, the torsion tensor of D is

$$\begin{aligned}T(X,Y)&=D_XY-D_YX-[X,Y]\\&=\pi_*(\bar{\nabla}_{X^{\mathrm{H}}}Y^{\mathrm{H}}-\bar{\nabla}_{Y^{\mathrm{H}}}X^{\mathrm{H}}-[X^{\mathrm{H}},Y^{\mathrm{H}}])\\&=0,\end{aligned}$$

and D is the Levi-Civita connection for g. □

3. Main results

3.1. Reduction by a normal subgroup of isometries

Let $(\bar{M},\bar{g})$ be a homogeneous Riemannian manifold. Let $\bar{G}$ be a group of isometries acting transitively on $\bar{M}$ and let $H\triangleleft\bar{G}$ be a normal subgroup acting freely on $\bar{M}$. The quotient $M=\bar{M}/H$ is thus endowed (see [**13**, Theorem 9.16]) with a smooth structure such that $\pi\colon\bar{M}\to M$ is an H-principal bundle. By definition, the bundle $\pi\colon\bar{M}\to M$ is equipped with the mechanical connection and M is Riemannian with the reduced metric g as in (2.5). Since H is normal, there is a well-defined action of the group $G=\bar{G}/H$ on M given by

$$\begin{aligned}\Phi\colon G\times M&\to M\\([\bar{a}],[\bar{x}])&\mapsto\Phi_{[\bar{a}]}([\bar{x}])=[\Phi_{\bar{a}}(\bar{x})],\end{aligned}\tag{3.1}$$

where $[\bar{a}]$ and $[\bar{x}]$ denotes the classes modulo H of $\bar{a}\in\bar{G}$ and $\bar{x}\in\bar{M}$, respectively, and $\Phi_{\bar{a}}$ denotes the action of $\bar{G}$ on $\bar{M}$. The action of G is obviously transitive, but need not be effective. If it is not, we replace G by G/N, where N is the kernel of the map $G\to\mathrm{Iso}(M)$, $a\mapsto\Phi_a$, $a\in G$.

Proposition 3.1. *The group G acts on (M,g) by isometries.*

Proof. The action (3.1) can be written as $\pi\circ\Phi_{\bar{a}}=\Phi_a\circ\pi$ for $a=[\bar{a}]$. This implies that $\bar{G}$ preserves vertical subspaces and, acting by isometries, also preserves their horizontal complements. Hence, the horizontal lift of $(\Phi_a)_*(X)$ is $(\Phi_{\bar{a}})_*(X^{\mathrm{H}})$ for all $X\in\mathfrak{X}(M)$. In addition, for $X,Y\in\mathfrak{X}(M)$,

$$\begin{aligned}g((\Phi_a)_*(X),(\Phi_a)_*(Y))\circ\pi&=\bar{g}((\Phi_a)_*(X)^{\mathrm{H}},(\Phi_a)_*(Y)^{\mathrm{H}})\\&=\bar{g}((\Phi_{\bar{a}})_*(X^{\mathrm{H}}),(\Phi_{\bar{a}})_*(Y^{\mathrm{H}}))\\&=\bar{g}(X^{\mathrm{H}},Y^{\mathrm{H}})\\&=g(X,Y)\circ\pi,\end{aligned}$$

and then Φ_a is an isometry. □

From this last proposition, the manifold (M,g) is homogeneous Riemannian. We call it the *reduced homogeneous Riemannian manifold*.

Remark 3.2. Note that Proposition 3.1 shows that the horizontal distribution is invariant by $\bar{G}$. This means that the mechanical connection is $\bar{G}$-invariant, an important fact that is used in § 3.3.

Let $\bar{x} \in \bar{M}$ and let $x = \pi(\bar{x}) \in M$. We denote by $\bar{K}$ the isotropy group of $\bar{x}$ under the action of $\bar{G}$, and by K the corresponding isotropy group of x under the action of G. We also denote their Lie algebras by $\bar{\mathfrak{k}}$ and $\mathfrak{k}$, respectively. We then have the following.

Lemma 3.3. *Let* $\tau\colon \bar{G} \to G$ *be the quotient homomorphism. Then,* $K = \tau(\bar{K})$, $H \cap \bar{K} = \{e\}$, *and the restriction* $\tau|_{\bar{K}}\colon \bar{K} \to K$ *is an isomorphism of groups.*

Proof. It is obvious from (3.1) that $\tau(\bar{K}) \subset K$. Now let $k \in K$ and take $\bar{a} \in \bar{G}$ such that $k = \tau(\bar{a})$. Then, for any $x \in M$, we have $x = \Phi_k(x) = \pi(\Phi_{\bar{a}}(\bar{x}))$, and then $\Phi_{\bar{a}}(\bar{x})$ is in the same fibre as $\bar{x}$. Hence, there exists $h \in H$ such that $\Phi_h \circ \Phi_{\bar{a}}(\bar{x}) = \bar{x}$, so $h\bar{a} \in \bar{K}$. Since $\tau(h\bar{a}) = \tau(\bar{a}) = k$, we have $k \in \tau(\bar{K})$. For the injectivity of $\tau|_{\bar{K}}$, let $\bar{k}_1, \bar{k}_2 \in \bar{K}$ such that $\tau(\bar{k}_1) = \tau(\bar{k}_2)$. There exists $h \in H$ such that $h\bar{k}_1 = \bar{k}_2$. Then $\bar{k}_1^{-1}h\bar{k}_1 = \bar{k}_1^{-1}\bar{k}_2$, so $\bar{k}_1^{-1}\bar{k}_2 \in \bar{K} \cap H$. But since H acts freely, $\bar{k}_1^{-1}\bar{k}_2 = \bar{e}$, and then $\bar{k}_1 = \bar{k}_2$. □

Theorem 3.4. *Let* $(\bar{M}, \bar{g})$ *be a connected homogeneous Riemannian manifold and let* $\bar{G}$ *be a group of isometries acting transitively and effectively in* $\bar{M}$*. Let* $H \lhd \bar{G}$ *be a normal subgroup acting freely in* $\bar{M}$*. Every homogeneous structure tensor* $\bar{S}$ *associated with* $\bar{G}$ *then induces a homogeneous structure tensor* S *associated with* $G = \bar{G}/H$ *in the reduced Riemannian manifold* $M = \bar{M}/H$.

Proof. Let $\bar{x} \in \bar{M}$ and $x = \pi(\bar{x}) \in M$, and let $\bar{\mathfrak{g}}$ be the Lie algebra of $\bar{G}$. For any reductive decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ associated with $\bar{S}$, the restriction isomorphism $\bar{\mu}\colon \bar{\mathfrak{m}} \to T_{\bar{x}}\bar{M}$ induces from $\bar{g}$ a positive definite bilinear form B in $\bar{\mathfrak{m}}$. Moreover, by the commutativity of (2.2) the bilinear form B is $\mathrm{Ad}(\bar{K})$-invariant, that is,

$$B(\mathrm{Ad}(\bar{k})\xi, \mathrm{Ad}(\bar{k})\eta) = B(\xi, \eta) \quad \forall \bar{k} \in \bar{K}.$$

Then, (2.4) induces an orthogonal and $\mathrm{Ad}(\bar{K})$-invariant decomposition

$$\bar{\mathfrak{m}} = \bar{\mathfrak{m}}^v \oplus \bar{\mathfrak{m}}^h,$$

i.e. $\mathrm{Ad}(\bar{K})(\bar{\mathfrak{m}}^v) \subset \bar{\mathfrak{m}}^v$ and $\mathrm{Ad}(\bar{K})(\bar{\mathfrak{m}}^h) \subset \bar{\mathfrak{m}}^h$.

Let $\mathfrak{g} = \bar{\mathfrak{g}}/\mathfrak{h}$ be the Lie algebra of G and let $\mu\colon \mathfrak{g} \to T_xM$ be the corresponding infinitesimal action at x. For any $\bar{\xi} \in \bar{\mathfrak{g}}$, by (3.1) we have that

$$\begin{aligned}
\pi_* \circ \bar{\mu}(\bar{\xi}) &= \pi_*\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{\exp(t\bar{\xi})}(\bar{x})\right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\pi \circ \Phi_{\exp(t\bar{\xi})})(\bar{x}) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{\tau(\exp(t\bar{\xi}))}(\pi(\bar{x}))
\end{aligned}$$

$$= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{\exp(t\tau_*(\bar{\xi}))}(x)$$
$$= \mu \circ \tau_*(\bar{\xi}),$$

which means that the following diagram is commutative:

$$\begin{array}{ccc} \bar{\mathfrak{g}} & \xrightarrow{\bar{\mu}} & T_{\bar{x}}\bar{M} \\ {\scriptstyle\tau_*}\downarrow & \circlearrowleft & \downarrow{\scriptstyle\pi_*} \\ \mathfrak{g} & \xrightarrow{\mu} & T_x M \end{array} \tag{3.2}$$

Restrictions to $\bar{\mathfrak{m}}^h$ and $\bar{\mathfrak{m}}^v$ give the commutative diagrams:

$$\begin{array}{ccc} \bar{\mathfrak{m}}^v & \xrightarrow{\bar{\mu}} & V_{\bar{x}}\bar{M} \\ {\scriptstyle\tau_*}\downarrow & \circlearrowleft & \downarrow{\scriptstyle\pi_*} \\ \tau_*(\bar{\mathfrak{m}}^v) & \xrightarrow{\mu} & \{0\} \end{array} \qquad \begin{array}{ccc} \bar{\mathfrak{m}}^h & \xrightarrow{\bar{\mu}} & H_{\bar{x}}\bar{M} \\ {\scriptstyle\tau_*}\downarrow & \circlearrowleft & \downarrow{\scriptstyle\pi_*} \\ \tau_*(\bar{\mathfrak{m}}^h) & \xrightarrow{\mu} & T_x M \end{array} \tag{3.3}$$

which shows that $\tau_* \colon \bar{\mathfrak{m}}^h \to \tau_*(\bar{\mathfrak{m}}^h)$ and $\mu \colon \tau_*(\bar{\mathfrak{m}}^h) \to T_x M$ are isomorphisms, and $\tau_*(\bar{\mathfrak{m}}^v) \subset \mathfrak{k}$. In addition, by Lemma 3.3 the restriction of $\tau_* \colon \bar{\mathfrak{g}} \to \mathfrak{g}$ to $\bar{\mathfrak{k}}$ is an isomorphism of Lie algebras from $\bar{\mathfrak{k}}$ to $\mathfrak{k}$. Therefore, denoting by $\mathfrak{m}$ the image $\tau_*(\bar{\mathfrak{m}}^h)$, we have the decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}. \tag{3.4}$$

Let $k \in K$ and let $\xi \in \mathfrak{m}$, and let $\bar{k} \in \bar{K}$ and $\bar{\xi} \in \bar{\mathfrak{m}}^h$ be such that $\tau(\bar{k}) = k$ and $\tau_*(\bar{\xi}) = \xi$. We have that

$$\begin{aligned} \mathrm{Ad}(k)(\xi) &= \mathrm{Ad}(\tau(\bar{k}))(\tau_*(\bar{\xi})) \\ &= \mu^{-1} \circ \Phi_{\tau(\bar{k})} \circ \mu(\tau_*(\bar{\xi})) \\ &= \mu^{-1} \circ \Phi_{\tau(\bar{k})} \circ \pi_*(\bar{\mu}(\bar{\xi})) \\ &= \mu^{-1} \circ \pi_* \circ \Phi_{\bar{k}}(\bar{\mu}(\bar{\xi})) \\ &= \mu^{-1} \circ \pi_* \circ \bar{\mu}(\mathrm{Ad}(\bar{k})(\bar{\xi})) \\ &= \mu^{-1} \circ \mu \circ \tau_*(\mathrm{Ad}(\bar{k})(\bar{\xi})) \\ &= \tau_*(\mathrm{Ad}(\bar{k})(\bar{\xi})). \end{aligned}$$

Since $\bar{\mathfrak{m}}^h$ is $\mathrm{Ad}(\bar{K})$-invariant we deduce that $\mathrm{Ad}(k)(\mathfrak{m}) \subset \tau_*(\bar{\mathfrak{m}}^h) = \mathfrak{m}$, which proves that (3.4) is a reductive decomposition.

The homogeneous structure tensor associated with (3.4) at x is given (see [**18**, p. 24]) by

$$(S_x)_X Y = (\nabla_Y \xi^*)_x, \quad X, Y \in T_x M,$$

https://doi.org/10.1017/S0013091513000679 Published online by Cambridge University Press

where ξ^* is the vector field given by the infinitesimal action of $\xi \in \mathfrak{m}$ with $\xi^*_x = \mu(\xi) = X$. Let $\bar{\xi} \in \bar{\mathfrak{m}}^h$ be such that $\tau_*(\bar{\xi}) = \xi$; then,

$$\begin{aligned}(S_x)_X Y &= (\nabla_Y \xi^*)_x \\ &= \pi_*((\bar{\nabla}_{Y^{\mathrm{H}}}(\xi^*)^{\mathrm{H}})_{\bar{x}}) \\ &= \pi_*((\bar{\nabla}_{Y^{\mathrm{H}}}\bar{\xi}^*)) - \pi_*((\bar{\nabla}_{Y^{\mathrm{H}}}(\bar{\xi}^*)^v)_{\bar{x}}).\end{aligned}$$

Now let $\bar{Z} \in T_{\bar{x}}\bar{M}$ be a horizontal vector; since $\bar{\xi}^*_{\bar{x}}$ is horizontal,

$$\bar{g}((\bar{\nabla}_{Y^{\mathrm{H}}}(\bar{\xi}^*)^v)_{\bar{x}}, \bar{Z}) = Y^{\mathrm{H}}\bar{g}((\bar{\xi}^*)^v, \bar{Z}) - \bar{g}((\bar{\xi}^*)^v_{\bar{x}}, \bar{\nabla}_{Y^{\mathrm{H}}}\bar{Z}) = 0.$$

Hence, by [**18**, p. 24] and (3.3),

$$(S_x)_X Y = \pi_*((\bar{S}_{\bar{x}})_{X^{\mathrm{H}}} Y^{\mathrm{H}}), \quad X, Y \in T_x M. \tag{3.5}$$

Finally, we extend S_x to the whole M with the action of G to obtain a homogeneous structure tensor S. □

We call the tensor field S the *reduced homogeneous structure tensor.*

Corollary 3.5. *The reduced homogeneous structure can be expressed as*

$$S_X Y = \pi_*(\bar{S}_{X^{\mathrm{H}}} Y^{\mathrm{H}}), \quad X, Y \in \mathfrak{X}(M). \tag{3.6}$$

Proof. Let $\bar{a} \in \bar{G}$ and $a = \tau(\bar{a}) \in G$. We have already proved that the horizontal lift of $(\Phi_a)_*(X)$ is $(\Phi_{\bar{a}})_*(X^{\mathrm{H}})$ for all $X \in \mathfrak{X}(M)$. This together with the invariance of $\bar{S}$ by $\bar{G}$ and the invariance of S by G gives (3.6). □

3.2. The space of tensors reducing to a given tensor

Suppose that we are now in the situation of Theorem 3.4 and we have a homogeneous structure tensor S associated with G in the reduced manifold M. Using diagram (3.2) we can define

$$\bar{\mathfrak{m}}^h = \tau_*^{-1}(\mathfrak{m}) \cap \bar{\mu}^{-1}(H_{\bar{x}}\bar{M}) \quad \text{and} \quad \bar{\mathfrak{m}}^v = \mathfrak{h}.$$

The decomposition

$$\bar{\mathfrak{g}} = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}, \quad \text{with } \bar{\mathfrak{m}} = \bar{\mathfrak{m}}^v \oplus \bar{\mathfrak{m}}^h, \tag{3.7}$$

is then a reductive decomposition. Indeed, since H is normal in $\bar{G}$, it is obvious that $\mathrm{Ad}(\bar{K})(\mathfrak{h}) \subset \mathfrak{h}$. On the other hand, for $\bar{k} \in \bar{K}$ and $\bar{\xi} \in \bar{\mathfrak{m}}^h$, as $\bar{\mu}(\mathrm{Ad}(\bar{k})(\bar{\xi})) = (\Phi_{\bar{k}})_*(\bar{\mu}(\bar{\xi}))$, we have $\bar{\mu}(\mathrm{Ad}(\bar{k})(\bar{\xi})) \in H_{\bar{x}}\bar{M}$ and $\tau_*(\mathrm{Ad}(\bar{k})(\bar{\xi})) \in \mathfrak{m}$, and then $\mathrm{Ad}(\bar{k})(\bar{\xi} \in \bar{\mathfrak{m}}^h$. The homogeneous structure tensor associated with this decomposition at $\bar{x}$ is (see, for example, [**9**])

$$(\bar{S}_{\bar{x}})_{\bar{X}\bar{Y}\bar{Z}} = \tfrac{1}{2}(B([\bar{\xi}, \bar{\eta}]_{\bar{\mathfrak{m}}}, \bar{\zeta}) - B([\bar{\eta}, \bar{\zeta}]_{\bar{\mathfrak{m}}}, \bar{\xi}) + B([\bar{\zeta}, \bar{\xi}]_{\bar{\mathfrak{m}}}, \bar{\eta})), \quad \bar{X}, \bar{Y}, \bar{Z} \in T_{\bar{x}}\bar{M}, \tag{3.8}$$

where $\bar{\xi}, \bar{\eta}, \bar{\zeta} \in \bar{\mathfrak{m}}$ are such that their images by $\bar{\mu}$ are $\bar{X}$, $\bar{Y}$, $\bar{Z}$, and B is the bilinear form induced on $\bar{\mathfrak{m}}$ from $T_{\bar{x}}\bar{M}$ by $\bar{\mu}$. Note that we have exactly the same situation in the proof of Theorem 3.4, so the homogeneous structure tensor $\bar{S}$ associated with (3.7) reduces to S.

We can construct all other homogeneous structures in $\bar{M}$ associated with $\bar{G}$ by changing $\bar{\mathfrak{m}}$ in (3.7) by the graph

$$\bar{\mathfrak{m}}^{\varphi} = \{X + \varphi(X)/X \in \bar{\mathfrak{m}}\}$$

of an $\mathrm{Ad}(\bar{K})$-equivariant map $\varphi\colon \mathfrak{h} \oplus \bar{\mathfrak{m}}^h \to \bar{\mathfrak{k}}$. The condition that the new homogeneous structure tensors reduce to S is equivalent to the condition $\varphi|_{\bar{\mathfrak{m}}^h} = 0$. So the family of homogeneous structure tensors that reduce to S is parametrized by the set of $\mathrm{Ad}(\bar{K})$-equivariant maps $\varphi\colon \mathfrak{h} \to \bar{\mathfrak{k}}$. For the sake of convenience we denote by the same φ both $\varphi\colon \mathfrak{h} \to \bar{\mathfrak{k}}$ and its extension by zero to $\bar{\mathfrak{m}} = \mathfrak{h} \oplus \bar{\mathfrak{m}}^h$. The expression of the homogeneous structure tensor $\bar{S}^{\varphi}$ associated with this map is the same as in (3.8) by changing $\bar{\mathfrak{m}}$ to $\bar{\mathfrak{m}}^{\varphi}$, B to the induced bilinear form B^{φ} in $\bar{\mathfrak{m}}^{\varphi}$, and the $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ to $\bar{\xi}' = \bar{\xi} + \varphi(\bar{\xi})$, $\bar{\eta}' = \bar{\eta} + \varphi(\bar{\eta})$, $\bar{\zeta}' = \bar{\zeta} + \varphi(\bar{\zeta}) \in \bar{\mathfrak{m}}^{\varphi}$. As

$$[\bar{\xi}', \bar{\eta}']_{\bar{\mathfrak{m}}^{\varphi}} = [\bar{\xi}, \bar{\eta}]_{\bar{\mathfrak{m}}^{\varphi}} + [\bar{\xi}, \varphi(\bar{\eta})]_{\bar{\mathfrak{m}}^{\varphi}} + [\varphi(\bar{\xi}), \bar{\eta}]_{\bar{\mathfrak{m}}^{\varphi}} + [\varphi(\bar{\xi}), \varphi(\bar{\eta})]_{\bar{\mathfrak{m}}^{\varphi}}$$

and $[\varphi(\bar{\xi}), \varphi(\bar{\eta})]_{\bar{\mathfrak{m}}^{\varphi}} = 0$, we have that

$$\begin{aligned} B^{\varphi}([\bar{\xi}', \bar{\eta}']_{\bar{\mathfrak{m}}^{\varphi}}, \bar{\zeta}') &= B^{\varphi}([\bar{\xi}, \bar{\eta}]_{\bar{\mathfrak{m}}^{\varphi}}, \bar{\zeta}') + B^{\varphi}([\bar{\xi}, \varphi(\bar{\eta})]_{\bar{\mathfrak{m}}^{\varphi}} + [\varphi(\bar{\xi}), \bar{\eta}]_{\bar{\mathfrak{m}}^{\varphi}}, \bar{\zeta}') \\ &= B([\bar{\xi}, \bar{\eta}]_{\bar{\mathfrak{m}}}, \bar{\zeta}) + B([\bar{\xi}, \varphi(\bar{\eta})] + [\varphi(\bar{\xi}), \bar{\eta}], \bar{\zeta}), \end{aligned}$$

where one has to take into account that the isomorphism $\bar{\mathfrak{m}} \to \bar{\mathfrak{m}}^{\varphi}$, $\bar{\xi} \mapsto \bar{\xi} + \varphi(\bar{\xi})$ is an isometry with respect to B and B^{φ}. Hence,

$$\begin{aligned} (\bar{S}^{\varphi}_{\bar{x}})_{\bar{X}\bar{Y}\bar{Z}} = (\bar{S}_{\bar{x}})_{\bar{X}\bar{Y}\bar{Z}} + \tfrac{1}{2}\{&B([\bar{\xi}, \varphi(\bar{\eta})] + [\varphi(\bar{\xi}), \bar{\eta}], \bar{\zeta}) \\ &- B([\bar{\eta}, \varphi(\bar{\zeta})] + [\varphi(\bar{\eta}), \bar{\zeta}], \bar{\xi}) + B([\bar{\zeta}, \varphi(\bar{\xi})] + [\varphi(\bar{\zeta}), \bar{\xi}], \bar{\eta})\}. \end{aligned} \tag{3.9}$$

The summands involving B define a tensor field P^{φ} globally defined in $\bar{M}$ by the left action of $\bar{G}$. More precisely, for any $\bar{y} \in \bar{M}$, with $\bar{y} = \Phi_{\bar{a}}(\bar{x})$, $\bar{a} \in \bar{G}$, this tensor is

$$\begin{aligned} (P^{\varphi}_{\bar{y}})_{\bar{X}\bar{Y}\bar{Z}} = \tfrac{1}{2}\{B_{\bar{y}}([\bar{\xi}, \varphi_{\bar{y}}(\bar{\eta})] + [\varphi_{\bar{y}}(\bar{\xi}), \bar{\eta}], \bar{\zeta}) - B_{\bar{y}}([\bar{\eta}, \varphi_{\bar{y}}(\bar{\zeta})] + [\varphi_{\bar{y}}(\bar{\eta}), \bar{\zeta}], \bar{\xi}) \\ + B_{\bar{y}}([\bar{\zeta}, \varphi_{\bar{y}}(\bar{\xi})] + [\varphi_{\bar{y}}(\bar{\zeta}), \bar{\xi}], \bar{\eta})\} \end{aligned} \tag{3.10}$$

for $\bar{X}, \bar{Y}, \bar{Z} \in T_{\bar{y}}\bar{M}$, where

$$\begin{aligned} \bar{\mathfrak{m}}_{\bar{y}} &:= \mathrm{Ad}(\bar{a})(\bar{\mathfrak{m}}), \qquad \bar{\mathfrak{k}}_{\bar{y}} := \mathrm{Ad}(\bar{a})(\bar{\mathfrak{k}}), \\ \varphi_{\bar{y}} &:= \mathrm{Ad}(\bar{a}) \circ \varphi \circ \mathrm{Ad}(\bar{a}^{-1})\colon \mathfrak{h} \to \bar{\mathfrak{k}}_{\bar{y}}, \end{aligned}$$

$B_{\bar{y}}$ is the bilinear form on $\bar{\mathfrak{m}}_{\bar{y}}$ induced from $\bar{g}_{\bar{y}}$ by

$$\bar{\mu}_{\bar{y}} := (\Phi_{\bar{a}})_* \circ \bar{\mu} \circ \mathrm{Ad}(\bar{a}^{-1})\colon \bar{\mathfrak{m}}_{\bar{y}} \to T_{\bar{y}}\bar{M},$$

and $\bar{\xi}, \bar{\eta}, \bar{\zeta} \in \bar{\mathfrak{m}}_{\bar{y}}$ are such that their images by $\bar{\mu}_{\bar{y}}$ are $\bar{X}$, $\bar{Y}$, $\bar{Z}$, respectively.

We have then proved the following.

Proposition 3.6. *In the situation of Theorem 3.4, let S be a homogeneous structure tensor in M associated with G. The space of homogeneous structure tensors in $\bar{M}$ associated with $\bar{G}$ and reducing to S is then a vector space isomorphic to the space of $\mathrm{Ad}(\bar{K})$-equivariant maps $\varphi\colon \mathfrak{h} \to \bar{\mathfrak{k}}$. Moreover, the isomorphism is given by*

$$\varphi \mapsto \bar{S}^{\varphi} = \bar{S} + P^{\varphi},$$

where $\bar{S}$ is the homogeneous structure associated with the decomposition (3.7) and P^{φ} is given in (3.10).

3.3. Reduction in a principal bundle

We have noted in Remark 3.2 that the normality of the group H gives the invariance of the mechanical connection. This implies that the connection form ω is $\mathrm{Ad}(\bar{G})$-equivariant, i.e.

$$\Phi_{\bar{a}}^{*}\omega = \mathrm{Ad}(\bar{a}) \cdot \omega \quad \forall \bar{a} \in \bar{G}, \tag{3.11}$$

where $\mathrm{Ad}(\bar{a}) \cdot \omega$ denotes the 1-form in $\bar{M}$ with values in $\mathfrak{h}$ given by

$$(\mathrm{Ad}(\bar{a}) \cdot \omega)(\bar{X}) = \mathrm{Ad}(\bar{a})(\omega(\bar{X})).$$

The canonical linear connection $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}$ of the reductive decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ at $\bar{x}$ is characterized by the following property: for every $\bar{\xi} \in \bar{\mathfrak{m}}$, the parallel displacement with respect to $\tilde{\bar{\nabla}}$ along the curve $\gamma(t) = \Phi_{\exp(t\bar{\xi})}(\bar{x})$, from $\bar{x}$ to $\gamma(t)$, is equal to $(\Phi_{\exp(t\bar{\xi})})_{*}$ (see [**12**, Chapter X, Corollary 2.5]). Hence, infinitesimally we have that

$$(\tilde{\bar{\nabla}}_{\bar{X}}\omega)_{\bar{x}} = \mathrm{ad}(\bar{\mu}^{-1}(\bar{X})) \cdot \omega_{\bar{x}} \quad \forall \bar{X} \in T_{\bar{x}}\bar{M},$$

and, by the invariance of $\tilde{\bar{\nabla}}$ by $\bar{G}$,

$$(\tilde{\bar{\nabla}}_{\bar{X}}\omega)_{\bar{y}} = \mathrm{ad}(\bar{\mu}_{\bar{y}}^{-1}(\bar{X})) \cdot \omega_{\bar{y}} \quad \forall \bar{y} \in \bar{M},\ \forall \bar{X} \in T_{\bar{y}}\bar{M}, \tag{3.12}$$

that is, the covariant derivative of ω by the connection $\tilde{\bar{\nabla}}$ is proportional to itself by a suitable linear operator. We note that, in particular, if H is contained in the centre of $\bar{G}$, the linear operator is null. Hence, ω is invariant by $\bar{G}$. If H is just a normal subgroup not contained in the centre, (3.12) follows from the equivariance of ω.

The preceding discussion suggests that we study the reduction of homogeneous structure tensors $\bar{S}$ in a principal bundle without the use of the group $\bar{G}$. More precisely, the group $\bar{G}$ (and its reductive decomposition) associated with the tensor $\bar{S}$ was a key ingredient in Theorem 3.4. We now begin with any tensor $\bar{S}$ in a manifold $(\bar{M}, \bar{g})$ where a group H acts by isometries (and such that $\bar{M} \to \bar{M}/H = M$ is a principal bundle) satisfying the Ambrose–Singer equations and an additional algebraic condition for the mechanical connection analogous to (3.12). The tensor $\bar{S}$ can then also be projected without using any reductive decomposition, as we see in the following result.

Theorem 3.7. *Let $(\bar{M}, \bar{g})$ be a Riemannian manifold. Let $\pi\colon \bar{M} \to M$ be a principal bundle with structure group H acting on $\bar{M}$ by isometries, and endowed with the mechanical connection ω. For every H-invariant homogeneous Riemannian structure tensor $\bar{S}$ with canonical linear connection $\tilde{\bar{\nabla}}$, if*

$$\tilde{\bar{\nabla}}\omega = \alpha \cdot \omega \tag{3.13}$$

for some 1-form α in $\bar{M}$ taking values in $\operatorname{End}(\mathfrak{h})$, *the tensor field S defined by*

$$S_X Y = \pi_*(\bar{S}_{X^{\mathrm{H}}} Y^{\mathrm{H}}), \quad X, Y \in \mathfrak{X}(M), \tag{3.14}$$

is a homogeneous Riemannian structure tensor in (M, g), where g is the reduced Riemannian metric.

Proof. First note that H-invariance of $\bar{S}$ implies that $\bar{S}_{X^{\mathrm{H}}} Y^{\mathrm{H}}$ is projectable, and then S is well defined. Since the structure group H acts by isometries, the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$ is H-invariant, which implies that $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}$ is also H-invariant. Now, from condition (3.13) we have that, for all $X, Y \in \mathfrak{X}(M)$,

$$\omega(\tilde{\bar{\nabla}}_{X^{\mathrm{H}}} Y^{\mathrm{H}}) = X^{\mathrm{H}}(\omega(Y^{\mathrm{H}})) - (\tilde{\bar{\nabla}}_{X^{\mathrm{H}}}\omega)(Y^{\mathrm{H}}) = -\alpha(X^{\mathrm{H}}) \cdot \omega(Y^{\mathrm{H}}) = 0,$$

so $\tilde{\bar{\nabla}}_{X^{\mathrm{H}}} Y^{\mathrm{H}}$ is horizontal. If we define $\tilde{\nabla} = \nabla - S$, ∇ being the Levi-Civita connection of g, then $\tilde{\bar{\nabla}}_{X^{\mathrm{H}}} Y^{\mathrm{H}}$ projects to $\tilde{\nabla}_{X^{\mathrm{H}}} Y^{\mathrm{H}}$. Hence, by H-invariance,

$$(\tilde{\nabla}_X Y)^{\mathrm{H}} = \tilde{\bar{\nabla}}_{X^{\mathrm{H}}} Y^{\mathrm{H}}. \tag{3.15}$$

We now prove that S satisfies the Ambrose–Singer equations (equivalent to those in (2.1)):

$$\tilde{\nabla} g = 0, \qquad \tilde{\nabla}\tilde{R} = 0, \qquad \tilde{\nabla} S = 0, \tag{3.16}$$

where $\tilde{R}$ is the curvature tensor of $\tilde{\nabla}$, and $\tilde{R}$ and S are seen to be $(0, 4)$ and $(0, 3)$ tensors, respectively, by lowering their contravariant index with respect to g.

For the first equation, taking into account (3.15), we have, for $U, X, Y \in \mathfrak{X}(M)$, that

$$\begin{aligned}
(\tilde{\nabla}_U g)(X, Y) \circ \pi &= U(g(X, Y)) \circ \pi - g(\tilde{\nabla}_U X, Y) \circ \pi - g(X, \tilde{\nabla}_U Y) \circ \pi \\
&= U^{\mathrm{H}}(\bar{g}(X^{\mathrm{H}}, Y^{\mathrm{H}})) - \bar{g}((\tilde{\nabla}_U X)^{\mathrm{H}}, Y^{\mathrm{H}}) - \bar{g}(X^{\mathrm{H}}, (\tilde{\nabla}_U Y)^{\mathrm{H}}) \\
&= U^{\mathrm{H}}(\bar{g}(X^{\mathrm{H}}, Y^{\mathrm{H}})) - \bar{g}(\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} X^{\mathrm{H}}, Y^{\mathrm{H}}) - \bar{g}(X^{\mathrm{H}}, \tilde{\bar{\nabla}}_{U^{\mathrm{H}}} Y^{\mathrm{H}}) \\
&= (\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} \bar{g})(X^{\mathrm{H}}, Y^{\mathrm{H}}),
\end{aligned}$$

and then, since $\tilde{\bar{\nabla}}\bar{g} = 0$, we have $\tilde{\nabla} g = 0$.

For the third equation, let $U, X, Y, Z \in \mathfrak{X}(M)$. Then, again by (3.15), we have that

$$\begin{aligned}
(\tilde{\nabla}_U S)_{XYZ} \circ \pi &= U(S_{XYZ}) \circ \pi - (S_{\tilde{\nabla}_U XYZ}) \circ \pi - (S_{X\tilde{\nabla}_U YZ}) \circ \pi - (S_{XY\tilde{\nabla}_U Z}) \circ \pi \\
&= U^{\mathrm{H}}(\bar{S}_{X^{\mathrm{H}} Y^{\mathrm{H}} Z^{\mathrm{H}}}) - \bar{S}_{(\tilde{\nabla}_U X)^{\mathrm{H}} Y^{\mathrm{H}} Z^{\mathrm{H}}} - \bar{S}_{X^{\mathrm{H}} (\tilde{\nabla}_U Y)^{\mathrm{H}} Z^{\mathrm{H}}} - \bar{S}_{X^{\mathrm{H}} Y^{\mathrm{H}} (\tilde{\nabla}_U Z)^{\mathrm{H}}} \\
&= U^{\mathrm{H}}(\bar{S}_{X^{\mathrm{H}} Y^{\mathrm{H}} Z^{\mathrm{H}}}) - \bar{S}_{\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} X^{\mathrm{H}} Y^{\mathrm{H}} Z^{\mathrm{H}}} - \bar{S}_{X^{\mathrm{H}} \tilde{\bar{\nabla}}_{U^{\mathrm{H}}} Y^{\mathrm{H}} Z^{\mathrm{H}}} - \bar{S}_{X^{\mathrm{H}} Y^{\mathrm{H}} \tilde{\bar{\nabla}}_{U^{\mathrm{H}}} Z^{\mathrm{H}}} \\
&= (\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} \bar{S})_{X^{\mathrm{H}} Y^{\mathrm{H}} Z^{\mathrm{H}}},
\end{aligned}$$

which vanishes as $\tilde{\bar{\nabla}}\bar{S} = 0$.

We now prove the second Ambrose–Singer equation. Let $\tilde{\bar{R}}$ be the curvature tensor of $\tilde{\bar{\nabla}}$. From (3.15), for $X, Y, Z \in \mathfrak{X}(M)$ we first have that

$$\begin{aligned}(\tilde{R}_{XY}Z)^{\mathrm{H}} &= \tilde{\bar{\nabla}}_{X^{\mathrm{H}}}(\tilde{\nabla}_Y Z)^{\mathrm{H}} - \tilde{\bar{\nabla}}_{Y^{\mathrm{H}}}(\tilde{\nabla}_X Z)^{\mathrm{H}} - \tilde{\bar{\nabla}}_{[X,Y]^{\mathrm{H}}} Z^{\mathrm{H}} \\ &= \tilde{\bar{\nabla}}_{X^{\mathrm{H}}}(\tilde{\bar{\nabla}}_{Y^{\mathrm{H}}} Z^{\mathrm{H}}) - \tilde{\bar{\nabla}}_{Y^{\mathrm{H}}}(\tilde{\bar{\nabla}}_{X^{\mathrm{H}}} Z^{\mathrm{H}}) - \tilde{\bar{\nabla}}_{[X^{\mathrm{H}},Y^{\mathrm{H}}]^h} Z^{\mathrm{H}} \\ &= \tilde{\bar{R}}_{X^{\mathrm{H}}Y^{\mathrm{H}}} Z^{\mathrm{H}} + \tilde{\bar{\nabla}}_{[X^{\mathrm{H}},Y^{\mathrm{H}}]^v} Z^{\mathrm{H}}.\end{aligned}$$

We also denote by $\tilde{\bar{R}}$ the $(0, 4)$ tensor field associated with $\tilde{\bar{R}}$ with respect to $\bar{g}$. Then, for $X, Y, Z, W \in \mathfrak{X}(M)$, one has that

$$\begin{aligned}\tilde{R}_{XYZW} \circ \pi &= \tilde{\bar{R}}_{X^{\mathrm{H}}Y^{\mathrm{H}}Z^{\mathrm{H}}W^{\mathrm{H}}} + \bar{g}(\tilde{\bar{\nabla}}_{[X^{\mathrm{H}},Y^{\mathrm{H}}]^v} Z^{\mathrm{H}}, W^{\mathrm{H}}) \\ &= \tilde{\bar{R}}_{X^{\mathrm{H}}Y^{\mathrm{H}}Z^{\mathrm{H}}W^{\mathrm{H}}} - \bar{g}(\tilde{\bar{\nabla}}_{\Omega(X^{\mathrm{H}},Y^{\mathrm{H}})^*} Z^{\mathrm{H}}, W^{\mathrm{H}}),\end{aligned} \tag{3.17}$$

where $\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}})^*$ is the fundamental vector field associated with $\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}) \in \mathfrak{h}$. For any $\bar{x} \in \bar{M}$, let $\mathbb{I}(\bar{x})$ be the bilinear form in $\mathfrak{h}$ defined as

$$\mathbb{I}(\bar{x})(\xi, \eta) = \bar{g}(\xi^*_{\bar{x}}, \eta^*_{\bar{x}}) \quad \forall \xi, \eta \in \mathfrak{h}.$$

Applying Koszul's formula for $\bar{\nabla}$ and taking into account that $[X^{\mathrm{H}}, \xi^*] = 0$ for any $X \in \mathfrak{X}(M)$, $\xi \in \mathfrak{h}$, we have that

$$\begin{aligned}\bar{g}(\tilde{\bar{\nabla}}_{\Omega(X^{\mathrm{H}},Y^{\mathrm{H}})^*} Z^{\mathrm{H}}, W^{\mathrm{H}}) &= \bar{g}(\bar{\nabla}_{\Omega(X^{\mathrm{H}},Y^{\mathrm{H}})^*} Z^{\mathrm{H}}, W^{\mathrm{H}}) - \bar{g}(\bar{S}_{\Omega(X^{\mathrm{H}},Y^{\mathrm{H}})^*} Z^{\mathrm{H}} W^{\mathrm{H}}) \\ &= \tfrac{1}{2}\mathbb{I}(\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), \Omega(Z^{\mathrm{H}}, W^{\mathrm{H}})) - \bar{S}_{\Omega(X^{\mathrm{H}},Y^{\mathrm{H}})^* Z^{\mathrm{H}} W^{\mathrm{H}}},\end{aligned}$$

where, as usual, $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}$. Applying the previous equation and (3.17), a direct computation then shows that

$$\begin{aligned}(\tilde{\nabla}_U \tilde{R})_{XYZW} \circ \pi = (\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} \tilde{\bar{R}})_{X^{\mathrm{H}}Y^{\mathrm{H}}Z^{\mathrm{H}}W^{\mathrm{H}}} &\\ - \tfrac{1}{2}U^{\mathrm{H}}(\mathbb{I}(\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), \Omega(Z^{\mathrm{H}}, W^{\mathrm{H}}))) &\\ + \tfrac{1}{2}\mathbb{I}(\Omega(\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} X^{\mathrm{H}}, Y^{\mathrm{H}}), \Omega(Z^{\mathrm{H}}, W^{\mathrm{H}})) &\\ + \tfrac{1}{2}\mathbb{I}(\Omega(X^{\mathrm{H}}, \tilde{\bar{\nabla}}_{U^{\mathrm{H}}} Y^{\mathrm{H}}), \Omega(Z^{\mathrm{H}}, W^{\mathrm{H}})) &\\ + \tfrac{1}{2}\mathbb{I}(\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), \Omega(\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} Z^{\mathrm{H}}, W^{\mathrm{H}})) &\\ + \tfrac{1}{2}\mathbb{I}(\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), \Omega(Z^{\mathrm{H}}, \tilde{\bar{\nabla}}_{U^{\mathrm{H}}} W^{\mathrm{H}})) &\\ + U^{\mathrm{H}}(\bar{S}_{\Omega(X^{\mathrm{H}},Y^{\mathrm{H}})^* Z^{\mathrm{H}} W^{\mathrm{H}}}) - \bar{S}_{\Omega(\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} X^{\mathrm{H}}, Y^{\mathrm{H}})^* Z^{\mathrm{H}} W^{\mathrm{H}}} &\\ - \bar{S}_{\Omega(X^{\mathrm{H}}, \tilde{\bar{\nabla}}_{U^{\mathrm{H}}} Y^{\mathrm{H}})^* Z^{\mathrm{H}} W^{\mathrm{H}}} - \bar{S}_{\Omega(X^{\mathrm{H}},Y^{\mathrm{H}})^* (\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} Z^{\mathrm{H}}) W^{\mathrm{H}}} &\\ - \bar{S}_{\Omega(X^{\mathrm{H}},Y^{\mathrm{H}})^* Z^{\mathrm{H}} (\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} W^{\mathrm{H}})}. &\end{aligned} \tag{3.18}$$

On the other hand, by (3.13),

$$0 = (\tilde{\bar{\nabla}}_{X^{\mathrm{H}}} \omega)(Y^{\mathrm{H}}) - (\tilde{\bar{\nabla}}_{Y^{\mathrm{H}}} \omega)(X^{\mathrm{H}}) = \mathrm{d}\omega(X^{\mathrm{H}}, Y^{\mathrm{H}}) - \omega(\tilde{\bar{T}}_{X^{\mathrm{H}}} Y^{\mathrm{H}}),$$

https://doi.org/10.1017/S0013091513000679 Published online by Cambridge University Press

where $\tilde{\bar{T}}$ is the torsion tensor field of $\tilde{\bar{\nabla}}$. Then, since by definition $\Omega(\bar{X}, \bar{Y}) = \mathrm{d}\omega(\bar{X}^h, \bar{Y}^h)$, we have that

$$\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}) = \omega(\tilde{\bar{T}}_{X^{\mathrm{H}}} Y^{\mathrm{H}}).$$

Using that

$$\tilde{\bar{T}}_{X^{\mathrm{H}}} Y^{\mathrm{H}} = \bar{S}_{Y^{\mathrm{H}}} X^{\mathrm{H}} - \bar{S}_{X^{\mathrm{H}}} Y^{\mathrm{H}},$$

and the conditions (3.13) and $\tilde{\bar{\nabla}} \bar{S} = 0$, one has that

$$(\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} \Omega)(X^{\mathrm{H}}, Y^{\mathrm{H}}) = \alpha(U^{\mathrm{H}}) \cdot \Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}). \tag{3.19}$$

Now, from $\omega([X^{\mathrm{H}}, Y^{\mathrm{H}}]^v) = -\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}})$ and (3.13) we get that

$$\omega(\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} [X^{\mathrm{H}}, Y^{\mathrm{H}}]^v) = -U^{\mathrm{H}}(\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}})) + \alpha(U^{\mathrm{H}}) \cdot \Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), \tag{3.20}$$

and hence we have that

$$\begin{aligned} U^{\mathrm{H}}(\mathbb{I}(\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), \Omega(Z^{\mathrm{H}}, W^{\mathrm{H}}))) &= \bar{g}(\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} [X^{\mathrm{H}}, Y^{\mathrm{H}}]^v, [Z^{\mathrm{H}}, W^{\mathrm{H}}]^v) \\ &\quad + \bar{g}([X^{\mathrm{H}}, Y^{\mathrm{H}}]^v, \tilde{\bar{\nabla}}_{U^{\mathrm{H}}} [Z^{\mathrm{H}}, W^{\mathrm{H}}]^v) \\ &= \mathbb{I}(U^{\mathrm{H}} \Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), \Omega(Z^{\mathrm{H}}, W^{\mathrm{H}})) \\ &\quad - \mathbb{I}(\alpha(U^{\mathrm{H}}) \cdot \Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), \Omega(Z^{\mathrm{H}}, W^{\mathrm{H}})) \\ &\quad + \mathbb{I}(\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), U^{\mathrm{H}} \Omega(Z^{\mathrm{H}}, W^{\mathrm{H}})) \\ &\quad - \mathbb{I}(\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), \alpha(U^{\mathrm{H}}) \cdot \Omega(Z^{\mathrm{H}}, W^{\mathrm{H}})). \end{aligned}$$

In addition, by (3.19) and (3.20),

$$\Omega(\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} X^{\mathrm{H}}, Y^{\mathrm{H}}) + \Omega(X^{\mathrm{H}}, \tilde{\bar{\nabla}}_{U^{\mathrm{H}}} Y^{\mathrm{H}}) = -\omega(\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} [X^{\mathrm{H}}, Y^{\mathrm{H}}]^v),$$

so

$$\Omega(\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} X^{\mathrm{H}}, Y^{\mathrm{H}})^* + \Omega(X^{\mathrm{H}}, \tilde{\bar{\nabla}}_{U^{\mathrm{H}}} Y^{\mathrm{H}})^* = \tilde{\bar{\nabla}}_{U^{\mathrm{H}}} \Omega(X^{\mathrm{H}}, Y^{\mathrm{H}})^*, \tag{3.21}$$

since $\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} [X^{\mathrm{H}}, Y^{\mathrm{H}}]^v$ is vertical. Making use of the preceding formulae and grouping terms, (3.18) becomes

$$\begin{aligned} (\tilde{\nabla}_U \tilde{R})_{XYZW} \circ \pi = (\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} \tilde{\bar{R}})_{X^{\mathrm{H}} Y^{\mathrm{H}} Z^{\mathrm{H}} W^{\mathrm{H}}} &\\ + \tfrac{1}{2} \mathbb{I}((\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} \Omega)(X^{\mathrm{H}}, Y^{\mathrm{H}}), \Omega(Z^{\mathrm{H}}, W^{\mathrm{H}})) &\\ - \tfrac{1}{2} \mathbb{I}(\alpha(U^{\mathrm{H}}) \cdot \Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), \Omega(Z^{\mathrm{H}}, W^{\mathrm{H}})) &\\ + \tfrac{1}{2} \mathbb{I}(\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), (\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} \Omega)(Z^{\mathrm{H}}, W^{\mathrm{H}})) &\\ - \tfrac{1}{2} \mathbb{I}(\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}}), \alpha(U^{\mathrm{H}}) \cdot \Omega(Z^{\mathrm{H}}, W^{\mathrm{H}})) &\\ - (\tilde{\bar{\nabla}}_{U^{\mathrm{H}}} \bar{S})_{\Omega(X^{\mathrm{H}}, Y^{\mathrm{H}})^* Z^{\mathrm{H}} W^{\mathrm{H}}}, & \end{aligned}$$

from which, taking into account (3.19) and (3.21), we deduce that $\tilde{\nabla}_U \tilde{R} = 0$. This completes the proof of Theorem 3.7. □

https://doi.org/10.1017/S0013091513000679 Published online by Cambridge University Press

Remark 3.8. In the situation of Theorem 3.7, in the case where $\bar{S}$ is a homogeneous structure tensor associated with a Lie group $\bar{G}$ acting by isometries in $\bar{M}$, one could ask if H can be seen to be a normal subgroup of $\bar{G}$ and if the projected tensor S is associated with the group $G = \bar{G}/H$. The answer is not necessarily affirmative. More precisely, for a connected, simply connected and complete manifold $\bar{M}$, if we construct the group $\bar{G}$ from $\bar{S}$ following the proof of the Ambrose–Singer theorem (as in [**18**]), one can see that the normality of H is not guaranteed and the group $\bar{G}$ need not project to the group G constructed in M from S by the same method. An example of this situation is shown in § 4.2.1 (the Hopf fibration case $\lambda = 0$).

Remark 3.9. The algebraic condition (3.13) for $\alpha = 0$ is an invariance condition and can be implemented in Ambrose–Singer conditions as in Kiričenko's theorem (see [**11**]). This situation can be found in the last section of the present paper in the framework of almost contact metric homogeneous structures, where this condition is automatically satisfied. Note that for non-trivial α, the situation would require an equivariant version of this theorem.

3.4. Reduction and homogeneous classes

In the situation of Theorem 3.7, we have the following.

Proposition 3.10. *The classes* $\{0\}$, $\mathcal{S}_1$, $\mathcal{S}_3$, $\mathcal{S}_1 \oplus \mathcal{S}_2$ *and* $\mathcal{S}_1 \oplus \mathcal{S}_3$ *are invariant under the reduction procedure.*

Proof. By the expression of the reduced structure tensor (3.14) it is obvious that if $\bar{S} = 0$, then $S = 0$. Let $\bar{S} \in \mathcal{S}_1$ be given by the expression

$$\bar{S}_{\bar{X}\bar{Y}\bar{Z}} = \bar{g}(\bar{X}, \bar{Y})\bar{g}(\bar{\xi}, \bar{Z}) - \bar{g}(\bar{Y}, \bar{\xi})\bar{g}(\bar{X}, \bar{Z}),$$

where $\bar{\xi}$ is a vector field parallel with respect to $\bar{\tilde{\nabla}}$. Since $\bar{S}$ is H-invariant, the vector field $\bar{\xi}$ is also H-invariant, and thus projectable. Let ξ be the projection of $\bar{\xi}$. We have $\xi^{\mathrm{H}} = \bar{\xi}^h$, and then

$$\begin{aligned} S_{XYZ} \circ \pi &= \bar{g}(X^{\mathrm{H}}, Y^{\mathrm{H}})\bar{g}(\bar{\xi}, Z^{\mathrm{H}}) - \bar{g}(Y^{\mathrm{H}}, \bar{\xi})\bar{g}(X^{\mathrm{H}}, Z^{\mathrm{H}}) \\ &= \bar{g}(X^{\mathrm{H}}, Y^{\mathrm{H}})\bar{g}(\xi^{\mathrm{H}}, Z^{\mathrm{H}}) - \bar{g}(Y^{\mathrm{H}}, \xi^{\mathrm{H}})\bar{g}(X^{\mathrm{H}}, Z^{\mathrm{H}}) \\ &= g(X, Y)g(\xi, Z) \circ \pi - g(Y, \xi)g(X, Z) \circ \pi; \end{aligned}$$

hence, $S \in \mathcal{S}_1$. With a similar argument, one proves that the class $\mathcal{S}_1 \oplus \mathcal{S}_2$ is also invariant. The classes $\mathcal{S}_3$ and $\mathcal{S}_1 \oplus \mathcal{S}_3$ are characterized by algebraic conditions clearly preserved by the reduction formula (3.14). □

The other two classes $\mathcal{S}_2$ and $\mathcal{S}_2 \oplus \mathcal{S}_3$ are characterized by the vanishing of the trace c_{12}. Let $x \in M$ and let $\{e_i\}_{i=1,\dots,n}$ be an orthonormal base of T_xM; then, for $X \in T_xM$,

$$c_{12}(S)(X) = \sum_i S_{e_i e_i X} = \sum_i \bar{S}_{e_i^{\mathrm{H}} e_i^{\mathrm{H}} X^{\mathrm{H}}} = c_{12}(\bar{S})(X^{\mathrm{H}}) - \sum_j \bar{S}_{V_j V_j X^{\mathrm{H}}}, \tag{3.22}$$

https://doi.org/10.1017/S0013091513000679 Published online by Cambridge University Press

where $\{V_j\}_{j=1,\dots,r}$ is an orthonormal basis of the vertical subspace $V_{\bar{x}}\bar{M}$, $\bar{x} \in \pi^{-1}(x)$. From $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}$, one has that

$$\bar{S}_{V_j V_j X^{\mathrm{H}}} = \bar{g}(\bar{\nabla}_{V_j} V_j, X^{\mathrm{H}}) - \bar{g}(\tilde{\bar{\nabla}}_{V_j} V_j, X^{\mathrm{H}}) = -\bar{g}(\bar{\nabla}_{V_j} X^{\mathrm{H}}, V_j) + \bar{g}(\tilde{\bar{\nabla}}_{V_j} X^{\mathrm{H}}, V_j),$$

where the vectors V_j, $j = 1, \dots, r$, are extended to unitary and, respectively, orthogonal vertical vector fields. As from (3.13) we have that

$$\omega(\tilde{\bar{\nabla}}_{V_j} X^{\mathrm{H}}) = V_j(\omega(X^{\mathrm{H}})) - \alpha(V_j) \cdot \omega(X^{\mathrm{H}}) = 0,$$

the second summand in the formula for $\bar{S}_{V_j V_j X^{\mathrm{H}}}$ is 0, and then

$$\bar{S}_{V_j V_j X^{\mathrm{H}}} = -\bar{g}(\bar{\nabla}_{V_j} X^{\mathrm{H}}, V_j) = \bar{g}(B(V_j, V_j), X^{\mathrm{H}}),$$

where B denotes the second fundamental form of the fibre $\pi^{-1}(x)$ at $\bar{x}$. Inserting this into (3.22), we obtain that

$$c_{12}(S)(X) = c_{12}(\bar{S})(X^{\mathrm{H}}) - \sum_j \bar{g}(B(V_j, V_j), X^{\mathrm{H}}) = c_{12}(\bar{S})(X^{\mathrm{H}}) - r\bar{g}(H, X^{\mathrm{H}}),$$

where H denotes the mean curvature operator of the fibre at $\bar{x}$. We have proved the following.

Proposition 3.11. *The classes $\mathcal{S}_2$ and $\mathcal{S}_2 \oplus \mathcal{S}_3$ are invariant under reduction if and only if the fibres of the principal bundle $\pi\colon (\bar{M}, \bar{g}) \to (M, g)$ are minimal Riemannian sub-manifolds of $(\bar{M}, \bar{g})$.*

Remark 3.12. Propositions 3.10 and 3.11 (when the fibres are minimal) do not exclude that a homogeneous structure tensor $\bar{S}$ in a class $\mathcal{S}_i \oplus \mathcal{S}_j$ reduces to a tensor S belonging to classes $\mathcal{S}_i$ or $\mathcal{S}_j$, or even to the null tensor. We show some examples of these situations in the next section.

4. Examples

4.1. Real hyperbolic space

The real n-dimensional hyperbolic space $(\mathbb{R}\mathrm{H}(n), \bar{g})$,

$$\mathbb{R}\mathrm{H}(n) = \{(\bar{y}^0, \bar{y}^1, \dots, \bar{y}^{n-1}) \in \mathbb{R}^n / \bar{y}^0 > 0\},$$
$$\bar{g} = \frac{1}{(\bar{y}^0)^2} \sum_{j=0}^{n-1} \mathrm{d}\bar{y}^j \otimes \mathrm{d}\bar{y}^j,$$

is a symmetric space, $\mathbb{R}\mathrm{H}(n) = \mathrm{SO}(n-1,1)/\mathrm{O}(n-1)$. If we consider the Iwasawa decomposition of its full Lie group of isometries,

$$\mathrm{SO}(1, n-1) = \mathrm{O}(n-1)AN,$$

https://doi.org/10.1017/S0013091513000679 Published online by Cambridge University Press

then we can identify $\mathbb{R}\mathrm{H}(n) \simeq AN$ such that the hyperbolic space has a solvable Lie group structure given by

$$(\bar{x}^0, \bar{x}^1, \dots, \bar{x}^{n-1}) \cdot (\bar{y}^0, \bar{y}^1, \dots, \bar{y}^{n-1}) = (\bar{x}^0\bar{y}^0, \bar{x}^0\bar{y}^1 + \bar{x}^1, \dots, \bar{x}^0\bar{y}^{n-1} + \bar{x}^{n-1}).$$

Hence, the real hyperbolic space acts freely, transitively and by isometries on itself by left translations. The homogeneous structure tensor $\bar{S}$ associated with this action (see [**18**]) is a $\mathcal{S}_1$ structure given by

$$\bar{S}_{\bar{X}\bar{Y}\bar{Z}} = \bar{g}(\bar{X}, \bar{Y})\bar{g}(\bar{\xi}, \bar{Y}) - \bar{g}(\bar{\xi}, \bar{Y})\bar{g}(\bar{X}, \bar{Z}), \quad \bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\mathbb{R}\mathrm{H}(n)),$$

where

$$\bar{\xi} = \bar{y}^0 \frac{\partial}{\partial \bar{y}^0}.$$

Let $H_i \simeq \mathbb{R}$, $i = 2, \dots, n-1$, be the normal subgroups of $\mathbb{R}\mathrm{H}(n)$ given by

$$H_i = \{(1, 0, \dots, \lambda, 0, \dots, 0)/\lambda \in \mathbb{R}\},$$

where λ is in the ith position. Reduction by the action of H_i gives the fibration

$$\begin{aligned} \mathbb{R}\mathrm{H}(n) &\to \mathbb{R}\mathrm{H}(n-1) \\ (\bar{y}^0, \dots, \bar{y}^{n-1}) &\mapsto (\bar{y}^0, \dots, \bar{y}^{i-1}, \bar{y}^{i+1} \dots, \bar{y}^{n-1}) \end{aligned}$$

with vertical and horizontal subspaces at $\bar{y} \in \mathbb{R}\mathrm{H}(n)$,

$$\begin{aligned} V_{\bar{y}}\mathbb{R}\mathrm{H}(n) &= \mathrm{span}\left\{\frac{\partial}{\partial \bar{y}^i}\right\}, \\ H_{\bar{y}}\mathbb{R}\mathrm{H}(n) &= \mathrm{span}\left\{\frac{\partial}{\partial \bar{y}^0}, \dots, \frac{\partial}{\partial \bar{y}^{i-1}}, \frac{\partial}{\partial \bar{y}^{i+1}}, \dots, \frac{\partial}{\partial \bar{y}^{n-1}}\right\}. \end{aligned}$$

Hence, the induced metric on $\mathbb{R}\mathrm{H}(n-1)$ is

$$g = \frac{1}{(y^0)^2} \sum_{j=0}^{n-2} \mathrm{d}y^j \otimes \mathrm{d}y^j,$$

where $(y^0, \dots, y^{n-2})$ are the natural coordinates of $\mathbb{R}\mathrm{H}(n-1)$. As a straightforward computation shows, the reduced homogeneous structure tensor S is

$$S_{XYZ} = g(X, Y)g(\xi, Z) - g(\xi, Y)g(X, Z), \quad X, Y, Z \in \mathfrak{X}(\mathbb{R}\mathrm{H}(n-1)),$$

where

$$\xi = y^0 \frac{\partial}{\partial y^0}.$$

We have proved that the reduction $\mathbb{R}\mathrm{H}(n) \to \mathbb{R}\mathrm{H}(n-1)$ sends the canonical tensor associated with the solvable structure of the n-dimensional hyperbolic space to the canonical tensor associated with the solvable structure of the $(n-1)$-dimensional hyperbolic space. The reduction procedure has then preserved the $\mathcal{S}_1$ class in this case.

We now confine ourselves to the four-dimensional hyperbolic space. Besides its symmetric description, all other groups of isometries acting transitively are of the type (see [**5**]) $\bar{G} = FN$, where F is a connected closed subgroup of $\mathrm{SO}(3)A$ with non-trivial projection to A. In particular, we now consider

$$\bar{G} = \mathrm{SO}(2)AN.$$

Geometrically, if we see $\mathrm{SO}(2)$ as the isotropy group of the point $\bar{x} = (1, 0, 0, 0)$, its Lie algebra $\bar{\mathfrak{k}}$ are infinitesimal rotations generated by

$$r = \bar{y}^2 \frac{\partial}{\partial \bar{y}^3} - \bar{y}^3 \frac{\partial}{\partial \bar{y}^2}.$$

The subspace $\bar{\mathfrak{m}} = \mathfrak{a} \oplus \mathfrak{n}$, which is the Lie algebra of the factor AN, gives a reductive decomposition

$$\bar{\mathfrak{g}} = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}.$$

Let $a \in \mathfrak{a}$, $n_1, n_2, n_3 \in \mathfrak{n}$ be the generators of $\mathfrak{a}$ and $\mathfrak{n}$, respectively, where n_i is the infinitesimal translation in $\mathbb{R}\mathrm{H}(4)$ in the direction of $\partial/\partial \bar{y}^i$. All other reductive decompositions $\bar{\mathfrak{g}} = \bar{\mathfrak{m}}^{\varphi} + \bar{\mathfrak{k}}$ associated with $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{k}}$ are given by the graph of any equivariant map $\varphi\colon \mathfrak{m} \to \mathfrak{k}$. As a computation shows, all these equivariant maps are

$$\begin{aligned} \varphi_{(\lambda_0,\lambda_1)}\colon \mathfrak{m} &\to \mathfrak{k} \\ a &\mapsto \lambda_0 r \\ n_1 &\mapsto \lambda_1 r \\ n_2, n_3 &\mapsto 0, \end{aligned}$$

with $\lambda_0, \lambda_1 \in \mathbb{R}$. The homogeneous structure tensors associated with this two-parameter family of reductive decompositions are

$$\bar{S}^{(\lambda_0,\lambda_1)} = \frac{1}{(\bar{y}^0)^3}\bigg(\sum_{k=1}^{3} \mathrm{d}\bar{y}^k \otimes \mathrm{d}\bar{y}^k \wedge \mathrm{d}\bar{y}^0 - \lambda_0\, \mathrm{d}\bar{y}^0 \otimes \mathrm{d}\bar{y}^2 \wedge \mathrm{d}\bar{y}^3 - \lambda_1\, \mathrm{d}\bar{y}^1 \otimes \mathrm{d}\bar{y}^2 \wedge \mathrm{d}\bar{y}^3\bigg),$$

and the canonical connection $\bar{\tilde{\nabla}} = \bar{\nabla} - \bar{S}^{(\lambda_0,\lambda_1)}$ (where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$) is then given by

$$\begin{aligned} &\bar{\tilde{\nabla}}_{\partial_0}\partial_0 = -\frac{1}{\bar{y}^0}\partial_0, && \bar{\tilde{\nabla}}_{\partial_0}\partial_1 = -\frac{1}{\bar{y}^0}\partial_1, && \bar{\tilde{\nabla}}_{\partial_0}\partial_2 = -\frac{1}{\bar{y}^0}\partial_2 + \frac{\lambda_0}{\bar{y}^0}\partial_3, \\ &\bar{\tilde{\nabla}}_{\partial_0}\partial_3 = -\frac{1}{\bar{y}^0}\partial_3 - \frac{\lambda_0}{\bar{y}^0}\partial_2, && \bar{\tilde{\nabla}}_{\partial_1}\partial_2 = \frac{\lambda_1}{\bar{y}^0}\partial_3, && \bar{\tilde{\nabla}}_{\partial_1}\partial_3 = -\frac{\lambda_1}{\bar{y}^0}\partial_2, \end{aligned}$$

where ∂_k stands for $\partial/\partial \bar{y}^k$. Let $H \simeq \mathbb{R}$ be the subgroup of $\mathbb{R}\mathrm{H}(4)$ given by

$$H = \{(1, \lambda, 0, 0)/\lambda \in \mathbb{R}\}.$$

We take the H-principal bundle

$$\begin{aligned} \mathbb{R}\mathrm{H}(4) &\to \mathbb{R}\mathrm{H}(3) \\ (\bar{y}^0, \bar{y}^1, \bar{y}^2, \bar{y}^3) &\mapsto (\bar{y}^0, \bar{y}^2, \bar{y}^3) \end{aligned}$$

https://doi.org/10.1017/S0013091513000679 Published online by Cambridge University Press

with mechanical connection form $\omega = \mathrm{d}\bar{y}^1$. We have that

$$\tilde{\bar{\nabla}}\omega = \left(\frac{1}{\bar{y}^0}\,\mathrm{d}\bar{y}^0\right)\cdot\omega,$$

where we have identified $\mathfrak{h} \simeq \mathbb{R}$ and $\mathrm{End}(\mathfrak{h}) \simeq \mathbb{R}$. From Theorem 3.7, the family of homogeneous structure tensors $\bar{S}^{(\lambda_0,\lambda_1)}$ can then be reduced to $\mathbb{R}\mathrm{H}(3)$. If (y^0, y^1, y^2) are the standard coordinates of $\mathbb{R}\mathrm{H}(3)$, these reduced homogeneous structure tensors form a one-parameter family

$$S^{\lambda_0} = \frac{1}{(y^0)^3}\left(\sum_{k=1}^{2}\mathrm{d}y^k\otimes\mathrm{d}y^k\wedge\mathrm{d}y^0 - \lambda_0\,\mathrm{d}y^0\otimes\mathrm{d}y^1\wedge\mathrm{d}y^2\right).$$

Note that in the expression of both $\bar{S}^{(\lambda_0,\lambda_1)}$ and S^{λ_0} the first summand is the standard $\mathcal{S}_1$ structure of $\mathbb{R}\mathrm{H}(4)$ and $\mathbb{R}\mathrm{H}(3)$, respectively. The other summands are of type $\mathcal{S}_2\oplus\mathcal{S}_3$ since they have null trace, which makes $\bar{S}^{(\lambda_0,\lambda_1)}$ and S^{λ_0} of type $\mathcal{S}_1\oplus\mathcal{S}_2\oplus\mathcal{S}_3$ in the generic case. In the special case $\lambda_0 = 0$ we have a reduction of the generic class $\mathcal{S}_1\oplus\mathcal{S}_2\oplus\mathcal{S}_3$ to the class $\mathcal{S}_1$. This example can be generalized to the principal bundle $\mathbb{R}\mathrm{H}(n)\to\mathbb{R}\mathrm{H}(n-1)$.

4.2. Hopf fibrations

4.2.1. *The fibration $S^3 \to S^2$*

Let $S^3 \subset \mathbb{R}^4 \simeq \mathbb{C}^2$ be the 3-sphere with its standard Riemannian metric with full isometry group $\mathrm{O}(4)$. The natural action of $\mathrm{U}(2)$ in $\mathbb{C}^2$ defines a transitive and effective action of $\mathrm{U}(2)$ on S^3 given by

$$\begin{gathered}\mathrm{U}(2)\hookrightarrow\mathrm{SO}(4)\\ \begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto\begin{pmatrix}\mathrm{Re}(a) & -\mathrm{Im}(a) & \mathrm{Re}(b) & -\mathrm{Im}(b)\\ \mathrm{Im}(a) & \mathrm{Re}(a) & \mathrm{Im}(b) & \mathrm{Re}(b)\\ \mathrm{Re}(c) & -\mathrm{Im}(c) & \mathrm{Re}(d) & -\mathrm{Im}(d)\\ \mathrm{Im}(c) & \mathrm{Re}(c) & \mathrm{Im}(d) & \mathrm{Re}(d)\end{pmatrix}.\end{gathered}$$

The isotropy group at $\bar{x} = (1,0,0,0)\in S^3$ is

$$\bar{K} = \left\{\begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix}\in\mathrm{U}(2)\Big/ z\in\mathrm{U}(1)\right\}$$

with Lie algebra

$$\bar{\mathfrak{k}} = \mathrm{span}\left\{\begin{pmatrix}0 & 0\\ 0 & \mathrm{i}\end{pmatrix}\right\}.$$

It is easy to see that the complement

$$\bar{\mathfrak{m}} = \mathrm{span}\left\{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \begin{pmatrix}\mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix}\right\}$$

https://doi.org/10.1017/S0013091513000679 Published online by Cambridge University Press

makes $\mathfrak{u}(2) = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ a reductive decomposition. The rest of the complements $\bar{\mathfrak{m}}'$ giving reductive decompositions $\mathfrak{u}(2) = \bar{\mathfrak{m}}' \oplus \bar{\mathfrak{k}}$ are obtained as the graph of $\mathrm{Ad}(\bar{K})$-equivariant maps $\varphi\colon \bar{\mathfrak{m}} \to \bar{\mathfrak{k}}$. One can check that these decompositions are exhausted by the following one-parameter family of complements:

$$\bar{\mathfrak{m}}_\lambda = \mathrm{span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i} \end{pmatrix} \right\}, \quad \lambda \in \mathbb{R}.$$

From (3.8), the expression of the homogeneous structure tensor $\bar{S}^\lambda$ associated with each reductive decomposition computed at $T_{\bar{x}}S^3$ is given by

$$(\bar{S}^\lambda)_{\bar{x}} = (\lambda - 1)\, \mathrm{d}\bar{x}^2 \otimes \mathrm{d}\bar{x}^3 \wedge \mathrm{d}\bar{x}^4 + \mathrm{d}\bar{x}^3 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^4 - \mathrm{d}\bar{x}^4 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^3, \tag{4.1}$$

where $(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4)$ is the natural system of coordinates in $\mathbb{R}^4$.

Let H be the subgroup of $\mathrm{U}(2)$ isomorphic to $\mathrm{U}(1)$ given by

$$H = \left\{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \Big/ z \in \mathrm{U}(1) \right\}.$$

It is easy to check that H is a normal subgroup of $\mathrm{U}(2)$ acting freely on S^3. Reduction by the action of H gives the Hopf fibration $S^3 \to S^2$ with vertical and horizontal subspaces at $\bar{x}$,

$$V_{\bar{x}}S^3 = \mathrm{span}\left\{ \frac{\partial}{\partial \bar{x}_2} \right\}, \qquad H_{\bar{x}}S^3 = \mathrm{span}\left\{ \frac{\partial}{\partial \bar{x}_3}, \frac{\partial}{\partial \bar{x}_4} \right\}.$$

Since all the terms of $\bar{S}^\lambda$ have the vertical factor $\mathrm{d}\bar{x}^2$, it is obvious that they all reduce to the structure tensor $S = 0$ on S^2, describing S^2 as a symmetric space. Note that this is what one can expect, since S^2 only admits the zero homogeneous structure tensor [**18**].

For the case $\lambda = 0$, one can follow the proof of the Ambrose–Singer theorem to construct the Lie algebra of a group acting transitively on S^3. As a computation shows, the holonomy of the connection $\tilde{\bar{\nabla}} = \bar{\nabla} - \bar{S}_0$ is trivial, and one obtains the reductive decomposition $T_eS^3 \oplus \{0\} \simeq \mathfrak{su}(2)$ that describes the action of $\mathrm{SU}(2) \simeq S^3$ on itself. We then have an example of a homogeneous Riemannian structure $\bar{S}^0$ satisfying $\tilde{\bar{\nabla}}\omega = \alpha \cdot \omega$ as in Theorem 3.7 (ω being the mechanical connection form of the Hopf fibration $S^3 \to S^2$), but for which the structure group of the fibration ($H = \mathrm{U}(1)$) can not be seen as a normal subgroup of the group ($\bar{G}' = \mathrm{SU}(2)$) obtained by the proof of the Ambrose–Singer theorem.

Remark 4.1. There are no more reducible tensors than those described above, as the other groups acting transitively on S^3 are $\mathrm{SO}(4)$, which has no normal subgroups, and $\mathrm{SU}(2) \simeq S^3$. In addition, this procedure can be adapted to the Berger 3-spheres, where a family of homogeneous structures is calculated in [**7**]. All reducible structures of this family reduce to $S = 0$ on S^2 as expected.

Remark 4.2. The groups acting isometrically and transitively on S^7 (see [**17**]) are $\mathrm{SO}(7)$, $\mathrm{SU}(4)$, $\mathrm{Sp}(2)\mathrm{Sp}(1)$, $\mathrm{U}(4)$ and $\mathrm{Sp}(2)\mathrm{U}(1)$. The first two groups do not have normal subgroups and, hence, do not fit in the reduction scheme. The group $\bar{G} = \mathrm{Sp}(2)\mathrm{Sp}(1)$ has the normal subgroup $H = \mathrm{Sp}(1) = \mathrm{SU}(2)$, which gives the Hopf fibration $S^7 \to S^4$. In this case, a similar computation to the fibration $S^3 \to S^2$ shows that the corresponding homogeneous Riemannian structures in the 7-sphere reduce to the null tensor on S^4, the only homogeneous structure in the four-dimensional sphere. The last two groups are analyzed in the following subsection.

4.2.2. *The fibration* $S^7 \to \mathbb{C}P^3$

Let Δ^i_j denote the 4×4 complex matrix with 1 in the ith row and the jth column, and all other entries equal to 0. Let S^7 be the standard 7-sphere as a Riemannian submanifold of $\mathbb{C}^4$ with the usual Hermitian inner product. The standard action of the unitary group $\mathrm{U}(4)$ on $\mathbb{C}^4$ gives a transitive and effective action on S^7 by isometries. The isotropy group $\bar{K}$ at $\bar{x} = (1, 0, 0, 0) \in \mathbb{S}^7$ is isomorphic to $\mathrm{U}(3)$ and we can decompose $\mathfrak{u}(4) = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$, where

$$\bar{\mathfrak{k}} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \Big/ A \in \mathfrak{u}(3) \right\}$$

and

$$\bar{\mathfrak{m}} = \mathrm{span}\{\mathrm{i}\Delta^1_1, \Delta^1_j - \Delta^j_1, \mathrm{i}(\Delta^1_j + \Delta^j_1),\ j = 1, 2, 3\}.$$

One can check that $\mathfrak{u}(4) = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ is the unique reductive decomposition of $\mathfrak{u}(4)$ with respect to $\bar{\mathfrak{k}}$. From (3.8), identifying $\mathbb{R}^8 \simeq \mathbb{C}^4$ and taking its natural coordinates $(\bar{x}^1, \dots, \bar{x}^8)$, the expression of the homogeneous structure tensor $\bar{S}$ associated with this decomposition at $T_{\bar{x}}S^7$ reads

$$\begin{aligned} \bar{S}_{\bar{x}} = {} & \mathrm{d}\bar{x}^3 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^4 - \mathrm{d}\bar{x}^4 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^3 + \mathrm{d}\bar{x}^5 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^6 \\ & - \mathrm{d}\bar{x}^6 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^5 + \mathrm{d}\bar{x}^7 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^8 - \mathrm{d}\bar{x}^8 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^7. \end{aligned} \tag{4.2}$$

As a simple computation shows, this tensor belongs to the class $\mathcal{S}_2 \oplus \mathcal{S}_3$.

Let H be the subgroup of $\mathrm{U}(4)$ isomorphic to $\mathrm{U}(1)$ given by

$$H = \{z \cdot I / z \in \mathrm{U}(1)\},$$

where I is the 4×4 identity matrix. It is obvious that H is a normal subgroup of $\mathrm{U}(4)$ and its action on S^7 is free. The reduction of S^7 by the action of H gives the Hopf fibration $S^7 \to \mathbb{C}P^3$ with which the complex projective space inherits the Fubini–Study metric. The vertical and horizontal subspaces at $\bar{x}$ are

$$V_{\bar{x}}S^7 = \mathrm{span}\left\{\frac{\partial}{\partial \bar{x}_2}\right\}, \qquad H_{\bar{x}}S^7 = \mathrm{span}\left\{\frac{\partial}{\partial \bar{x}_3}, \dots, \frac{\partial}{\partial \bar{x}_8}\right\}.$$

As in the Hopf fibration $S^3 \to S^2$, the homogeneous structure tensor $\bar{S}$ reduces to $S = 0$, describing

$$\mathbb{C}P^3 = \frac{U(4)}{U(3) \times \mathrm{U}(1)}$$

as a symmetric space.

If $\mathbb{H}$ denotes the quaternion algebra, we now see the 7-sphere

$$S^7 = \left\{ \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathbb{H}^2 \middle/ |q_1|^2 + |q_2|^2 = 1 \right\}$$

as a Riemannian sub-manifold of $\mathbb{H}^2$ with the standard quaternion inner product. The group $\mathrm{Sp}(2)\mathrm{U}(1) = \mathrm{Sp}(2) \times_{\mathbb{Z}_2} \mathrm{U}(1)$ acts on $\mathbb{H}^2$ by

$$(A, z) \cdot \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = A \begin{pmatrix} q_1 \bar{z} \\ q_2 \bar{z} \end{pmatrix}, \quad \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathbb{H}^2,\ A \in \mathrm{Sp}(2),\ z \in \mathrm{U}(1),$$

where $\bar{z}$ stands for the complex conjugation. This action restricts to a transitive and effective action by isometries on S^7. The isotropy group at $\bar{x} = (1, 0) \in S^7$ is

$$\bar{K} = \left\{ \left(\begin{pmatrix} z & 0 \\ 0 & q \end{pmatrix}, z \right) \middle/ q \in \mathrm{Sp}(1),\ z \in \mathrm{U}(1) \right\} / \mathbb{Z}_2,$$

which is isomorphic to $\mathrm{Sp}(1)\mathrm{U}(1)$. Let i, j, k be the imaginary quaternion units and let i be the imaginary complex unit. The Lie algebra of $\mathrm{Sp}(2)\mathrm{U}(1)$ is then $\mathfrak{sp}(2) \oplus \mathfrak{u}(1)$, where

$$\mathfrak{sp}(2) = \mathrm{span} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{i} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{j} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{j} \\ \mathrm{j} & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \mathrm{k} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{k} \\ \mathrm{k} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{j} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{k} \end{pmatrix} \right\}$$

and $\mathfrak{u}(1) = \mathrm{span}\{\mathrm{i}\}$; the isotropy algebra is then

$$\bar{\mathfrak{k}} = \mathrm{span} \left\{ \begin{pmatrix} \mathrm{i} & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{i}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{j} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{k} \end{pmatrix} \right\}.$$

Taking

$$\bar{\mathfrak{m}} = \mathrm{span} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{i} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{j} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{j} \\ \mathrm{j} & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{k} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{k} \\ \mathrm{k} & 0 \end{pmatrix} \right\},$$

we have that $\mathfrak{sp}(2) \oplus \mathfrak{u}(1) = \bar{\mathfrak{m}} \oplus \bar{\mathfrak{k}}$ is a reductive decomposition. All other reductive decompositions associated with $\mathfrak{sp}(2) \oplus \mathfrak{u}(1)$ and $\bar{\mathfrak{k}}$ are given by a one-parameter family of complements $\bar{\mathfrak{m}}_\lambda$, $\lambda \in \mathbb{R}$, which are the graph of the $\mathrm{Ad}(\bar{K})$-equivariant maps $\varphi_\lambda \colon \bar{\mathfrak{m}} \to \bar{\mathfrak{k}}$, where φ_λ maps

$$\begin{pmatrix} \mathrm{i} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{to} \quad \lambda \begin{pmatrix} \mathrm{i} & 0 \\ 0 & 0 \end{pmatrix} + \lambda \mathrm{i},$$

and the rest of the elements of the basis to 0. Identifying $\mathbb{H}^2 \equiv \mathbb{R}^8$, the homogeneous structure tensor $\bar{S}^\lambda$ associated with each reductive decomposition $\mathfrak{sp}(2) \oplus \mathfrak{u}(1) = \bar{\mathfrak{m}}_\lambda \oplus \bar{\mathfrak{k}}$

is computed at $T_{\bar{x}}S^7$ as

$$\begin{aligned}(\bar{S}^\lambda)_{\bar{x}} = {} & \mathrm{d}\bar{x}^5 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^6 + \mathrm{d}\bar{x}^5 \otimes \mathrm{d}\bar{x}^3 \wedge \mathrm{d}\bar{x}^7 + \mathrm{d}\bar{x}^5 \otimes \mathrm{d}\bar{x}^4 \wedge \mathrm{d}\bar{x}^8 \\ & - \lambda\, \mathrm{d}\bar{x}^2 \otimes \mathrm{d}\bar{x}^5 \wedge \mathrm{d}\bar{x}^6 + (1+2\lambda)\, \mathrm{d}\bar{x}^2 \otimes \mathrm{d}\bar{x}^3 \wedge \mathrm{d}\bar{x}^4 + \lambda\, \mathrm{d}\bar{x}^2 \otimes \mathrm{d}\bar{x}^7 \wedge \mathrm{d}\bar{x}^8 \\ & + \mathrm{d}\bar{x}^6 \otimes \mathrm{d}\bar{x}^5 \wedge \mathrm{d}\bar{x}^2 + \mathrm{d}\bar{x}^6 \otimes \mathrm{d}\bar{x}^3 \wedge \mathrm{d}\bar{x}^8 - \mathrm{d}\bar{x}^6 \otimes \mathrm{d}\bar{x}^4 \wedge \mathrm{d}\bar{x}^7 \\ & + \mathrm{d}\bar{x}^3 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^4 + \mathrm{d}\bar{x}^4 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^3 \\ & - \mathrm{d}\bar{x}^7 \otimes \mathrm{d}\bar{x}^3 \wedge \mathrm{d}\bar{x}^5 - \mathrm{d}\bar{x}^7 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^8 + \mathrm{d}\bar{x}^7 \otimes \mathrm{d}\bar{x}^4 \wedge \mathrm{d}\bar{x}^6 \\ & - \mathrm{d}\bar{x}^8 \otimes \mathrm{d}\bar{x}^4 \wedge \mathrm{d}\bar{x}^5 + \mathrm{d}\bar{x}^8 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^7 - \mathrm{d}\bar{x}^8 \otimes \mathrm{d}\bar{x}^3 \wedge \mathrm{d}\bar{x}^6.\end{aligned}$$

Let $H = \{(\mathrm{Id}, w)/w \in \mathrm{U}(1)\} \subset \mathrm{Sp}(2)\mathrm{U}(1)$, where Id is the identity of Sp(2); it is easy to see that H is a normal subgroup of Sp(2)U(1) isomorphic to U(1). Reduction by the action of H again gives the Hopf fibration $\pi\colon S^7 \to \mathbb{C}P^3$ with $\pi(\bar{x}) = [1:0:0:0] \in \mathbb{C}P^3$. The vertical and horizontal subspaces of π at $\bar{x}$ are

$$V_{\bar{x}}S^7 = \mathrm{span}\left\{\frac{\partial}{\partial \bar{x}_2}\right\}, \qquad H_{\bar{x}}S^7 = \mathrm{span}\left\{\frac{\partial}{\partial \bar{x}_3}, \dots, \frac{\partial}{\partial \bar{x}_8}\right\}.$$

Let $(t^1, \dots, t^6)\colon \mathbb{C}P^3 - \{z_0 = 0\} \to \mathbb{R}^6$ be the coordinate system around $x = [1:0:0:0]$ given by

$$[z_0\colon z_1\colon z_2\colon z_3] \mapsto \left(\mathrm{Re}\left(\frac{z_1}{z_0}\right), \mathrm{Im}\left(\frac{z_1}{z_0}\right), \mathrm{Re}\left(\frac{z_2}{z_0}\right), \mathrm{Im}\left(\frac{z_2}{z_0}\right), \mathrm{Re}\left(\frac{z_3}{z_0}\right), \mathrm{Im}\left(\frac{z_3}{z_0}\right)\right).$$

The reduced homogeneous structure tensor S is computed at $T_x\mathbb{C}P^3$ as

$$\begin{aligned} S_x = {} & \mathrm{d}t^3 \otimes \mathrm{d}t^1 \wedge \mathrm{d}t^5 + \mathrm{d}t^3 \otimes \mathrm{d}t^2 \wedge \mathrm{d}t^6 \\ & + \mathrm{d}t^4 \otimes \mathrm{d}t^1 \wedge \mathrm{d}t^6 - \mathrm{d}t^4 \otimes \mathrm{d}t^2 \wedge \mathrm{d}t^5 \\ & + \mathrm{d}t^5 \otimes \mathrm{d}t^2 \wedge \mathrm{d}t^4 - \mathrm{d}t^5 \otimes \mathrm{d}t^1 \wedge \mathrm{d}t^3 \\ & - \mathrm{d}t^6 \otimes \mathrm{d}t^2 \wedge \mathrm{d}t^3 - \mathrm{d}t^6 \otimes \mathrm{d}t^1 \wedge \mathrm{d}t^4. \end{aligned}$$

It is easy to check that $\bar{S}^\lambda$ is an $\mathcal{S}_2 \oplus \mathcal{S}_3$ structure for all $\lambda \in \mathbb{R}$, and not $\mathcal{S}_2$ nor $\mathcal{S}_3$ for any λ, and S is also a strict $\mathcal{S}_2 \oplus \mathcal{S}_3$ structure. Note that in the latter and the previous example the class $\mathcal{S}_2 \oplus \mathcal{S}_3$ is preserved by the reduction procedure. This fact is expected from Proposition 3.11 since the fibres of the Hopf fibration are totally geodesic and, in particular, minimal Riemannian sub-manifolds of S^7.

5. Almost contact metric–almost Hermitian and Sasakian–Kähler reduction

An *almost contact structure* on a manifold $\bar{M}$ is a triple (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ is a vector field and η is a 1-form satisfying

$$\phi(\xi) = 0, \qquad \eta(\phi(\bar{X})) = 0, \qquad \eta(\xi) = 1,$$
$$\phi^2 = -\mathrm{Id} + \eta \otimes \xi$$

for all $\bar{X} \in \mathfrak{X}(\bar{M})$. The almost contact structure is said to be *strictly regular* if ξ is a regular vector field whose integral curves are homeomorphic, and *invariant* if ϕ and η are invariant by the action of the one-parameter group of ξ. In the following almost all contact structures are supposed to be invariant and strictly regular. The following results were proved in [**16**].

Theorem 5.1. *Let (ϕ, ξ, η) be an almost contact structure and let M be the space of orbits given by ξ. Then, M is endowed with a smooth structure such that $\pi\colon \bar{M} \to M$ is a principal bundle and η is a connection form.*

Theorem 5.2. *In the situation of the previous theorem, the $(1,1)$-tensor field J defined in M by*

$$J_x X = \pi_*(\phi_{\bar{x}} X^{\mathrm{H}}), \quad x \in M,\ X \in \mathfrak{X}(M),$$

where $\bar{x} \in \pi^{-1}(x) \subset \bar{M}$ and X^{H} is the horizontal lift of X with respect to η, is an almost complex structure.

If $\bar{M}$ is equipped with a Riemannian metric $\bar{g}$, an almost contact structure (ϕ, ξ, η) is said to be *metric* if the following conditions hold:

$$\bar{g}(\xi, \bar{X}) = \eta(\bar{X}), \qquad \bar{g}(\phi\bar{X}, \phi\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y}).$$

Note that this implies that η defines the mechanical connection in $(\bar{M}, \bar{g}) \to M$ and induces a Riemannian metric g in M. In this situation it can be proved [**16**] that (J, g) is almost Hermitian. Let $\Phi(\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \phi\bar{Y})$ be the fundamental or Sasaki 2-form of the almost contact metric structure; then, (ϕ, ξ, η, g) is called an *almost Sasakian* structure if $\mathrm{d}\eta = 2\Phi$. If, moreover, $\bar{\nabla}\phi = \bar{g} \otimes \xi - \mathrm{Id} \otimes \eta$, where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$, then it is called a *Sasakian* structure. It can be proved [**16**] that if (ϕ, ξ, η, g) is (almost) Sasakian, then (J, g) is (almost) Kähler.

An almost contact metric manifold is called homogeneous almost contact metric if there exists a transitive group of isometries such that ϕ is invariant (and then so are ξ and η). If the manifold is (almost) Sasakian, then it is called (almost) Sasakian homogeneous. A homogeneous structure tensor $\bar{S}$ on $\bar{M}$ is called a homogeneous almost contact metric structure if $\tilde{\bar{\nabla}}\phi = 0$ (and then $\tilde{\bar{\nabla}}\xi = 0$ and $\tilde{\bar{\nabla}}\eta = 0$). From the result of Kiričenko [**11**] we have that a connected, simply connected and complete Riemannian manifold is a homogeneous almost contact metric manifold if and only if it admits a homogeneous almost contact metric structure. If the manifold is (almost) Sasakian, then it is homogeneous (almost) Sasakian if and only if it admits a homogeneous (almost) Sasakian structure.

We now assume that $\bar{S}$ is an almost contact metric homogeneous structure invariant by the one-parameter group of ξ. Since $\tilde{\bar{\nabla}}\eta = 0$, we are in the situation of Theorem 3.7, and then the tensor $S_X Y = \pi_*(\bar{S}_{X^{\mathrm{H}}} Y^{\mathrm{H}})$ defines a homogenous structure on M.

Proposition 5.3. *The reduced homogeneous structure S in M is a homogeneous almost Hermitian structure on M. Moreover, if $\bar{S}$ is a homogeneous (almost) Sasakian structure, then the reduced homogeneous structure S is a homogeneous (almost) Kähler structure on M.*

Proof. Let $\tilde{\nabla} = \nabla - S$, where ∇ is the Levi-Civita connection of g. Then $\tilde{\nabla}_X Y = \pi_*(\tilde{\bar{\nabla}}_{X^{\mathrm{H}}} Y^{\mathrm{H}})$. Since $\eta(\phi(\bar{X})) = 0$, we have that $\phi(\bar{X})$ is horizontal for all $\bar{X} \in \mathfrak{X}(\bar{M})$. For any $X, Y \in \mathfrak{X}(M)$ we have that

$$\begin{aligned}(\tilde{\nabla}_X J)Y &= \tilde{\nabla}_X(JY) - J(\tilde{\nabla}_X Y) \\ &= \pi_*(\tilde{\bar{\nabla}}_{X^{\mathrm{H}}}(JY)^{\mathrm{H}}) - \pi_*(\phi(\tilde{\bar{\nabla}}_{X^{\mathrm{H}}} Y^{\mathrm{H}})) \\ &= \pi_*(\tilde{\bar{\nabla}}_{X^{\mathrm{H}}}(\phi Y^{\mathrm{H}}) - \phi(\tilde{\bar{\nabla}}_{X^{\mathrm{H}}} Y^{\mathrm{H}})) \\ &= \pi_*((\tilde{\bar{\nabla}}_{X^{\mathrm{H}}}\phi) Y^{\mathrm{H}}) \\ &= 0,\end{aligned}$$

and hence $\tilde{\nabla} J = 0$. □

We now apply Proposition 5.3 to the Hopf fibrations $S^3 \to S^2$ and $S^7 \to \mathbb{C}P^3$ and check that the Sasakian–Kähler reduction procedure gives the null Kähler structures of the reduced spaces, the only homogeneous Kähler structures existing on S^2 and $\mathbb{C}P^3$. For the first case, let $(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4)$ be the natural coordinates of $\mathbb{R}^4$ and let

$$\alpha = -\bar{x}^2\,\mathrm{d}\bar{x}^1 + \bar{x}^1\,\mathrm{d}\bar{x}^2 - \bar{x}^4\,\mathrm{d}\bar{x}^3 + \bar{x}^3\,\mathrm{d}\bar{x}^4.$$

If $i\colon S^3 \to \mathbb{R}^4$ is the natural immersion of the Euclidean 3-sphere in $\mathbb{R}^4$, the form $\eta = i^*\alpha$ defines an almost contact metric structure on S^3 that is, moreover, a Sasakian structure [**3**]. One can check (see [**7**]) that the homogeneous Sasakian structures on S^3 with respect to η are those given in (4.1) after the isometry

$$\begin{aligned}\varphi\colon S^3 &\to S^3 \\ (\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4) &\mapsto (\bar{x}^1, -\bar{x}^2, -\bar{x}^3, -\bar{x}^4),\end{aligned}$$

namely,

$$(\bar{S}^\lambda)_{\bar{x}} = (1-\lambda)\,\mathrm{d}\bar{x}^2 \otimes \mathrm{d}\bar{x}^3 \wedge \mathrm{d}\bar{x}^4 - \mathrm{d}\bar{x}^3 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^4 + \mathrm{d}\bar{x}^4 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^3. \tag{5.1}$$

This homogeneous structures are obtained from the group of isometries

$$G = \{\varphi \circ \Phi_a \circ \varphi^{-1} / a \in \mathrm{U}(2)\},$$

where Φ_a denotes the standard action of $\mathrm{U}(2)$ on S^3. The subgroup

$$H = \{\varphi \circ \Phi_z \circ \varphi^{-1} / z \in \mathrm{U}(1)\}$$

is a normal subgroup of G, where $z \in \mathrm{U}(1)$ is seen in $\mathrm{U}(2)$ as the matrix

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}.$$

https://doi.org/10.1017/S0013091513000679 Published online by Cambridge University Press

Reduction by the action of H gives the fibration

$$S^3 \to S^2$$
$$(z_1, z_2) \mapsto (2z_1 z_2, |z_1|^2 - |z_2|^2),$$

which is precisely the fibration given by the Sasakian structure η in the sense of Theorem 5.1. The reduction described in Proposition 5.3 by the action of H of the family of homogeneous structures (5.1) is (as in § 4.2.1) the tensor $S = 0$.

As for the second fibration, we take $(\bar{x}^1, \dots, \bar{x}^8)$ as the coordinates of $\mathbb{R}^8$, and

$$\alpha = -\bar{x}^2\,\mathrm{d}\bar{x}^1 + \bar{x}^1\,\mathrm{d}\bar{x}^2 - \bar{x}^4\,\mathrm{d}\bar{x}^3 + \bar{x}^3\,\mathrm{d}\bar{x}^4 - \bar{x}^6\,\mathrm{d}\bar{x}^5 + \bar{x}^5\,\mathrm{d}\bar{x}^6 - \bar{x}^8\,\mathrm{d}\bar{x}^7 + \bar{x}^7\,\mathrm{d}\bar{x}^8.$$

The form $\eta = i^*\alpha$, where $i\colon S^7 \to \mathbb{R}^8$ is the natural immersion of the Euclidean 7-sphere, defines an almost contact metric structure on S^7 that is, moreover, Sasakian (see [**3**]). A homogeneous Sasakian structure on S^7 with respect to η is obtained by transforming (4.2) with respect to the isometry

$$\varphi\colon S^7 \to S^7$$
$$(\bar{x}^1, \dots, \bar{x}^8) \mapsto (\bar{x}^1, -\bar{x}^2, \dots, -\bar{x}^8),$$

and reads

$$\begin{aligned}\bar{S}_{\bar{x}} = &-\mathrm{d}\bar{x}^3 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^4 + \mathrm{d}\bar{x}^4 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^3 - \mathrm{d}\bar{x}^5 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^6 \\ &+ \mathrm{d}\bar{x}^6 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^5 - \mathrm{d}\bar{x}^7 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^8 + \mathrm{d}\bar{x}^8 \otimes \mathrm{d}\bar{x}^2 \wedge \mathrm{d}\bar{x}^7.\end{aligned} \tag{5.2}$$

This family of homogeneous structure tensors is also obtained from the action of the group of isometries

$$G = \{\varphi \circ \Phi_a \circ \varphi^{-1} / a \in \mathrm{U}(4)\},$$

where Φ_a denotes the standard action of U(4) on S^7. The subgroup

$$H = \{\varphi \circ \Phi_z \circ \varphi^{-1} / z \in \mathrm{U}(1)\}$$

is a normal subgroup of G, and reduction by the action of H provides the fibration given by the Sasakian structure η in the sense of Theorem 5.1. Again, the family (5.2) reduces to $S = 0$.

A non-trivial projection of homogeneous Sasakian structure tensors can be found in the following situation. Let $\pi\colon \bar{M} \to \mathbb{C}\mathrm{H}(n)$ be a principal line bundle endowed with the Sasakian structure $(\phi, \xi, \eta, \bar{g})$ given by an invariant metric $\bar{g}$ and its corresponding mechanical connection η in $\bar{M}$ (see [**8**]). Every homogeneous Kähler structure tensor S in $\mathbb{C}\mathrm{H}(n)$ can then be obtained as the reduction of the Sasakian homogeneous structure tensor

$$\bar{S}_{X^{\mathrm{H}}} Y^{\mathrm{H}} = (S_X Y)^{\mathrm{H}} - \bar{g}(X^{\mathrm{H}}, \phi Y^{\mathrm{H}})\xi, \qquad \bar{S}_{X^{\mathrm{H}}}\xi = -\phi X^{\mathrm{H}} = \bar{S}_\xi X^{\mathrm{H}}, \qquad \bar{S}_\xi \xi = 0,$$

in the sense of Proposition 5.3. The description of all these tensors was previously studied in [**8**]. Nevertheless, it is interesting to point out that the goal of that reference is to lift structures from $\mathbb{C}\mathrm{H}(n)$ to $\bar{M}$. The result given in Proposition 5.3 thus gives a reverse procedure of that particular situation.

Acknowledgements. The authors are deeply indebted to Andrew Swann and P. M. Gadea for useful conversations about the topics of this paper.

This work was partly funded by the Ministerio de Ciencia e Innovación, Spain (Project MTM2011-22528).

References

1. E. Abbena and S. Garbiero, Almost Hermitian homogeneous structures, *Proc. Edinb. Math. Soc.* **31** (1988), 375–395.
2. W. Ambrose and I. M. Singer, On homogeneous Riemannian manifolds, *Duke Math. J.* **25** (1958), 647–669.
3. D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, Volume 203 (Birkhäuser, 2002).
4. M. Castrillón López, P. M. Gadea and A. F. Swann, Homogeneous quaternionic Kähler structures and quaternionic hyperbolic space, *Transform. Groups* **11**(4) (2006), 575–608.
5. M. Castrillón López, P. M. Gadea and A. F. Swann, Homogeneous structures on real and complex hyperbolic spaces, *Illinois J. Math.* **53**(2) (2009), 561–574.
6. A. Fino, Intrinsic torsion and weak holonomy, *Math. J. Toyama Univ.* **21** (1998), 1–22.
7. P. M. Gadea and J. A. Oubiña, Homogeneous Riemannian structures on Berger 3-spheres, *Proc. Edinb. Math. Soc.* **48** (2005), 375–387.
8. P. M. Gadea and J. A. Oubiña, Homogeneous Kähler and Sasakian structures related to complex hyperbolic spaces, *Proc. Edinb. Math. Soc.* **53**(2) (2010), 393–413.
9. P. M. Gadea and J. A. Oubiña, Reductive homogeneous pseudo-Riemannian manifolds, *Monatsh. Math.* **124** (1997), 17–34.
10. P. M. Gadea, A. Montesinos Amilibia and J. Muñoz Masqué, Characterizing the complex hyperbolic space by Kähler homogeneous structures, *Math. Proc. Camb. Phil. Soc.* **128**(1) (2000), 87–94.
11. V. F. Kiričenko, On homogeneous Riemannian spaces with an invariant structure tensor, *Dokl. Akad. Nauk SSSR* **252**(2) (1980), 291–293.
12. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Volume II (Wiley, 1969).
13. J. Lee, *Introduction to smooth manifolds*, Graduate Texts in Mathematics, Volume 218 (Springer, 2003).
14. J. E. Marsden and J. Ostrowski, Symmetries in motion: geometric foundations of motion control, *Nonlin. Sci. Today* (1998), available at http://www.cds.caltech.edu/~marsden/bib/1998/13-MaOs1998/MaOs1998.pdf.
15. R. Montgomery, Isoholonomic problems and some applications, *Commun. Math. Phys.* **128**(3) (1990), 565–592.
16. K. Ogiue, On fiberings of almost contact manifolds, *Kodai Math. Sem. Rep.* **17** (1965), 53–62.
17. S. M. Salamon, *Riemannian geometry and Holonomy groups*, Pitman Research Notes in Mathematics, Volume 201 (Longman, New York, 1989).
18. F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Mathematical Society Lecture Note Series, Volume 83 (Cambridge University Press, 1983).