NOTE ON EPI IN $\mathcal{T}_o$

S. Baron

Burgess [1] has pointed out that in the categories $\mathcal{T}$ and $\mathcal{T}_1$ (where $\mathcal{T}_i$ is the category of T_i spaces), epi means onto. In this paper, Burgess' technique will be used to show that epi has a different meaning in $\mathcal{T}_o$ and that this meaning reduces to onto when the range is a T_1 space.

THEOREM. In $\mathcal{T}_o$, a map $e:A \to B$ is epi if and only if for each $b \in B$, every neighborhood of b intersects $\{b\}^- \cap e(A)$.

Proof. $\Longrightarrow$ Let $e:A \to B$ be epi and let $C = \{b/b \in B$ and every neighborhood of b intersects $\{b\}^- \cap e(A)\}$. Let $D = B_1 \cup B_2/\sim$ where B_1 and B_2 are copies of B and $\sim$ is defined as follows: $b_1 \sim b_2$ if and only if $b_1 = b_2$ or b_1 and b_2 are copies of the same $b \in C$. Let $f:B_1 \cup B_2 \to D$ be the quotient map and let $g_i:B \to B_1 \cup B_2$ be the canonical injection to the i-th copy. Each $f \circ g_i$ is 1 - 1. For any $d \in D$, it is clear that $(f \circ g_i)^{-1}(d)$ is nonempty for at least one i. When both $(f \circ g_1)^{-1}(d)$ and $(f \circ g_2)^{-1}(d)$ are nonempty, they coincide. Thus we may define a function $h:D \to B$ such that $h \circ f \circ g_i = 1_B$.

We now show that D is T_o. Let $d_1, d_2 \in D$. If $h(d_1) \neq h(d_2)$, then we may assume without loss that $h(d_1)$ has a neighborhood V that does not contain $h(d_2)$. $V' = f(g_1(V) \cup g_2(V))$ is then a neighborhood of d_1 that does not contain d_2.

Otherwise if $h(d_1) = h(d_2) = b$, we must have for suitable renumbering of d_1 and d_2, $d_i = f \circ g_i(b)$. We know $b \notin C$, since $d_1 \neq d_2$. Let V be a neighborhood of b that does not intersect $\{b\}^- \cap e(A)$. It follows that $f(g_1(\mathscr{C}(\{b\}^-)) \cup g_2(V))$ is an open set that contains d_2 but does not contain d_1. Thus, D is T_o.

https://doi.org/10.4153/CMB-1968-061-6 Published online by Cambridge University Press

$f \circ g_1$ and $f \circ g_2$ agree on $C \supseteq e(A)$; therefore, $f \circ g_1 \circ e = f \circ g_2 \circ e$. Since e is $\tilde{\mathcal{J}}_0$-epi, it follows that $f \circ g_1 = f \circ g_2$. Thus $C = B$ and we have the desired implication.

$\Leftarrow$ Suppose for each $b \in B$, each neighborhood of b intersects $\{b\}^- \cap e(A)$ and that for maps $h_1, h_2 : B \to E$, $h_1 \circ e = h_2 \circ e$. Now, suppose that for some $b \in B$, $h_1(b) \neq h_2(b)$. Since E is T_0, say $h_1(b) \notin \{h_2(b)\}^-$. Then $h_1^{-1}(\mathcal{C}\{h_2(b)\}^-)$ is an open neighborhood of b. Thus there must be an $e(y) \in \{b\}^- \cap h_1^{-1}(\mathcal{C}\{h_2(b)\}^-)$. But then $h_1 \circ e(y) = h_2 \circ e(y) \in h_2(\{b\}^-) \cap \mathcal{C}(\{h_2(b)\}^-) = \phi$. Thus for all $b \in B$, $h_1(b) = h_2(b)$ and e is epi.

COROLLARY. *If* B *is* T_1, *then* $\{b\}^- = \{b\} \subseteq e(A)$ *and* e *is* *onto*.

REFERENCE

1. W. Burgess, The meaning of mono and epi in some familiar categories. Can. Math. Bull., 8 (1965) 759-769.

McGill University

https://doi.org/10.4153/CMB-1968-061-6 Published online by Cambridge University Press