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A NOTE ON $M/G/1$ VACATION SYSTEMS WITH WAITING TIME LIMITS

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Abstract

We consider two variants of $M/G/1$ queues with exhaustive service and multiple vacations; (1) customers cannot wait for their services longer than an interval of length T, and (2) customers cannot stay in the system longer than an interval of length T. We show that the probability distribution functions of the waiting times for the two systems are given in terms of those for the corresponding $M/G/1$ vacation systems without any residence-time limits.

STEADY STATE DISTRIBUTION; ACTUAL WAITING TIME; VIRTUAL WAITING TIME; UP- AND DOWNCROSSINGS

1. Introduction

We consider $M/G/1$ queues with exhaustive service and multiple vacations. The server takes a vacation when he finds the queue empty. At the end of the vacation, the server scans the queue. If there are some customers, he serves continuously until the queue becomes empty, and then takes the next vacation. If there are no customers at the end of the vacation, the server takes another vacation. This system is studied by Levy and Yechiali (1975) and Keilson and Servi (1987).

The systems under consideration impose one of the following residence-time limits on customers: (1) no customers can wait for their services longer than an interval of length T, or (2) no customers can stay in the system longer than an interval of length T. Hereafter, the former is called the vacation system with the waiting time limit and the latter is called the vacation system with the sojourn time limit. In both systems, customers whose waiting times (or sojourn times) reach T should leave the system immediately.

In the case of no vacation, queueing systems with residence-time limits are studied by Daley (1964), Takács (1967), (1974), Cohen (1982) and Rubin and Ouaily (1988). On the other hand, the process of the unfinished work in the $M/G/1$ vacation system with the sojourn time limit is analyzed by van der Duyn Schouten (1978).

We derive the steady state probability distribution functions (PDFs) of the waiting times for both the $M/G/1$ vacation systems with the waiting time and the sojourn time limits in terms of those for the corresponding $M/G/1$ vacation systems without any limits in terms of those for the corresponding $M/G/1$ vacation systems without any residence-time limits. The analytical approach used in this paper is the level crossing argument which is developed by Brill and Posner (1977), Cohen (1977), Shanthikumar (1980).

We describe the mathematical model in detail. Customers arrive at the system in accordance with a Poisson process of density λ and they are served by a single server in order of arrival. The service time of a customer is independent and identically distributed in accordance with a general PDF $B(x)$ whose mean is denoted by $E[B]$. The length of a

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vacation is independent and identically distributed in accordance with a general PDF $V(x)$ whose mean is denoted by $E[V]$. For simplicity in description, let ρ denote $\lambda E[B]$.

Throughout this paper, we assume that the system is in equilibrium. In the following two sections, it is assumed that $\rho < 1$. The cases $\rho \geqq 1$ are considered in Section 4.

2. Waiting time limit

In this section, we consider the vacation system with the waiting time limit. Let $w(x)$ denote the steady state probability density function (p.d.f.) of the virtual waiting time. With the level crossing argument by Shanthikumar (1980), we have the following equation for $x < T$:

$$E[C_W]w(x) = 1 - V(x) + \lambda E[C_W]\int_0^x \{1 - B(x-y)\}w(y)\,dy, \qquad x < T, \tag{1}$$

where $E[C_W]$ denotes the expected length of a cycle which is defined as an interval between successive starts of vacations. The first term $1 - V(x)$ in the right-hand side of (1) represents the probability of an upcrossing of level x at the beginning of the cycle, and the second term represents the expected number of upcrossings of level x due to arrivals of customers during the cycle. The expected number of upcrossings of level x must be equal to the expected number of downcrossings of level x, and the latter is given by $E[C_W]w(x)$.

On the other hand, for $x \geqq T$, we have

$$E[C_W]w(x) = 1 - V(x) + \lambda E[C_W]\int_0^T \{1 - B(x-y)\}w(y)\,dy, \qquad x \geqq T. \tag{2}$$

Since customers who found the virtual waiting time greater than T at arrival instants leave the system without being served, the integration in the second term of the right-hand side of (2) is performed from 0 to T.

We note here that when T goes to infinity, i.e., the system has no residence-time limits, the p.d.f. $w_\infty(x)$ of the virtual waiting time satisfies (see Shanthikumar (1980))

$$\frac{E[V]}{1-\rho}w_\infty(x) = 1 - V(x) + \frac{\lambda E[V]}{1-\rho}\int_0^x \{1 - B(x-y)\}w_\infty(y)\,dy, \tag{3}$$

and $w_\infty(x)$ is given by (see Doshi (1986))

$$w_\infty(x) = \sum_{k=0}^{\infty} (1-\rho)\rho^k \hat{b}_{(k)}(x) * \hat{v}(x), \tag{4}$$

where $\hat{b}_{(k)}(x)$ denotes the k-fold convolution of the p.d.f. of the residual service time with itself, $\hat{v}(x)$ denotes the p.d.f. of the residual vacation time, and $*$ denotes the convolution operator. Because (1) and (3) have the same form except for the constant multiplier, $w(x)$ should be for $x < T$

$$w(x) = \frac{E[V]}{(1-\rho)E[C_W]}w_\infty(x), \qquad x < T. \tag{5}$$

Thus (2) becomes

$$w(x) = \frac{1 - V(x)}{E[C_W]} + \frac{\lambda E[V]}{(1-\rho)E[C_W]}\int_0^T \{1 - B(x-y)\}w_\infty(y)\,dy, \qquad x \geqq T. \tag{6}$$

Note that only $E[C_W]$ is the unknown value. We proceed to the derivation of $E[C_W]$.

Let p denote the blocking probability that an arbitrary customer leaves the system without being served. By the definition, we have

$$p = \int_T^\infty w(x)\,dx. \tag{7}$$

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Substituting $w(x)$ in (6) for that in (7) and manipulating with (3), we get

$$p=\frac{E[V]}{E[C_W]}\{1-W_\infty(T)\}, \tag{8}$$

where $W_\infty(x)$ denotes the PDF of the waiting time for the corresponding vacation system without any residence-time limits, i.e.,

$$W_\infty(x)=\int_0^x w_\infty(y)\,dy. \tag{9}$$

On the other hand, with (5), we have

$$\begin{aligned} 1-p&=\int_0^T w(x)\,dx \\ &=\frac{E[V]}{(1-\rho)E[C_W]}W_\infty(T). \end{aligned} \tag{10}$$

Thus, from (8) and (10), we get

$$E[C_W]=\left\{1+\frac{\rho}{1-\rho}W_\infty(T)\right\}E[V]. \tag{11}$$

Hence, $w(x)$ is completely determined by (5), (6) and (11).

We provide formulas with respect to individual customers in the vacation system with the waiting time limit. When the virtual waiting time is less than T, the virtual and the actual waiting times have the same p.d.f. because of the Poison arrival process. Therefore, from (5) and (11), the PDF $W(x)$ of the actual waiting time for an arbitrary customer is found to be

$$W(x)=\begin{cases}\dfrac{W_\infty(x)}{1-\rho+\rho W_\infty(T)}, & x<T\\ 1, & x\geqq T.\end{cases} \tag{12}$$

The blocking probability p is given by

$$p=\frac{(1-\rho)(1-W_\infty(T))}{1-\rho+\rho W_\infty(T)}. \tag{13}$$

Lastly, the PDF $\tilde{W}(x)$ of the waiting time of an arbitrary customer who receives his service is given by

$$\begin{aligned} \tilde{W}(x)&=\frac{1}{1-p}\int_0^x w(y)\,dy \\ &=\frac{W_\infty(x)}{W_\infty(T)}, \qquad x<T. \end{aligned} \tag{14}$$

3. Sojourn time limit

In this section, we consider the vacation system with the sojourn time limit. Note that a customer whose sojourn time reaches T should leave the system immediately even if he is being served. Thus, the steady state p.d.f. $w(x)$ of the virtual waiting time satisfies

$$E[C_S]w(x)=1-V(x)+\lambda E[C_S]\int_0^x\{1-B(x-y)\}w(y)\,dy, \qquad x<T, \tag{15}$$

and

$$E[C_S]w(x)=1-V(x), \qquad x\geqq T, \tag{16}$$

where $E[C_S]$ denotes the expected length of a cycle. By the same approach as in Section 2, $w(x)$ can be expressed as for $x < T$

$$w(x) = \frac{E[V]}{(1-\rho)E[C_S]} w_\infty(x), \qquad x < T. \tag{17}$$

Since $w(x)$ is the p.d.f., we have

$$\int_0^\infty w(x)\, dx = 1. \tag{18}$$

It follows from (16), (17) and (18) that

$$E[C_S] = \frac{E[V]}{1-\rho} W_\infty(T) + E[V] - \int_0^T \{1 - V(y)\}\, dy. \tag{19}$$

Thus, the p.d.f. $w(x)$ of the virtual waiting time for the vacation system is determined by (16), (17) and (19).

We provide the formulas with respect to individual customers in the vacation system with the sojourn time limit. From (17) and (19), the PDF $W(x)$ of the actual waiting time of an arbitrary customer is found to be

$$W(x) = \begin{cases} \dfrac{E[V]W_\infty(x)}{E[V]W_\infty(T) + (1-\rho)\left(E[V] - \int_0^T \{1 - V(y)\}\, dy\right)}, & x < T, \\ 1, & x \geqq T. \end{cases}$$

Let q denote the blocking probability that an arbitrary customer leaves the system without being completely served. Then we have

$$\begin{aligned} q &= \int_0^T \{1 - B(T-y)\} w(y)\, dy + \int_T^\infty w(y)\, dy \\ &= 1 - \frac{E[V]D_\infty(T)}{E[V]W_\infty(T) + (1-\rho)\left(E[V] - \int_0^T \{1 - V(y)\}\, dy\right)}, \end{aligned}$$

where $D_\infty(x)$ denotes the PDF of the sojourn time in the corresponding vacation system without any residence-time limits, i.e.,

$$D_\infty(x) = \int_0^x B(x-y) w_\infty(y)\, dy.$$

Lastly, the PDF $\tilde{W}(x)$ of the waiting time of an arbitrary customer who was completely served is found to be

$$\begin{aligned} \tilde{W}(x) &= \frac{1}{1-q} \int_0^x B(T-y) w(y)\, dy \\ &= \frac{1}{D_\infty(T)} \left\{ W_\infty(x)B(T-x) + \int_0^x W_\infty(T-y)\, dB(y) \right\}, \qquad x < T. \end{aligned}$$

4. General cases

In this section, we consider the cases $\rho \geqq 1$ for the two systems. The Laplace–Stieltjes transform (LST) $F^*(s)$ of a function $F(x)$ which satisfies

$$\frac{dF(x)}{dx} = 1 - V(x) + \lambda \int_0^x \{1 - B(x-y)\}\, dF(y),$$

is found to be

$$F^*(s) = \frac{1 - V^*(s)}{s - \lambda + \lambda B^*(s)}, \qquad \operatorname{Re}(s) > \lambda - \lambda\omega^*,$$

where $B^*(s)$ and $V^*(s)$ denote the LSTs of $B(x)$ and $V(x)$, respectively, and $\omega = \omega^*$ is the smallest positive real root of the equation

$$\omega = B^*(\lambda - \lambda\omega).$$

We can obtain the formulas for the vacation systems with the waiting time and the sojourn time limits in terms of $F(x)$. The derivation of the following is the same as in the cases $\rho < 1$.

For the vacation system with the waiting time limit:

$$E[C_W] = E[V] + \rho F(T),$$

$$W(x) = \begin{cases} \dfrac{F(x)}{E[V] + \rho F(T)}, & x < T, \\ 1, & x \geqq T, \end{cases}$$

$$p = 1 - \frac{F(T)}{E[V] + \rho F(T)},$$

$$\tilde{W}(x) = \frac{F(x)}{F(T)}, \qquad x < T,$$

and for the vacation system with the sojourn time limit:

$$E[C_S] = F(T) + E[V] - \int_0^T \{1 - V(y)\}\, dy,$$

$$W(x) = \begin{cases} \dfrac{F(x)}{F(T) + E[V] - \int_0^T \{1 - V(y)\}\, dy}, & x < T, \\ 1, & x \geqq T, \end{cases}$$

$$q = 1 - \frac{\int_0^T F(T-y)\, dB(y)}{F(T) + E[V] - \int_0^T \{1 - V(y)\}\, dy},$$

$$\tilde{W}(x) = \frac{F(x)B(T-x) + \int_0^x F(T-y)\, dB(y)}{\int_0^T F(T-y)\, dB(y)}, \qquad x < T.$$

Note that when $\rho < 1$, ω^* is equal to 1 and $F(x)$ becomes $E[V]W_\infty(x)/(1-\rho)$. Thus, the above formulas are also valid for $\rho < 1$ and they are equivalent to those in the previous sections.

References

BRILL, P. H. AND POSNER, M. J. M. (1977) Level crossings in point processes applied to queues: single-server case. *Operat. Res.* **25,** 662–674.

COHEN, J. W. (1977) On up- and downcrossings. *J. Appl. Prob.* **14,** 405–410.

COHEN, J. W. (1982) *The Single Server Queue,* 2nd edn, Chaps. III.4 and III.5. North-Holland, Amsterdam.

DALEY, D. J. (1964) Single-server queueing systems with uniformly limited queueing time. *J. Austral. Math. Soc.* **4,** 489–505.

DOSHI, B. T. (1986) Queueing systems with vacation—a survey. *Queueing Sys.* **1,** 29–66.

KEILSON, J. AND SERVI, L. D. (1987) Dynamics of the $M/G/1$ vacation model. *Operat. Res.* **35,** 575–582.

KEILSON, Y. AND YECHIALI, U. (1989) Blocking probability for $M/G/1$ vacation systems with occupancy level dependent schedules. *Operat. Res.* **37,** 134–140.

LEVY, Y. AND YECHIALI, U. (1975) Utilization of idle time in an $M/G/1$ queueing system. *Management Sci.* **22,** 202–211.

RUBIN, I. AND OUAILY, M. (1988) Performance of communication and queueing processors under message delay limits. *Proc. IEEE GLOBECOM* '88, Hollywood FL, 501–505.

SHANTHIKUMAR, J. G. (1980) Some analysis on the control of queues using level crossings of regenerative processes. *J. Appl. Prob.* **17,** 814–821.

TAKÁCS, L. (1967) The distribution of the content of finite dams. *J. Appl. Prob.* **4,** 151–161.

TAKÁCS, L. (1974) A single server queue with limited virtual waiting time. *J. Appl. Prob.* **11,** 612–617.

VAN DER DUYN SCHOUTEN, F. A. (1978) An $M/G/1$ queueing model with vacation times. *Z. Operat. Res.* **22,** 95–105.