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ON PSEUDO-LUCAS NUMBERS OF THE FORM $2S^2$.

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I [1] have shown that

$$u_1 = 1 \quad \text{and} \quad u_{10} = 225$$

are the only square pseudo-Lucas numbers in the set of pseudo-Lucas numbers defined by

(1) $$u_1 = 1, \quad u_2 = 6 \quad \text{and} \quad u_{n+2} = u_{n+1} + u_n \quad \text{for} \quad n > 0.$$

In this paper, it is shown that none of the pseudo-Lucas numbers are of the form $2S^2$, where S is an integer.

The following congruence holds (See e.g. [1]):

(2) $$u_{n+2r} \equiv (-1)^{r+1} u_n \,(\text{mod } L_r 2^{-s}),$$

where $S = 0$ or 1.

We need the following tables of values:–

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
u_n	5	1	6	7	13	20	33	53	86	139	225	364	589	953
t	7													
L_t	29													

Let

(3) $$2x^2 = u_n,$$

where x is an integer.

The proof is now accomplished in fourteen stages:

(a) (3) is impossible if $n \equiv 0 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_0 \,(\text{mod } L_7).$$

Thus we find that

$$\frac{u_n}{2} \equiv 17 \,(\text{mod } 29), \quad \text{since} \quad (2, 29) = 1$$

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and since

$$\left(\frac{17}{29}\right) = -1,$$

(3) is impossible.

(b) (3) is impossible if $n \equiv 1 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_1 \pmod{L_7}.$$

Thus,

$$\frac{u_n}{2} \equiv 15 \pmod{29}, \quad \text{since} \quad (2, 29) = 1$$

and since

$$\left(\frac{15}{29}\right) = -1,$$

(3) is impossible.

(c) (3) is impossible if $n \equiv 2 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_2 \pmod{L_7}.$$

Thus we find that

$$\frac{u_n}{2} \equiv 3 \pmod{29}, \quad \text{since} \quad (2, 29) = 1$$

and since

$$\left(\frac{3}{29}\right) = -1,$$

(3) is impossible.

(d) (3) is impossible if $n \equiv 3 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_3 \pmod{L_7}.$$

Thus,

$$\frac{u_n}{2} \equiv 18 \pmod{29}, \quad \text{since} \quad (2, 29) = 1$$

and since

$$\left(\frac{18}{29}\right) = -1,$$

(3) is impossible.

(e) (3) is impossible if $n \equiv 4 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_4 \pmod{L_7}.$$

Thus

$$\frac{u_n}{2} \equiv -8 \pmod{29}, \quad \text{since} \quad (2, 29) = 1$$

and since

$$\left(\frac{-8}{29}\right) = -1,$$

(3) is impossible.

(f) (3) is impossible if $n \equiv 5 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_5 \pmod{L_7}.$$

Thus we find that

$$\frac{u_n}{2} \equiv 10 \pmod{29}, \quad \text{since} \quad (2, 29) = 1$$

and since

$$\left(\frac{10}{29}\right) = -1,$$

(3) is impossible.

(g) (3) is impossible if $n \equiv 6 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_6 \pmod{L_7}.$$

Thus,

$$\frac{u_n}{2} \equiv 2 \pmod{29}, \quad \text{since} \quad (2, 29) = 1$$

and since

$$\left(\frac{2}{29}\right) = -1,$$

(3) is impossible.

(h) (3) is impossible if $n \equiv \pmod{14}$.

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For, using (2) we find that

$$u_n \equiv u_7 \pmod{L_7}.$$

Thus,

$$\frac{u_n}{2} \equiv 12 \pmod{29}, \quad \text{since} \quad (2, 29) = 1$$

and since

$$\left(\frac{12}{29}\right) = -1,$$

(3) is impossible.

(i) (3) is impossible if $n \equiv 8 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_8 \pmod{L_7}.$$

Thus,

$$\frac{u_n}{2} \equiv 43 \pmod{29}, \quad \text{since} \quad (2, 29) = 1$$

and since

$$\left(\frac{43}{29}\right) = -1,$$

(3) is impossible.

(j) (3) is impossible if $n \equiv 9 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_9 \pmod{L_7}.$$

Thus,

$$\frac{u_n}{2} \equiv -3 \pmod{29}, \quad \text{since} \quad (2, 29) = 1$$

and since

$$\left(\frac{-3}{29}\right) = -1,$$

(3) is impossible.

(k) (3) is impossible if $n \equiv 10 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_{10} \pmod{L_7}.$$

Thus,

$$\frac{u_n}{2} \equiv 98 \pmod{29}, \quad \text{since} \quad (2, 29) = 1$$

and since

$$\left(\frac{98}{29}\right) = -1,$$

(3) is impossible.

(l) (3) is impossible if $n \equiv 11 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_{11} \pmod{L_7}.$$

Thus,

$$\frac{u_n}{2} \equiv 182 \pmod{29}, \quad \text{since} \quad (2, 29) = 1$$

and since

$$\left(\frac{182}{29}\right) = -1,$$

(3) is impossible.

(m) (3) is impossible if $n \equiv 12 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_{12} \pmod{L_7}.$$

Thus,

$$\frac{u_n}{2} \equiv 280 \pmod{29}, \quad \text{since} \quad (2, 29) = 1$$

and since

$$\left(\frac{280}{29}\right) = -1,$$

(3) is impossible.

(n) (3) is impossible if $n \equiv 13 \pmod{14}$.

For, using (2) we find that

$$u_n \equiv u_{13} \pmod{L_7}.$$

Thus,

$$\frac{u_n}{2} \equiv -2 \pmod{29}, \quad \text{since } (2, 29) = 1$$

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and since

$$\left(\frac{-2}{29}\right) = -1,$$

(3) is impossible.

Hence none of the pseudo-Lucas numbers are of the form $2S^2$, where S is an integer.

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REFERENCE

1. A. Eswarathasan, *On Square Pseudo-Lucas numbers*, Canadian Mathematical Bulletin, **21** (1978), pp. 297–304.

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