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Abstract

We consider the problem of preservation of stability under the Fourier–Mukai transform $\mathcal{F}_{\mathcal{E}} : \mathbf{D}(X) \to \mathbf{D}(Y)$ on an abelian surface and a $K3$ surface. If Y is the moduli space of μ-stable sheaves on X with respect to a polarization H, we have a canonical polarization $\widehat{H}$ on Y and we have a correspondence between (X, H) and $(Y, \widehat{H})$. We show that the stability with respect to these polarizations is preserved under $\mathcal{F}_{\mathcal{E}}$, if the degree of stable sheaves on X is sufficiently large.

Introduction

Let X be an abelian or a $K3$ surface defined over $\mathbb{C}$. For a smooth projective variety Z, $\mathbf{D}(Z)$ denotes the bounded derived category of coherent sheaves on Z. For a variety Y and an object $\mathcal{E} \in \mathbf{D}(X \times Y)$, an integral functor

$$\begin{array}{rcl} \mathcal{F}_{\mathcal{E}} : \mathbf{D}(X) & \to & \mathbf{D}(Y) \\ x & \mapsto & \mathbf{R}p_{Y*}(p_X^*(x) \otimes \mathcal{E}) \end{array}$$

is called the Fourier–Mukai transform, if $\mathcal{F}_{\mathcal{E}}$ is an equivalence of categories, where p_X and p_Y are projections from $X \times Y$ to X and Y, respectively (Y is then an abelian surface or a $K3$ surface). The Fourier–Mukai transform is a very useful tool for analyzing moduli spaces of sheaves on X. In order to apply the Fourier–Mukai transform to an actual problem, it is important to study the problem of preservation of stability under the Fourier–Mukai transform. We assume that Y is a fine moduli space of stable sheaves on X with respect to H and $\mathcal{E}$ is the universal family. Under a suitable condition on H, there is a natural polarization $\widehat{H}$ on Y. For a stable sheaf E on X with respect to H, we would like to study the stability of the transform $\mathcal{F}_{\mathcal{E}}(E)$ with respect to $\widehat{H}$. For this problem, we introduced the twisted degree $\deg_G(E)$ and the G-twisted stability in [Yos01a, Yos03a], and under some conditions we showed that $\mathcal{F}_{\mathcal{E}}(E)$ or its dual is G_2-twisted stable up to the action of the shift functor, for a G_1-twisted stable sheaf E with $\deg_{G_1}(E) = 0, 1$, where $G_1 = \mathcal{E}^{\vee}_{|X \times \{y\}}$ and $G_2 = \mathcal{E}_{|\{x\} \times Y}$ (see also a generalization by Huybrechts [Huy06]). On the other hand, we showed that the Fourier–Mukai transform does not always preserve the stability, even for a μ-stable vector bundle [Yos03b].

In this paper, we shall provide positive results on this problem (Theorem 1.7). Let H be an ample divisor on X. For a coherent sheaf E on X, $R^i p_{Y*}(p_X^*(E(mH)) \otimes \mathcal{E}) = 0$, $i > 0$, for $m \gg 0$. Hence, the Fourier–Mukai transform of $E(mH)$, $m \gg 0$, is a sheaf. Since the G_1-twisted stability is defined by using the asymptotic behavior of $\chi(E(mH) \otimes G_1^{\vee})$, $m \gg 0$, we may expect the stability of $\mathcal{F}_{\mathcal{E}}(E(mH))$. Let us explain an observation on this expectation. For simplicity, we assume that X is a principally polarized abelian surface and $\mathcal{E}$ the Poincaré line bundle on $X \times X$, where we have identified the dual abelian surface Y with X. In this case, we have (1)

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$\mathrm{ch}(\mathcal{F}_{\mathcal{E}}(E)) = a - \xi + r\varrho_X$ if $\mathrm{ch}(E) = r + \xi + a\varrho_X$ where $\xi \in \mathrm{NS}(X)$, $\int_X \varrho_X = 1$, (2) $\widehat{H}$ coincides with H and (3) the twisted stability coincides with the usual stability. For a semi-stable sheaf E on X with $\mathrm{ch}(E) = r + \xi + a\varrho_X$, $\xi \in \mathrm{NS}(X)$, and a subsheaf E_1 with $\mathrm{ch}(E_1) = r_1 + \xi_1 + a_1\varrho_X$, $\xi_1 \in \mathrm{NS}(X)$, we see that

$$\begin{aligned}&\frac{\deg(\mathcal{F}_{\mathcal{E}}(E_1(mH)))}{\mathrm{rk}(\mathcal{F}_{\mathcal{E}}(E_1(mH)))} - \frac{\deg(\mathcal{F}_{\mathcal{E}}(E(mH)))}{\mathrm{rk}(\mathcal{F}_{\mathcal{E}}(E(mH)))}\\ &\quad = \frac{-(\xi_1 + mr_1H, H)}{\chi(E_1(mH))} - \frac{-(\xi + mrH, H)}{\chi(E(mH))}\\ &\quad = \frac{(r\xi_1 - r_1\xi, H)m^2(H^2)/2 + (ra_1 - r_1a)m(H^2) + ((\xi, H)a_1 - (\xi_1, H)a)}{\chi(E_1(mH))\chi(E(mH))}\end{aligned}$$

and

$$\frac{\chi(\mathcal{F}_{\mathcal{E}}(E_1(mH)))}{\mathrm{rk}(\mathcal{F}_{\mathcal{E}}(E_1(mH)))} - \frac{\chi(\mathcal{F}_{\mathcal{E}}(E(mH)))}{\mathrm{rk}(\mathcal{F}_{\mathcal{E}}(E(mH)))} = \frac{r_1\chi(E(mH)) - r\chi(E_1(mH))}{\chi(E_1(mH))\chi(E(mH))}.$$

Hence, if m is sufficiently large, then E_1 does not induce a destabilizing subsheaf of $\mathcal{F}_{\mathcal{E}}(E(mH))$. The choice of m depends on E_1. Moreover, for a subsheaf F_1 of $\mathcal{F}_{\mathcal{E}}(E(mH))$, $E_1^\bullet = \mathcal{F}_{\mathcal{E}}^{-1}(F_1)(-mH)$ may not be a subsheaf of E. Hence, in order to show the stability of $\mathcal{F}_{\mathcal{E}}(E(mH))$, this observation is not sufficient. We also need to study the complex $E_1^\bullet$ or its cohomology sheaves. This will be done in this paper.

The organization of this paper is as follows. In § 1, we first explain some background to state the main result such as twisted stability, the Fourier–Mukai transform and a canonical polarization on Y. Then we state our main result (Theorem 1.7). In § 2, we explain key results to prove the main result. We first collect two results of Huybrechts [Huy06] on the Fourier–Mukai transform and their variants in § 2.1, and then we prepare two propositions (Propositions 2.8 and 2.11) in § 2.2, which will be used to analyse $E_1^\bullet$ above. In § 3, we discuss the problem of preservation of stability. We first prove the main result in §§ 3.1 and 3.2. We next treat a special case in § 3.3. We assume that X is an abelian surface with $\mathrm{NS}(X) = \mathbb{Z}$ and Y is the dual of X. Then we can give a more precise result (Theorem 3.7). We also add a remark on the birational correspondence of moduli spaces induced by the Fourier–Mukai transform (Theorem 3.14).

This is a revised version of the second half of [Yos01b]. In that paper, we proved Theorem 1.7 under some technical conditions. In particular, we assumed the stability of $\mathcal{E}_{|\{x\}\times Y}$. Recently, in an important paper by Huybrechts [Huy06], the stability of $\mathcal{E}_{|\{x\}\times Y}$ is proved. Moreover, he found a natural abelian subcategory of the derived category which is preserved under the Fourier–Mukai transform. With these results, we can not only simplify the proof of the WIT properties in [Yos01b] but also complete the proof of the main result.

1. Some background and the main theorem

1.1 Notation

Let X be a $K3$ surface or an abelian surface defined over $\mathbb{C}$. We define a lattice structure $\langle\cdot,\cdot\rangle$ on $H^{ev}(X,\mathbb{Z}) := \bigoplus_{i=0}^2 H^{2i}(X,\mathbb{Z})$ by

$$\begin{aligned}\langle x, y\rangle &:= -\int_X x^\vee \cup y\\ &= \int_X (x_1 \cup y_1 - x_0 \cup y_2 - x_2 \cup y_0),\end{aligned}$$

where $x_i \in H^{2i}(X, \mathbb{Z})$ (respectively $y_i \in H^{2i}(X, \mathbb{Z})$) is the $2i$th component of x (respectively y) and $x^\vee = x_0 - x_1 + x_2$. It is now called the *Mukai lattice* [Muk87]. The Mukai lattice has a weight-2 Hodge structure such that the (p, q)-part is $\bigoplus_i H^{p+i,q+i}(X)$. For a coherent sheaf E on X,

$$\begin{aligned}v(E) &:= \mathrm{ch}(E)\sqrt{\mathrm{td}_X}\\ &= \mathrm{rk}(E) + c_1(E) + (\chi(E) - \epsilon\,\mathrm{rk}(E))\varrho_X \in H^{ev}(X, \mathbb{Z})\end{aligned}$$

is called the *Mukai vector* of E, where $\epsilon = 0, 1$ according to whether X is an abelian surface or a $K3$ surface, and ϱ_X is the fundamental class of X.

In [Yos01a], we introduced the notion of twisted stability. Let $K(X)$ be the Grothendieck group of X. We fix an ample divisor H on X. For $G \in K(X) \otimes \mathbb{Q}$ with $\mathrm{rk}\, G > 0$, we define the G-twisted rank, degree, and Euler characteristic of $x \in K(X) \otimes \mathbb{Q}$ by

$$\begin{aligned}\mathrm{rk}_G(x) &:= \mathrm{rk}(G^\vee \otimes x),\\ \deg_G(x) &:= \deg(G^\vee \otimes x) = (c_1(G^\vee \otimes x), H),\\ \chi_G(x) &:= \chi(G^\vee \otimes x).\end{aligned}$$

For a coherent sheaf E, we set

$$\mu_G(E) := \begin{cases} \dfrac{\deg_G(E)}{\mathrm{rk}_G(E)}, & \mathrm{rk}\, E > 0,\\ \infty, & \mathrm{rk}\, E = 0.\end{cases}$$

We define the G-twisted stability as follows.

Definition 1.1.

(1) A torsion-free sheaf E on X is *G-twisted semi-stable* (respectively *G-twisted stable*) with respect to H, if

$$\frac{\chi_G(F(nH))}{\mathrm{rk}_G(F)} \le \frac{\chi_G(E(nH))}{\mathrm{rk}_G(E)}, \quad n \gg 0$$

for $0 \subsetneq F \subsetneq E$ (respectively the inequality is strict).

(2) A purely one-dimensional sheaf E on X is *G-twisted semi-stable* (respectively *G-twisted stable*) with respect to H, if

$$\frac{\chi_G(F(nH))}{\deg_G(F)} \le \frac{\chi_G(E(nH))}{\deg_G(E)}, \quad n \gg 0$$

for $0 \subsetneq F \subsetneq E$ (respectively the inequality is strict).

Definition 1.2. For a Mukai vector v, we denote the moduli stack of G-twisted semi-stable sheaves E with $v(E) = v$ by $\mathcal{M}_H^G(v)^{ss}$ and the open substack consisting of G-twisted stable sheaves by $\mathcal{M}_H^G(v)^s$. Let $\overline{M}_H^G(v)$ (respectively $M_H^G(v)$) be the moduli space of S-equivalence classes of G-twisted semi-stable sheaves (respectively G-twisted stable sheaves) E with $v(E) = v$. If $G = \mathcal{O}_X$, then we omit the symbol G (e.g. we abbreviate $\mathcal{M}_H^{\mathcal{O}_X}(v)^{ss}$ to $\mathcal{M}_H(v)^{ss}$).

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DEFINITION 1.3. For a coherent sheaf $E \neq 0$ on X, we set

$$\mu_{\max,G}(E) := \max_{0 \neq F \subset E} \mu_G(F),$$
$$\mu_{\min,G}(E) := \min_{F \subsetneq E} \mu_G(E/F).$$

The following easy lemma shows that $\mu_{\max,G}(E)$ and $\mu_{\min,G}(E)$ are well-defined.

LEMMA 1.1.

(1) *For a torsion-free sheaf E on X, let $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$ be the Harder–Narasimhan filtration of E with respect to the μ-semi-stability. Then*

$$\mu_{\max,G}(E) = \frac{\deg_G(F_1)}{\operatorname{rk}_G(F_1)} = \frac{\deg(F_1)}{\operatorname{rk} F_1} - \frac{\deg G}{\operatorname{rk} G},$$
$$\mu_{\min,G}(E) = \frac{\deg_G(F_s/F_{s-1})}{\operatorname{rk}_G(F_s/F_{s-1})} = \frac{\deg(F_s/F_{s-1})}{\operatorname{rk}(F_s/F_{s-1})} - \frac{\deg G}{\operatorname{rk} G}.$$

(2) *If E($\operatorname{rk} E \neq 0$) has a torsion, then $\mu_{\max,G}(E) = \infty$ and $\mu_{\min,G}(E) = \mu_{\min,G}(E/T)$, where T is the torsion submodule of E.*

(3) *If $E \neq 0$ is a torsion sheaf, then $\mu_{\max,G}(E) = \mu_{\min,G}(E) = \infty$.*

Proof. Part (1) follows from properties of the Harder–Narasimhan filtration. Parts (2) and (3) are obvious. □

DEFINITION 1.4. Let v be a Mukai vector with $\operatorname{rk} v > 0$. A polarization H on X is general with respect to v if, for every μ-semi-stable sheaf E with $v(E) = v$ and a subsheaf $F \neq 0$ of E,

$$\frac{(c_1(F), H)}{\operatorname{rk} F} = \frac{(c_1(E), H)}{\operatorname{rk} E} \quad \text{if and only if} \quad \frac{c_1(F)}{\operatorname{rk} F} = \frac{c_1(E)}{\operatorname{rk} E}.$$

1.2 Fourier–Mukai transform

Let $v_0 := r_0 + \xi_0 + a_0\varrho_X$, $r_0 > 0$, $\xi_0 \in \operatorname{NS}(X)$ be a primitive isotropic Mukai vector on X. We take a general ample divisor H with respect to v_0. We set $Y := M_H(v_0)$. Then Y is an abelian surface (respectively a $K3$ surface), if X is an abelian surface (respectively a $K3$ surface). By the proof of [Yos99b, Lemma 2.1], the following lemma holds.

LEMMA 1.2. *We write $v_0 = l(r + \xi) + a\varrho_X$, $\xi \in \operatorname{NS}(X)$, where $r + \xi$ is primitive and $\gcd(l, a) = 1$. Assume that H is general with respect to v_0.*

(1) *If X is a $K3$ surface and $r|(\xi^2)/2 + 1$, then there is a μ-stable vector bundle E_0 such that $\langle v(E_0)^2\rangle = -2$ and $v_0 = \operatorname{rk}(E_0)v(E_0^\vee) - \varrho_X$. Moreover, $Y \cong X$ and a universal family is given by*

$$\mathcal{E} := \ker(E_0^\vee \boxtimes E_0 \xrightarrow{\phi} \mathcal{O}_\Delta), \tag{1.1}$$

where ϕ is the composition of the restriction map $E_0^\vee \boxtimes E_0 \to E_0^\vee \boxtimes E_{0|\Delta}$ with $E_0^\vee \otimes E_0 \to \mathcal{O}_X$.

(2) *If X is an abelian surface or $r \nmid (\xi^2)/2 + 1$, then Y consists of μ-stable locally free sheaves.*

Proof. For the convenience of the reader, we give a proof.

(1) Assume that X is a $K3$ surface and $r|(\xi^2)/2 + 1$. We set $b := ((\xi^2)/2 + 1)/r \in \mathbb{Z}$. Then $u_0 := r + \xi + b\varrho_X$ satisfies $\langle u_0^2\rangle = -2$. We take an ample $\mathbb{Q}$-divisor H' such that H' is sufficiently

close to H and general with respect to u_0. By [Yos01a, Theorem 8.1], there is a stable sheaf F_0 with $v(F_0)=u_0$, where we consider the stability with respect to H'. By our choice of H', F_0 is μ-semi-stable with respect to H. We shall prove that F_0 is a μ-stable locally free sheaf with respect to H. We first note that F_0 is rigid, and hence F_0 is locally free. Since $v_0 = lv(F_0)+(a-lb)\varrho_X$ and $\langle v_0^2\rangle = -2l^2-2lr(a-lb)=0$, we have $r|l$ and $l/r=lb-a$. Then, by the primitivity of v_0, we have $l=r$ and

$$v_0 = \mathrm{rk}(F_0)v(F_0)-\varrho_X.$$

We set $E_0 := F_0^{\vee}$ and $\mathcal{E} := \ker(F_0 \boxtimes E_0 \to \mathcal{O}_\Delta)$. Then $\mathcal{E}_{|X\times\{x\}} = \ker(\mathrm{Hom}(F_0,\mathbb{C}_x)\otimes F_0 \to \mathbb{C}_x)$ is a μ-semi-stable sheaf with $v(\mathcal{E}_{|X\times\{x\}})=v_0$. Since H is general with respect to v_0, we see that H is general with respect to u_0. Then, by the primitivity of $r+\xi$, F_0 and E_0 are μ-stable sheaves with respect to H. We shall prove that $\mathcal{E}_{|X\times\{x\}}$ is stable with respect to H. Let F be a subsheaf of $\mathcal{E}_{|X\times\{x\}}$ such that $(\mathrm{rk}\,F, c_1(F)) = k(r,\xi)$, $k<r$. Since $\mathrm{Hom}(F_0,\mathbb{C}_x)\otimes F_0$ is semi-stable, $\chi(F)/\mathrm{rk}\,F \le \chi(F_0)/\mathrm{rk}\,F_0$ and if the equality holds, then $F\cong F_0^{\oplus m}$. Since $\mathrm{Hom}(F_0,\mathcal{E}_{|X\times\{x\}})=0$, this is impossible. Therefore, we get

$$\frac{\chi(F)}{\mathrm{rk}\,F} \le \frac{\chi(F_0)}{\mathrm{rk}\,F_0} - \frac{1}{\mathrm{rk}\,F} < \frac{\chi(\mathcal{E}_{|X\times\{x\}})}{\mathrm{rk}(\mathcal{E}_{|X\times\{x\}})},$$

which implies that $\mathcal{E}_{|X\times\{x\}}$ is stable. Hence, (1.1) gives a family of stable sheaves with the Mukai vector v_0. By the irreducibility of $M_H(v_0)$, $E\cong \mathcal{E}_{|X\times\{x\}}$ for a point $x\in X$.

(2) We first assume that X is a $K3$ surface. Then $r_0=lr>1$. Assume that $E\in M_H(v_0)$ is S-equivalent to $\bigoplus_{i=1}^s E_i$ with respect to the μ-stability, where E_i are μ-stable sheaves with respect to H. Since H is general with respect to v_0, we may set $v(E_i):=l_i(r+\xi)+a_i\varrho_X$. Since $\langle v(E_i)^2\rangle = l_i(l_i(\xi^2)-2ra_i)\ge -2$, we have $l_i(\xi^2)-2ra_i\ge 0$, or $l_i=1$ and $(\xi^2)-2ra_i=-2$. By our assumption, the latter case does not hold. Hence, $l_i(\xi^2)-2ra_i\ge 0$ for all i. Then we have $l(\xi^2)=\sum_i l_i(\xi^2)\ge 2\sum_i a_i r = 2ar$, which implies that $l_i(\xi^2)=2a_ir$ for all i. This means that $lv(E_i)=l_iv(E)$. By the primitivity of $v(E)$, we get $s=1$. Thus E is μ-stable. Assume that E is not locally free. Then $E^{\vee\vee}$ is a μ-stable locally free sheaf with $\langle v(E^{\vee\vee})^2\rangle \le -2lr < -2$. Therefore, E is locally free.

Assume that X is an abelian surface. Then $\langle v(F)^2\rangle \ge 0$ for any μ-semi-stable sheaf F. Hence, a similar argument shows the claim. □

Remark 1.1. If X is an abelian surface, then every simple sheaf E with $v(E)=v_0$ is μ-stable with respect to all H (cf. [Muk78]). Hence, Y does not depend on H and consists of μ-stable vector bundles. In particular, every H is general with respect to v_0.

Definition 1.5. Assume that there is a universal family $\mathcal{E}$ on $X\times Y$. Let $p_X: X\times Y\to X$ (respectively $p_Y: X\times Y\to Y$) be the projection. We define $\mathcal{F}_{\mathcal{E}}: \mathbf{D}(X)\to\mathbf{D}(Y)$ by

$$\mathcal{F}_{\mathcal{E}}(x) := \mathbf{R}p_{Y*}(\mathcal{E}\otimes p_X^*(x)), \quad x\in\mathbf{D}(X),$$

and $\widehat{\mathcal{F}}_{\mathcal{E}}: \mathbf{D}(Y)\to\mathbf{D}(X)$ by

$$\widehat{\mathcal{F}}_{\mathcal{E}}(y) := \mathbf{R}\,\mathrm{Hom}_{p_X}(\mathcal{E}, p_Y^*(y)), \quad y\in\mathbf{D}(Y),$$

where $\mathrm{Hom}_{p_X}(-,-) = p_{X*}\mathcal{H}om_{\mathcal{O}_{X\times Y}}(-,-)$ is the sheaf of relative homomorphisms.

Orlov [Orl97] and Bridgeland [Bri99] showed that $\mathcal{F}_{\mathcal{E}}$ is an equivalence of categories and the inverse is given by $\widehat{\mathcal{F}}_{\mathcal{E}}[2]$. $\mathcal{F}_{\mathcal{E}}$ is now called the *Fourier–Mukai functor*. We denote the ith cohomology sheaf $H^i(\mathcal{F}_{\mathcal{E}}(x))$ by $\mathcal{F}^i_{\mathcal{E}}(x)$. $\mathcal{F}_{\mathcal{E}}$ also induces a Hodge isometry of the Mukai lattices

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$$\begin{array}{rcl}\mathcal{F}_{\mathcal{E}}: H^{ev}(X,\mathbb{Z}) & \to & H^{ev}(Y,\mathbb{Z}) \\ v & \mapsto & p_{Y*}(\mathrm{ch}(\mathcal{E})\sqrt{\mathrm{td}_{X\times Y}}\,p_X^*(v)).\end{array}$$

By the Grothendieck–Riemann–Roch theorem, we have the following commutative diagram.

$$\begin{array}{ccc}\mathbf{D}(X) & \xrightarrow{\mathcal{F}_{\mathcal{E}}} & \mathbf{D}(Y) \\ \downarrow & & \downarrow \\ H^{ev}(X,\mathbb{Z}) & \xrightarrow{\mathcal{F}_{\mathcal{E}}} & H^{ev}(Y,\mathbb{Z})\end{array}$$

We are also interested in the composition of $\mathcal{F}_{\mathcal{E}}$ and the 'taking-dual' functor $\mathcal{D}_Y : \mathbf{D}(Y) \to \mathbf{D}(Y)_{\mathrm{op}}$ sending $x \in \mathbf{D}(Y)$ to $\mathbf{R}\,\mathcal{H}om(x, \mathcal{O}_Y)$, where $\mathbf{D}(Y)_{\mathrm{op}}$ is the opposite category of $\mathbf{D}(Y)$. By Grothendieck–Serre duality, $\mathcal{G}_{\mathcal{E}} := (\mathcal{D}_Y \circ \mathcal{F}_{\mathcal{E}})[2]_{\mathrm{op}}$ is defined by

$$\mathcal{G}_{\mathcal{E}}(x) := \mathbf{R}\,\mathrm{Hom}_{p_Y}(\mathcal{E} \otimes p_X^*(x), \mathcal{O}_{X\times Y}), \quad x \in \mathbf{D}(X),$$

where $H^i(E[n]_{\mathrm{op}})$ means $H^{i-n}(E)$. Let $\widehat{\mathcal{G}}_{\mathcal{E}} : \mathbf{D}(Y)_{\mathrm{op}} \to \mathbf{D}(X)$ be the inverse of $\mathcal{G}_{\mathcal{E}}$:

$$\widehat{\mathcal{G}}_{\mathcal{E}}(y) := \mathbf{R}\,\mathrm{Hom}_{p_X}(\mathcal{E} \otimes p_Y^*(y), \mathcal{O}_{X\times Y}), \quad y \in \mathbf{D}(Y).$$

Definition 1.6. Let E be a coherent sheaf on X. We say that WIT_i holds for E with respect to $\mathcal{F}_{\mathcal{E}}$, if $\mathcal{F}_{\mathcal{E}}^j(E) = 0$ for $j \neq i$. Moreover, if $H^j(\mathcal{F}_{\mathcal{E}}(E) \otimes k(t)) = 0$, $j \neq i$ for all $t \in Y$, then we say that IT_i holds for E with respect to $\mathcal{F}_{\mathcal{E}}$. Similarly, we define WIT_i, IT_i for $\widehat{\mathcal{F}}_{\mathcal{E}}$, $\mathcal{G}_{\mathcal{E}}$ and $\widehat{\mathcal{G}}_{\mathcal{E}}$.

Since $\widehat{\mathcal{F}}_{\mathcal{E}}[2]$ is the inverse of $\mathcal{F}_{\mathcal{E}}$, we get the following.

Lemma 1.3. *We have spectral sequences*

$$E_2^{p,q} = \mathcal{F}_{\mathcal{E}}^p(\widehat{\mathcal{F}}_{\mathcal{E}}^q(E)) \Rightarrow E_\infty^{p+q} = \begin{cases} E, & p+q=2, \\ 0, & p+q \neq 2, \end{cases} \quad E \in \mathrm{Coh}(Y),$$

$$E_2^{p,q} = \widehat{\mathcal{F}}_{\mathcal{E}}^p(\mathcal{F}_{\mathcal{E}}^q(F)) \Rightarrow E_\infty^{p+q} = \begin{cases} F, & p+q=2, \\ 0, & p+q \neq 2, \end{cases} \quad F \in \mathrm{Coh}(X).$$

In particular,

(i) $\mathcal{F}_{\mathcal{E}}^p(\widehat{\mathcal{F}}_{\mathcal{E}}^0(E)) = 0$, $p = 0, 1$;

(ii) $\mathcal{F}_{\mathcal{E}}^p(\widehat{\mathcal{F}}_{\mathcal{E}}^2(E)) = 0$, $p = 1, 2$;

(iii) *there is an injective homomorphism* $\mathcal{F}_{\mathcal{E}}^0(\widehat{\mathcal{F}}_{\mathcal{E}}^1(E)) \to \mathcal{F}_{\mathcal{E}}^2(\widehat{\mathcal{F}}_{\mathcal{E}}^0(E))$;

(iv) *there is a surjective homomorphism* $\mathcal{F}_{\mathcal{E}}^0(\widehat{\mathcal{F}}_{\mathcal{E}}^2(E)) \to \mathcal{F}_{\mathcal{E}}^2(\widehat{\mathcal{F}}_{\mathcal{E}}^1(E))$.

1.3 A polarization of Y

We shall define a canonical polarization of Y. We set

$$G_1 := \mathcal{E}^{\vee}_{|X\times\{y\}}, \qquad G_2 := \mathcal{E}_{|\{x\}\times Y} \tag{1.2}$$

for some $x \in X$ and $y \in Y$. We also set

$$w_0 := v(\mathcal{E}_{|\{x\}\times Y}) = r_0 + \widetilde{\xi}_0 + \widetilde{a}_0\varrho_Y, \quad \widetilde{\xi}_0 \in \mathrm{NS}(Y). \tag{1.3}$$

Since $\mathcal{F}_{\mathcal{E}}(\mathcal{E}^{\vee}_{|X\times\{y\}}) = \mathbb{C}_y[-2]$ and $\mathcal{F}_{\mathcal{E}}(\mathbb{C}_x) = \mathcal{E}_{|\{x\}\times Y}$, we have

$$\mathcal{F}_{\mathcal{E}}(v_0^{\vee}) = \varrho_Y, \quad \mathcal{F}_{\mathcal{E}}(\varrho_X) = w_0$$

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and $\mathcal{F}_{\mathcal{E}}$ induces an isometry

$$v_0^{\vee\perp} \cap \varrho_X^{\perp} \to w_0^{\perp} \cap \varrho_Y^{\perp}.$$

We note that there are Hodge isometries

$$\begin{array}{rcl} f_{v_0^{\vee}} : H^2(X,\mathbb{Q}) & \to & (v_0^{\vee\perp} \cap \varrho_X^{\perp}) \otimes \mathbb{Q} \\ D & \mapsto & D + \dfrac{(D,-\xi_0)}{r_0}\varrho_X \\ f_{w_0} : H^2(Y,\mathbb{Q}) & \to & (w_0^{\perp} \cap \varrho_Y^{\perp}) \otimes \mathbb{Q} \\ D & \mapsto & D + \dfrac{(D,\widetilde{\xi}_0)}{r_0}\varrho_Y \end{array}$$

and the following commutative diagram.

$$\begin{array}{ccc} H^2(X,\mathbb{Q}) & \xrightarrow{\nu} & H^2(Y,\mathbb{Q}) \\ \downarrow{\scriptstyle Vf_{v_0}} & & \downarrow{\scriptstyle f_{w_0}} \\ (v_0^{\vee\perp} \cap \varrho_X^{\perp}) \otimes \mathbb{Q} & \xrightarrow{\mathcal{F}_{\mathcal{E}}} & (w_0^{\perp} \cap \varrho_Y^{\perp}) \otimes \mathbb{Q} \end{array}$$

For $D \in H^2(X,\mathbb{Q})$, we set

$$\begin{aligned} \widehat{D} &:= -\nu(D) \\ &= -\left[\mathcal{F}_{\mathcal{E}}\left(D + \frac{(D,-\xi_0)}{r_0}\varrho_X\right)\right]_1 \\ &= \left[p_{Y*}\left(\left(c_2(\mathcal{E}) - \frac{r_0-1}{2r_0}(c_1(\mathcal{E})^2)\right) \cup p_X^*(D)\right)\right]_1 \in H^2(Y,\mathbb{Q}), \end{aligned} \tag{1.4}$$

where $[\cdot]_1$ means the projection to $H^2(Y,\mathbb{Q})$.

Lemma 1.4.

(1) *For a divisor D on X (up to numerical equivalence), we take an element $E \in K(X)$ with $v(E) = -r_0 D + (c_1(L),\xi_0)\varrho_X$. Then the determinant line bundle $\det(p_{Y!}(\mathcal{F}_{\mathcal{E}}(E)))$ on Y satisfies*

$$c_1(\det(p_{Y!}(\mathcal{F}_{\mathcal{E}}(E)))) = r_0\widehat{D}. \tag{1.5}$$

(2) *$r_0\widehat{H}$ is represented by an ample divisor on Y.*

Proof. Part (1) is a consequence of the Grothendieck–Riemann–Roch theorem.

For part (2), since H is general with respect to v_0, G-twisted semi-stability does not depend on the choice of G. In particular, $\overline{M}_H^{G_1}(v_0) = Y$. Assume that X is a $K3$ surface. Then in [OY03, Proposition 1.3], we constructed an ample line bundle $\mathcal{L}$ on $\overline{M}_H^{G_1}(v_0)$ such that

$$c_1(\mathcal{L}) = [p_{Y*}(\mathrm{ch}(\mathcal{E})p_X^*(\sqrt{\mathrm{td}_X}(r_0H + (H,\xi_0)\varrho_X)^{\vee}))]_1. \tag{1.6}$$

Indeed we proved that Simpson's polarization of Y satisfies (1.6). It is easy to see that the same construction works for an abelian surface X. By using (1.4), we get

$$c_1(\mathcal{L}) = r_0\widehat{H}.$$

Therefore, the claim holds. □

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Remark 1.2.

(i) $-\nu$ is the same as the Donaldson μ-map.

(ii) Lemma 1.4(2) is proved by Bartocci *et al.* in [BBH97], if X is a $K3$ surface.

By its definition, $\widehat{H} \notin H^2(Y,\mathbb{Z})$ in general. In Appendix Appendix A, we shall study the generator of $\mathbb{Q}\widehat{H} \cap H^2(Y,\mathbb{Z})$.

Definition 1.7. We use the $\mathbb{Q}$-divisor $\widehat{H}$ as a polarization of Y. Hence, for $x \in K(Y) \otimes \mathbb{Q}$, $\deg_{G_2}(x) = (c_1(x \otimes G_2^{\vee}), \widehat{H})$. We also study G_2-twisted stability of $F \in \mathrm{Coh}(Y)$ with respect to $\widehat{H}$.

Remark 1.3. If $\mathcal{E}$ is not locally free, that is, the case of Lemma 1.2(1), then $\widehat{H} = H$.

By using (1.4), we get

$$-\mathcal{F}_{\mathcal{E}}\left(D + \frac{(D,-\xi_0)}{r_0}\varrho_X\right) = \widehat{D} + \frac{(\widehat{D},\widetilde{\xi}_0)}{r_0}\varrho_Y.$$

1.4 The cohomological correspondence

By using § 1.3, we can describe the cohomological Fourier–Mukai transform $\mathcal{F}_{\mathcal{E}} : H^2(X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$ more explicitly.

Proposition 1.5.

(1) *Every element* $v \in H^{ev}(X,\mathbb{Z})$ *can be uniquely written as*

$$v = l v_0^{\vee} + a\varrho_X + d\left(H - \frac{1}{r_0}(H,\xi_0)\varrho_X\right) + \left(D - \frac{1}{r_0}(D,\xi_0)\varrho_X\right),$$

where

$$\begin{aligned} l &= \frac{\mathrm{rk}\, v}{\mathrm{rk}\, v_0} = -\frac{\langle v, \varrho_X\rangle}{\mathrm{rk}\, v_0} \in \frac{1}{r_0}\mathbb{Z},\\ a &= -\frac{\langle v, v_0^{\vee}\rangle}{\mathrm{rk}\, v_0} \in \frac{1}{r_0}\mathbb{Z},\\ d &= \frac{\deg_{G_1}(v)}{\mathrm{rk}\, v_0 (H^2)} \in \frac{1}{r_0(H^2)}\mathbb{Z} \end{aligned} \tag{1.7}$$

and $D \in H^2(X,\mathbb{Q}) \cap H^{\perp}$. *Moreover,* $v \in v(\mathbf{D}(X))$ *if and only if* $D \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp}$.

(2)

$$\begin{aligned} &\mathcal{F}_{\mathcal{E}}\left(l v_0^{\vee} + a\varrho_X + \left(dH + D - \frac{1}{r_0}(dH + D,\xi_0)\varrho_X\right)\right)\\ &\quad = l\varrho_Y + a w_0 - \left(d\widehat{H} + \widehat{D} + \frac{1}{r_0}(d\widehat{H} + \widehat{D},\widetilde{\xi}_0)\varrho_Y\right) \end{aligned} \tag{1.8}$$

where $D \in H^2(X,\mathbb{Q}) \cap H^{\perp}$.

Hence we have the following.

Corollary 1.6.

$$\deg_{G_1}(v) = -\deg_{G_2}(\mathcal{F}_{\mathcal{E}}(v)).$$

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In particular, $\deg_{G_2}(w) \in \mathbb{Z}$ *for* $w \in H^{ev}(Y, \mathbb{Z})$ *and*

$$\min\{\deg_{G_1}(E) > 0 \mid E \in K(X)\} = \min\{\deg_{G_2}(F) > 0 \mid F \in K(Y)\}.$$

1.5 Main theorem

We can now state our main theorem. For more details, see Theorem 3.1 and Proposition 3.2.

Theorem 1.7 (cf. Theorem 3.1, Proposition 3.2). *Let* $Y := M_H(v_0)$ *be the moduli space of stable sheaves on* X *with the isotropic Mukai vector* v_0 *of* $\operatorname{rk} v_0 > 0$. *Assume that* H *is a general polarization with respect to* v_0 *and there is a universal family* $\mathcal{E}$. *We set* $G_1 := \mathcal{E}^{\vee}_{|X \times \{y\}}$ *and* $G_2 := \mathcal{E}_{|\{x\} \times Y}$, *where* $x \in X$ *and* $y \in Y$. *For a Mukai vector* $v \in v(\mathbf{D}(X))$, *we write*

$$v := l v_0^{\vee} + a \varrho_X + (dH + D) - (dH + D, \xi_0)\varrho_X / r_0,$$

where $l \geq 0$, $a > 0$ *and* $D \in \operatorname{NS}(X) \otimes \mathbb{Q} \cap H^{\perp}$.

(1) *Assume that* $l > 0$. *If*

$$d > \max\{(4l^2 r_0^3 + 1/(H^2))\epsilon, (1+\epsilon) r_0^2 l (\langle v^2 \rangle - (D^2))\},$$

then $\mathcal{F}_{\mathcal{E}}$ *induces an isomorphism*

$$\mathcal{M}_H^{G_1}(v)^{ss} \to \mathcal{M}_{\widehat{H}}^{G_2}(\mathcal{F}_{\mathcal{E}}(v))^{ss}$$

which preserves the S*-equivalence classes.*

(2) *Assume that* $l = 0$. *If*

$$a > \max\{(2r_0 + 1)\epsilon, (\langle v^2 \rangle - (D^2))/2 + \epsilon\},$$

then $\mathcal{F}_{\mathcal{E}}$ *induces an isomorphism*

$$\mathcal{M}_H^{G_1}(v)^{ss} \to \mathcal{M}_{\widehat{H}}^{G_2}(\mathcal{F}_{\mathcal{E}}(v))^{ss}$$

which preserves the S*-equivalence classes.*

Remark 1.4. We note that $\langle v^2 \rangle - \langle D^2 \rangle$ is invariant under $v \mapsto v e^{mH}$. Hence, for $E \in \mathcal{M}_H^{G_1}(v)^{ss}$, $\mathcal{F}_{\mathcal{E}}(E(mH)) \in \mathcal{M}_{\widehat{H}}^{G_2}(\mathcal{F}_{\mathcal{E}}(v))^{ss}$ if

$$m > \begin{cases} (\max\{(4l^2 r_0^3 + 1/(H^2))\epsilon, (1+\epsilon) r_0^2 l (\langle v^2 \rangle - (D^2))\} - d)/l r_0, & l > 0, \\ (\max\{(2r_0+1)\epsilon, (\langle v^2 \rangle - (D^2))/2 + \epsilon\} - a)/(d(H^2)), & l = 0. \end{cases}$$

Remark 1.5. If E is G_1-twisted stable, then by using Corollary 2.14 we get the μ-stability of $\mathcal{F}_{\mathcal{E}}(E(mH))$. Thus the twisted stability (cf. Definition 1.1) is as important as the μ-stability.

Let us briefly explain an outline of the proof of Theorem 1.7 (or Theorem 3.1 and Proposition 3.2). For the proof, we use two results. The first one is due to Huybrechts (see Theorem 2.1). We assume that $\mathcal{E}$ is locally free and set $X_1 := X$, $X_2 := Y$. Then Huybrechts finds a natural abelian subcategory $\mathfrak{A}_i$ of $\mathbf{D}(X_i)$ such that $\mathcal{F}_{\mathcal{E}}[1]$ induces an equivalence $\mathfrak{A}_1 \to \mathfrak{A}_2$. This abelian category $\mathfrak{A}_i$ is a tilting of a torsion pair in $\operatorname{Coh}(X_i)$, and naturally appears in Bridgeland's stability conditions [Bri03]. By the definition of $\mathfrak{A}_1$, we have $E \in \mathfrak{A}_1$ for $E \in \mathcal{M}_H^{G_1}(v)^{ss}$. Then, applying Theorem 2.1 to E, we have $\mathcal{F}_{\mathcal{E}}(E)[1] \in \mathfrak{A}_2$, which means that $\mu_{\max,G_2}(\mathcal{F}_{\mathcal{E}}^0(E)) \leq 0$ and $\mu_{\min,G_2}(\mathcal{F}_{\mathcal{E}}^1(E)) > 0$. We next prepare two results (Propositions 2.8 and 2.11) on the constraint of the Mukai vector

$$v(F_1) = a w_1 + l_1 \varrho_Y - ((d_1 \widehat{H} + \widehat{D}_1) + (d_1 \widehat{H} + \widehat{D}_1, \widehat{\xi}_0)\varrho_Y / r_0)$$

of a G_2-twisted stable subsheaf F_1 of $\mathcal{F}^i_{\mathcal{E}}(E)$, which is roughly a consequence of Bogomolov's inequality. If $\mathcal{F}_{\mathcal{E}}(E) \notin \mathcal{M}^{G_2}_{\widehat{H}}(\mathcal{F}_{\mathcal{E}}(v))^{ss}$, then we have an exact sequence in $\mathfrak{A}_2$:

$$0 \to F_1[1] \to \mathcal{F}_{\mathcal{E}}(E)[1] \to F_2^\bullet \to 0.$$

Applying Theorem 2.1, we have an exact sequence in $\mathfrak{A}_1$:

$$0 \to \widehat{\mathcal{F}}_{\mathcal{E}}(F_1)[2] \to E \to \widehat{\mathcal{F}}_{\mathcal{E}}(F_2^\bullet)[1] \to 0.$$

By using Proposition 2.8 or 2.11, we get a contradiction, which complete the proof of Theorem 1.7.

Remark 1.6. Let $X = \bigcup_i U_i$ be an analytic open covering of X and $\alpha = \{\alpha_{ijk}\}$ a Cech 2-cocycle of $\mathcal{O}_X^\times$ representing a torsion element $[\alpha] \in H^2(X, \mathcal{O}_X^\times)$. An α-twisted sheaf $E := \{(E_i, \varphi_{ij})\}$ is a collection of sheaves E_i on U_i and isomorphisms $\varphi_{ij} : E_{i|U_i\cap U_j} \to E_{j|U_i\cap U_j}$ such that $\varphi_{ii} = \mathrm{id}_{E_i}$, $\varphi_{ji} = \varphi_{ij}^{-1}$ and $\varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij} = \alpha_{ijk}\,\mathrm{id}_{E_i}$. In [HS05] and [Yos04], we studied α-twisted sheaves on an abelian or a $K3$ surface. In particular, we defined the basic notions such as the Mukai lattice and the Mukai vector of twisted sheaves, and we constructed the moduli spaces $\overline{M}_H^G(v)$ of semi-stable α-twisted sheaves E with the Mukai vector v, where G is a locally free α-twisted sheaf on X. Assume that H is general with respect to an isotropic Mukai vector v_0. Then $Y = M_H^G(v_0)$ is an abelian or a $K3$ surface and there is a universal family $\mathcal{E}$ as an $p_X^*(\alpha) \times p_Y^*(\beta)$-twisted sheaf on $X \times Y$, where $\beta = \{\beta_{ijk}\}$ is a suitable Cech 2-cocycle of $\mathcal{O}_Y^\times$. We also constructed the Fourier–Mukai transform $\mathcal{F}_{\mathcal{E}} : \mathbf{D}^\alpha(X) \to \mathbf{D}^\beta(Y)$ for the derived categories of twisted coherent sheaves.

It is easy to see that we have a similar polarization $\widehat{H}$ on Y and Proposition 1.5 holds. Thus the statements of Theorem 1.7 are well-defined. Moreover, Lemma 1.2, Theorems 2.1 and 2.2 and Propositions 2.8 and 2.11 hold for this case. Then by the same proof as in the untwisted case, we get Theorem 1.7 for the twisted case.

2. Basic results

We use the notation in § 1. From now on, we assume that there is a universal family $\mathcal{E}$ on $X \times Y$. Thus we have the Fourier–Mukai transform

$$\mathcal{F}_{\mathcal{E}} : \mathbf{D}(X) \to \mathbf{D}(Y).$$

2.1 Some results of Huybrechts

Based on Bridgeland's paper [Bri03], Huybrechts proved the following important results [Huy06].

Theorem 2.1 (Huybrechts). *We set*

$$\begin{aligned}
\mathfrak{T}_i &:= \{E \in \mathrm{Coh}(X_i) \mid \ E = 0 \text{ or } \mu_{\min,G_i}(E) > 0\},\\
\mathfrak{F}_i &:= \{E \in \mathrm{Coh}(X_i) \mid \ E = 0 \text{ or } \mu_{\max,G_i}(E) \le 0\},\\
\mathfrak{A}_i &:= \{V^\bullet \in \mathbf{D}(X_i) \mid H^j(V^\bullet) = 0, j \ne -1, 0,\ H^{-1}(V^\bullet) \in \mathfrak{F}_i, H^0(V^\bullet) \in \mathfrak{T}_i\},
\end{aligned}$$

where $X_1 = X$ and $X_2 = Y$. Assume that $\mathcal{E}$ is locally free. Then $\mathcal{F}_{\mathcal{E}}[1]$ induces an equivalence $\mathfrak{A}_1 \to \mathfrak{A}_2$.

In particular, the following easily follows from [Huy06, Proposition 2.2].

Theorem 2.2. *Assume that $\mathcal{E}$ is locally free.*

(1) $\mathcal{E}^{\vee}_{|\{x\}\times Y}$ *is μ-stable with respect to $\widehat{H}$ for all $x \in X$. Let $M_{\widehat{H}}(w_0^{\vee})$ be the moduli space of stable sheaves on Y with the Mukai vector $w_0^{\vee}$. Then we have an isomorphism $X \to M_{\widehat{H}}(w_0^{\vee})$ by the correspondence $x \mapsto \mathcal{E}^{\vee}_{|\{x\}\times Y}$. In particular, $M_{\widehat{H}}(w_0^{\vee})$ consists of μ-stable vector bundles.*

(2) *By the identification $M_{\widehat{H}}(w_0^{\vee}) = X$, $\mathcal{F}_{\mathcal{E}^{\vee}} : \mathbf{D}(Y) \to \mathbf{D}(M_{\widehat{H}}(w_0^{\vee}))$ coincides with $\widehat{\mathcal{F}}_{\mathcal{E}} : \mathbf{D}(Y) \to \mathbf{D}(X)$. Moreover, the canonical polarization $\widehat{(\widehat{H})}$ on $M_{\widehat{H}}(w_0^{\vee})$ coincides with H.*

Remark 2.1. Theorem 2.2(1) for a special $K3$ surface was first proved by Bartocci *et al.* [BBH97], and this gave the first example of the Fourier–Mukai transform on a $K3$ surface.

Remark 2.2. By the proof of these results below, we see that the claims of Theorem 2.1 and Theorem 2.2 hold for the twisted version of the Fourier–Mukai transform.

These results are crucial for our proof of the main result. For the sake of convenience, we give other proofs of Theorem 2.1 (see Lemma 2.5) and Theorem 2.2. We also treat the case where $\mathcal{E}$ is not locally free. We start with the following lemmas.

Lemma 2.3. *If E is a μ-semi-stable sheaf on X with $\deg_{G_1}(E) = 0$, then $\mathcal{F}^0_{\mathcal{E}}(E) = 0$.*

Proof. We may assume that E is μ-stable. Lemma 1.2 implies that $\mathcal{E}_{|X\times\{y\}}$ is a μ-stable vector bundle for any $y \in Y$ or $\mathcal{E} = \ker(E_0^{\vee} \boxtimes E_0 \to \mathcal{O}_{\Delta})$. In the first case, $\operatorname{Hom}(\mathcal{E}^{\vee}_{|X\times\{y\}}, E) \neq 0$ implies that $\mathcal{E}^{\vee}_{|X\times\{y\}} \cong E$. Since $\mathcal{F}^0_{\mathcal{E}}(E)$ is torsion free, we get $\mathcal{F}^0_{\mathcal{E}}(E) = 0$. In the second case, $\operatorname{Hom}(E_0, E) \otimes E_0 \to E$ is injective. Hence $\mathcal{F}^0_{\mathcal{E}}(E) = 0$. □

Lemma 2.4. *Assume that $\mathcal{E}$ is locally free. Let E be a coherent sheaf on X.*

(1) *If WIT_0 holds for E with respect to $\mathcal{F}_{\mathcal{E}}$, then $E \in \mathfrak{T}_1$.*

(2) *If WIT_2 holds for E with respect to $\mathcal{F}_{\mathcal{E}}$, then $E \in \mathfrak{F}_1$. In particular, E is torsion free. Moreover, if $\mathcal{F}^2_{\mathcal{E}}(E)$ does not contain a zero-dimensional subsheaf, then $\mu_{\max,G_1}(E) < 0$.*

Proof. For a coherent sheaf E on X, there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

such that $E_1 \in \mathfrak{T}_1$ and $E_2 \in \mathfrak{F}_1$. Applying $\mathcal{F}_{\mathcal{E}}$ to this exact sequence, we get a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathcal{F}^0_{\mathcal{E}}(E_1) \longrightarrow \mathcal{F}^0_{\mathcal{E}}(E) \longrightarrow \mathcal{F}^0_{\mathcal{E}}(E_2) \\ &\longrightarrow \mathcal{F}^1_{\mathcal{E}}(E_1) \longrightarrow \mathcal{F}^1_{\mathcal{E}}(E) \longrightarrow \mathcal{F}^1_{\mathcal{E}}(E_2) \\ &\longrightarrow \mathcal{F}^2_{\mathcal{E}}(E_1) \longrightarrow \mathcal{F}^2_{\mathcal{E}}(E) \longrightarrow \mathcal{F}^2_{\mathcal{E}}(E_2) \longrightarrow 0. \end{aligned} \tag{2.1}$$

Since $\mathcal{E}_{|X\times\{y\}}$ is μ-stable, $E_1 \in \mathfrak{T}_1$ implies that $\mathcal{F}^2_{\mathcal{E}}(E_1) = 0$. By using Lemma 2.3, we also see that $\mathcal{F}^0_{\mathcal{E}}(E_2) = 0$ from $E_2 \in \mathfrak{F}_1$. If WIT_0 holds for E, then we get $\mathcal{F}_{\mathcal{E}}(E_2) = 0$. Hence part (1) holds. If WIT_2 holds for E, then we get $\mathcal{F}_{\mathcal{E}}(E_1) = 0$. Thus the first part of (2) holds. Assume that WIT_2 holds for E and $\mathcal{F}^2_{\mathcal{E}}(E)$ does not contain a zero-dimensional subsheaf. If there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

such that E_1 is a μ-semi-stable sheaf with $\mu_{G_1}(E_1) = 0$ and E_2 is a torsion-free sheaf with $\mu_{\max,G_1}(E_2) < 0$ or $E_2 = 0$, then we see that WIT_2 holds for E_1, $\mathcal{F}^2_{\mathcal{E}}(E_1)$ is zero-dimensional

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and $\mathcal{F}^1_{\mathcal{E}}(E_2) \cong \mathcal{F}^2_{\mathcal{E}}(E_1)$. Since $\widehat{\mathcal{F}}^0_{\mathcal{E}}(\mathcal{F}^1_{\mathcal{E}}(E_2)) \to \widehat{\mathcal{F}}^2_{\mathcal{E}}(\mathcal{F}^0_{\mathcal{E}}(E_2)) = 0$ is injective, $\mathcal{F}^2_{\mathcal{E}}(E_1) = 0$, and hence $E_1 = 0$. Therefore, the claim holds. □

Proof of Theorem 2.2. (1) In order to prove the μ-stability of $\mathcal{E}^{\vee}_{|\{x\}\times Y}$, it is sufficient to prove the μ-stability of $\mathcal{E}_{|\{x\}\times Y}$. Assume that there is an exact sequence

$$0 \to F_1 \to \mathcal{E}_{|\{x\}\times Y} \to F_2 \to 0$$

such that $F_1 \neq 0$ is a torsion-free sheaf with $\mu_{\min,G_2}(F_1) \geq 0$ and $F_2 \neq 0$ is a torsion-free sheaf with $\mu_{\max,G_2}(F_2) \leq 0$. Applying $\widehat{\mathcal{F}}_{\mathcal{E}}$ to this exact sequence, we get a long exact sequence

$$\begin{aligned} 0 \longrightarrow \widehat{\mathcal{F}}^0_{\mathcal{E}}(F_1) &\longrightarrow 0 \longrightarrow \widehat{\mathcal{F}}^0_{\mathcal{E}}(F_2) \\ \longrightarrow \widehat{\mathcal{F}}^1_{\mathcal{E}}(F_1) &\longrightarrow 0 \longrightarrow \widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2) \\ \longrightarrow \widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1) &\longrightarrow \mathbb{C}_x \longrightarrow \widehat{\mathcal{F}}^2_{\mathcal{E}}(F_2) \longrightarrow 0. \end{aligned} \tag{2.2}$$

By Lemma 1.3, WIT_0 holds for $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1)$. Hence $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1) \in \mathfrak{T}_1$, which means that $\deg_{G_1}(\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1)) \geq 0$. If the equality holds, then $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1)$ is zero-dimensional. By Lemma 1.3, WIT_2 holds for $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_1) \cong \widehat{\mathcal{F}}^0_{\mathcal{E}}(F_2)$. Hence $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_1) \in \mathfrak{F}_1$, which implies that $\deg_{G_1}(\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_1)) \leq 0$. Therefore, $\deg_{G_1}(\widehat{\mathcal{F}}_{\mathcal{E}}(F_1)) \geq 0$. On the other hand, $\deg_{G_1}(\widehat{\mathcal{F}}_{\mathcal{E}}(F_1)) = -\deg_{G_2}(F_1) \leq 0$. Hence $\deg_{G_1}(\widehat{\mathcal{F}}_{\mathcal{E}}(F_1)) = \deg_{G_2}(F_1) = 0$. Then we see that $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_1)$ is a μ-semi-stable sheaf with $\deg_{G_1}(\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_1)) = 0$ and $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1)$ is a zero-dimensional sheaf. Hence $\mathcal{F}^2_{\mathcal{E}}(\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_1))$ is zero-dimensional. Since $\widehat{\mathcal{F}}^0_{\mathcal{E}}(F_1) = 0$ and $\mathcal{F}^1_{\mathcal{E}}(\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_1)) = 0$, Lemma 1.3 implies that there is an exact sequence

$$0 \to F_1 \to \mathcal{F}^0_{\mathcal{E}}(\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1)) \to \mathcal{F}^2_{\mathcal{E}}(\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_1)) \to 0.$$

Hence we see that $\operatorname{rk} F_1 \geq \operatorname{rk} \mathcal{E}_{|\{x\}\times Y}$, which is a contradiction. Therefore, $\mathcal{E}_{|\{x\}\times Y}$ is μ-stable. Then we have an injective morphism $\phi : X \to \overline{M}_{\widehat{H}}(w_0^{\vee})$ by sending $x \in X$ to $\mathcal{E}^{\vee}_{|\{x\}\times Y}$. We shall show that ϕ is surjective. Let F be a μ-semi-stable sheaf on Y with $v(F) = w_0^{\vee}$. Since $\widehat{\mathcal{F}}_{\mathcal{E}}(\mathbf{R}\,\mathcal{H}om(F, \mathcal{O}_Y)) = \widehat{\mathcal{G}}_{\mathcal{E}}(F) \neq 0$, there is a point $x \in X$ such that $\operatorname{Hom}(F, \mathcal{E}^{\vee}_{|\{x\}\times Y}) \neq 0$ or $\operatorname{Hom}(\mathcal{E}^{\vee}_{|\{x\}\times Y}, F) \neq 0$. Then we see that $F \cong \mathcal{E}^{\vee}_{|\{x\}\times Y}$. Therefore, ϕ is a surjective morphism and $\widehat{H}$ is general with respect to $w_0^{\vee}$.

(2) The first claim follows from the definition of $\widehat{\mathcal{F}}_{\mathcal{E}}$. Then $\widehat{(\widehat{H})} = H$ follows from Proposition 1.5(2). □

By Theorem 2.2, the role of (X, H) and $(Y, \widehat{H})$ are the same, if $\mathcal{E}$ is locally free. Theorem 2.1 follows from the following claims.

Lemma 2.5. *Assume that $\mathcal{E}$ is locally free. Let E be a coherent sheaf on X.*

(1) *If $E \in \mathfrak{T}_1$, then we have the following assertions:*
   - (a) $\mathcal{F}^2_{\mathcal{E}}(E) = 0$;
   - (b) $\mathcal{F}^0_{\mathcal{E}}(E) \in \mathfrak{F}_2$; *moreover, if E does not contain a zero-dimensional subsheaf, then we have* $\mu_{\max,G_2}(\mathcal{F}^0_{\mathcal{E}}(E)) < 0$ *or* $\mathcal{F}^0_{\mathcal{E}}(E) = 0$;
   - (c) $\mathcal{F}^1_{\mathcal{E}}(E) \in \mathfrak{T}_2$.

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(2) *If $E \in \mathfrak{F}_1$, then we have the following assertions:*
(a) $\mathcal{F}^0_{\mathcal{E}}(E) = 0$;
(b) $\mathcal{F}^1_{\mathcal{E}}(E) \in \mathfrak{F}_2$*; in particular,* $\mathcal{F}^1_{\mathcal{E}}(E)$ *is torsion free;*
(c) $\mathcal{F}^2_{\mathcal{E}}(E) \in \mathfrak{T}_2$.

Proof. (1) Claim (a) is obvious. Since WIT_2 holds for $\mathcal{F}^0_{\mathcal{E}}(E)$, the first part of (b) is a consequence of Lemma 2.4. For the second claim, it is sufficient to show that $\widehat{\mathcal{F}}^2_{\mathcal{E}}(\mathcal{F}^0_{\mathcal{E}}(E))$ does not contain a zero-dimensional sheaf. Since $\widehat{\mathcal{F}}^0_{\mathcal{E}}(\mathcal{F}^1_{\mathcal{E}}(E))$ is torsion free, by using the exact sequence

$$0 \to \widehat{\mathcal{F}}^0_{\mathcal{E}}(\mathcal{F}^1_{\mathcal{E}}(E)) \to \widehat{\mathcal{F}}^2_{\mathcal{E}}(\mathcal{F}^0_{\mathcal{E}}(E)) \to E \to 0$$

coming from Lemma 1.3, we get the assertion. To prove part (c), if $\mathcal{F}^1_{\mathcal{E}}(E) \notin \mathfrak{T}_2$, then we have an exact sequence

$$0 \to F_1 \to \mathcal{F}^1_{\mathcal{E}}(E) \to F_2 \to 0$$

such that $F_1 \in \mathfrak{T}_2$ and $F_2 \neq 0$ is a torsion-free sheaf with $\mu_{\max,G_2}(F_2) \leq 0$. We apply $\widehat{\mathcal{F}}_{\mathcal{E}}$ to this exact sequence. By our assumptions, $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1) = 0$ and $\widehat{\mathcal{F}}^0_{\mathcal{E}}(F_2) = 0$. Since $\mathcal{F}^2_{\mathcal{E}}(E) = 0$, Lemma 1.3 implies that $\widehat{\mathcal{F}}^2_{\mathcal{E}}(\mathcal{F}^1_{\mathcal{E}}(E)) = 0$. Thus WIT_1 holds for F_2 and we have a surjective homomorphism

$$E \to \widehat{\mathcal{F}}^1_{\mathcal{E}}(\mathcal{F}^1_{\mathcal{E}}(E)) \to \widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2)$$

with $\deg_{G_1} \widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2) = \deg_{G_2}(F_2)$. Then $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2) \in \mathfrak{T}_1$ and $\deg_{G_1} \widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2) \leq 0$, which implies that $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2)$ is zero-dimensional. Then $F_2 = \mathcal{F}^1_{\mathcal{E}}(\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2)) = 0$, which is a contradiction. Thus the claim holds.

(2) Claim (a) follows from Lemma 2.3, and claim (c) follows from Lemma 1.3 and Lemma 2.4(1). For the proof of part (b), assume that there is an exact sequence

$$0 \to F_1 \to \mathcal{F}^1_{\mathcal{E}}(E) \to F_2 \to 0$$

such that $0 \neq F_1 \in \mathfrak{T}_2$ and $F_2 \in \mathfrak{F}_2$. Since $\mathcal{F}^0_{\mathcal{E}}(E) = 0$ and $\widehat{\mathcal{F}}^0_{\mathcal{E}}(F_2) = \widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1) = 0$, by using Lemma 1.3, we see that $\widehat{\mathcal{F}}^0_{\mathcal{E}}(\mathcal{F}^1_{\mathcal{E}}(E)) = 0$, WIT_1 holds for F_1 and we have an injective homomorphism

$$\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_1) \to \widehat{\mathcal{F}}^1_{\mathcal{E}}(\mathcal{F}^1_{\mathcal{E}}(E)) \to E.$$

In particular, F_1 does not contain a zero-dimensional subsheaf. Since $\deg_{G_1}(\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_1)) = \deg_{G_2}(F_1) > 0$, this is impossible. Therefore, $\mathcal{F}_{\mathcal{E}}(E) \in \mathfrak{F}_2$. □

If $\mathcal{E}$ is not locally free, then $(Y, \widehat{H}) = (X, H)$ and $\mathcal{E} = \ker(E_0^{\vee} \boxtimes E_0 \to \mathcal{O}_{\Delta})$. In this case, since $\mathcal{E}^{\vee} \in \mathbf{D}(X \times X)$ is not a sheaf, we need to treat $\mathcal{F}_{\mathcal{E}}$ and $\widehat{\mathcal{F}}_{\mathcal{E}}$ separately. For a coherent sheaf E on X, we have exact sequences:

$$\begin{aligned} 0 &\longrightarrow \mathcal{F}^0_{\mathcal{E}}(E) \longrightarrow E_0 \otimes \mathrm{Hom}(E_0, E) \xrightarrow{\text{ev}} E \\ &\longrightarrow \mathcal{F}^1_{\mathcal{E}}(E) \longrightarrow E_0 \otimes \mathrm{Ext}^1(E_0, E) \longrightarrow 0 \\ &\longrightarrow \mathcal{F}^2_{\mathcal{E}}(E) \longrightarrow E_0 \otimes \mathrm{Ext}^2(E_0, E) \longrightarrow 0, \end{aligned} \tag{2.3}$$

$$\begin{aligned} 0 &\longrightarrow E_0 \otimes \mathrm{Hom}(E_0, E) \longrightarrow \widehat{\mathcal{F}}^0_{\mathcal{E}}(E) \longrightarrow \\ 0 &\longrightarrow E_0 \otimes \mathrm{Ext}^1(E_0, E) \longrightarrow \widehat{\mathcal{F}}^1_{\mathcal{E}}(E) \longrightarrow \\ E &\longrightarrow E_0 \otimes \mathrm{Ext}^2(E_0, E) \longrightarrow \widehat{\mathcal{F}}^2_{\mathcal{E}}(E) \longrightarrow 0. \end{aligned} \tag{2.4}$$

The following is a modification of Lemma 2.5.

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LEMMA 2.6. *Assume that $\mathcal{E}$ is not locally free. Let $E \neq 0$ be a coherent sheaf on X.*

(1) *Assume that $E \in \mathfrak{T}_1$. Then*
   (a) $\mathcal{F}^2_{\mathcal{E}}(E) = 0$;
   (b) $\mathcal{F}^0_{\mathcal{E}}(E) \in \mathfrak{F}_2$ *and* $\operatorname{Hom}(E_0, \mathcal{F}^0_{\mathcal{E}}(E)) = 0$; *moreover, if E does not contain a zero-dimensional subsheaf, then* $\mu_{\max,G_2}(\mathcal{F}^0_{\mathcal{E}}(E)) < 0$ *or* $\mathcal{F}^0_{\mathcal{E}}(E) = 0$;
   (c) *there is an exact sequence*

$$0 \to F_1 \to \mathcal{F}^1_{\mathcal{E}}(E) \to E_0 \otimes \operatorname{Hom}(\mathcal{F}^1_{\mathcal{E}}(E), E_0)^{\vee} \to 0$$

   such that $F_1 \in \mathfrak{T}_2$.

(2) *Assume that $E \in \mathfrak{F}_1$. Then*
   (a) $\mathcal{F}^0_{\mathcal{E}}(E) = 0$;
   (b) $\mathcal{F}^1_{\mathcal{E}}(E) \in \mathfrak{F}_2$ *and* $\operatorname{Hom}(E_0, \mathcal{F}^1_{\mathcal{E}}(E)) = 0$, *in particular $\mathcal{F}^1_{\mathcal{E}}(E)$ is torsion free;*
   (c) $\mathcal{F}^2_{\mathcal{E}}(E) \cong E_0 \otimes \operatorname{Hom}(E, E_0)^{\vee}$.

(3) *For a coherent sheaf E fitting in an exact sequence*

$$0 \to E_1 \to E \to E_0^{\oplus n} \to 0 \tag{2.5}$$

   such that $E_1 \in \mathfrak{T}_2$, we have $\widehat{\mathcal{F}}^0_{\mathcal{E}}(E) \cong E_0 \otimes \operatorname{Hom}(E_0, E)$, $\widehat{\mathcal{F}}^1_{\mathcal{E}}(E) \in \mathfrak{T}_1$ and $\widehat{\mathcal{F}}^2_{\mathcal{E}}(E) = 0$.

(4) *If $E \in \mathfrak{F}_2$, then we also have $\widehat{\mathcal{F}}^0_{\mathcal{E}}(E) \cong E_0 \otimes \operatorname{Hom}(E_0, E)$, $\widehat{\mathcal{F}}^1_{\mathcal{E}}(E) \in \mathfrak{F}_1$, and $\widehat{\mathcal{F}}^2_{\mathcal{E}}(E) \in \mathfrak{T}_1$.*

Proof. The proof of part (1)(a) is clear. For claim (b), since $\mathcal{F}_{\mathcal{E}}(E_0) = E_0[-2]$, we have

$$\operatorname{Hom}(E_0, \mathcal{F}^0_{\mathcal{E}}(E)) = \operatorname{Hom}(E_0, \mathcal{F}_{\mathcal{E}}(E)) = \operatorname{Hom}(E_0, E[-2]) = 0.$$

By (2.3), we get $\mathcal{F}^0_{\mathcal{E}}(E) \in \mathfrak{F}_2$. If E does not contain a zero-dimensional subsheaf, then $\ker(\mathrm{ev})$ is locally free. Since $\operatorname{Hom}(E_0, \ker(\mathrm{ev})) = 0$, $\ker(\mathrm{ev})$ does not contain a μ-stable subsheaf F with $\mu(F) \geq 0$. Thus $\mu_{\max,G_2}(\mathcal{F}^0_{\mathcal{E}}(E)) < 0$ or $\mathcal{F}^0_{\mathcal{E}}(E) = 0$. For claim (c), we set $F_1 := \operatorname{coker}(E_0 \otimes \operatorname{Hom}(E_0, E) \to E)$. Since $\mu_{\min,G_2}(E) > 0$, we have $F_1 \in \mathfrak{T}_2$. By (2.3), $\operatorname{Hom}(\mathcal{F}^1_{\mathcal{E}}(E), E_0) \cong \operatorname{Ext}^1(E_0, E)^{\vee}$, and hence we get our claim.

The proof of part (2)(a) is a consequence of Lemma 2.3. Since $\mathcal{F}^0_{\mathcal{E}}(E) = 0$,

$$\operatorname{Hom}(E_0, \mathcal{F}^1_{\mathcal{E}}(E)) = \operatorname{Hom}(E_0[-1], \mathcal{F}_{\mathcal{E}}(E)) = \operatorname{Hom}(E_0[-1], E[-2]) = 0.$$

The other claims follow from (2.3).

For the proof of part (3), we use (2.4). It is easy to see that $\widehat{\mathcal{F}}^2_{\mathcal{E}}(E_1) = \widehat{\mathcal{F}}^2_{\mathcal{E}}(E_0) = 0$. Hence $\widehat{\mathcal{F}}^2_{\mathcal{E}}(E) = 0$. Then we see that $\operatorname{Hom}(\widehat{\mathcal{F}}^1_{\mathcal{E}}(E), E_0) = \operatorname{Hom}(\widehat{\mathcal{F}}_{\mathcal{E}}(E), E_0[-1]) = 0$. The other claims also follow.

For the proof of part (4), since $\operatorname{Hom}(\widehat{\mathcal{F}}^2_{\mathcal{E}}(E), E_0) = \operatorname{Hom}(E, E_0[-2]) = 0$, we have $\widehat{\mathcal{F}}^2_{\mathcal{E}}(E) \in \mathfrak{T}_1$. By (2.4), we get $\operatorname{im}(\widehat{\mathcal{F}}^1_{\mathcal{E}}(E) \to E) \in \mathfrak{F}_1$, which implies that $\widehat{\mathcal{F}}^1_{\mathcal{E}}(E) \in \mathfrak{F}_1$. □

Then we have a similar result to Theorem 2.1.

PROPOSITION 2.7. *Let $\mathfrak{A}'_2$ be the subcategory of $\mathbf{D}(X)$ such that an object $V^{\bullet} \in \mathbf{D}(X)$ belongs to $\mathfrak{A}'_2$, if*

(i) $H^j(V^{\bullet}) = 0$ *for* $j \neq -1, 0$;
(ii) $H^{-1}(V^{\bullet}) = 0 \in \mathfrak{F}_2$ *and* $\operatorname{Hom}(E_0, H^{-1}(V^{\bullet})) = 0$;
(iii) *there is an exact sequence* $0 \to F \to H^0(V^{\bullet}) \to E_0^{\oplus n} \to 0$ *such that* $F \in \mathfrak{T}_2$.

Then $\mathcal{F}_{\mathcal{E}}[1]$ induces an equivalence $\mathfrak{A}_1 \to \mathfrak{A}'_2$.

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Proof. $\widehat{\mathcal{F}}_{\mathcal{E}}(\mathfrak{A}'_2) \subset \mathfrak{A}_1$ is a consequence of Lemma 2.6. If $V^\bullet \in \mathbf{D}(X)$ belongs to $\mathfrak{A}_1$, then by Lemma 2.6 we see that $H^{-1}(\mathcal{F}_{\mathcal{E}}(V^\bullet)[1])[1] \in \mathfrak{A}'_2$ and we have an exact sequence

$$E_0^{\oplus k} \to H^0(\mathcal{F}_{\mathcal{E}}(V^\bullet)[1]) \xrightarrow{\lambda} E' \to 0$$

where $\mathcal{F}_{\mathcal{E}}^2(H^{-1}(V^\bullet)) = E_0^{\oplus k}$ and $E' := \mathcal{F}_{\mathcal{E}}^1(H^0(V^\bullet))$ fits in an exact sequence

$$0 \to E'_1 \to E' \to E_0^{\oplus n} \to 0$$

with $E'_1 \in \mathfrak{T}_2$. Hence, in order to prove $\mathcal{F}_{\mathcal{E}}(V^\bullet) \in \mathfrak{A}'_2$, we shall show that $\phi : E \to E_0 \otimes \operatorname{Hom}(E, E_0)^\vee$ is surjective and $\ker \phi \in \mathfrak{T}_2$, where $E := H^0(\mathcal{F}_{\mathcal{E}}(V^\bullet)[1])$.

The surjectivity of ϕ is a consequence of the exact sequence

$$E_0^{\oplus k} \xrightarrow{\psi} E_0 \otimes \operatorname{Hom}(E, E_0)^\vee \to E_0^{\oplus n} \to 0,$$

where ψ is the composite $E_0^{\oplus k} \to E \to E_0 \otimes \operatorname{Hom}(E, E_0)^\vee$. We note that $\ker \psi = E_0^{\oplus k'}$, $k' \le k$. Then we have an exact sequence

$$E_0^{\oplus k'} \to \ker \phi \to E'_1 \to 0.$$

Since $\operatorname{Ext}^1(E_0, E_0) = 0$, we have $\operatorname{Hom}(\ker \phi, E_0) = 0$. Then we see that $\operatorname{Hom}(\ker \phi, F) = 0$ for all μ-stable sheaf F with $\mu_{G_2}(F) \le 0$. □

2.2 Useful facts

In the remaining part of this section, we give some facts which will play important roles in the next section. For $v \in v(\mathbf{D}(X))$, we write

$$v = l v_0^\vee + a \varrho_X + (dH + D) - (dH + D, \xi_0)\varrho_X / r_0, \tag{2.6}$$

as in Proposition 1.5.

Proposition 2.8. *Assume that $l, a > 0$. We set*

$$N := \max\{(4r_0^3 l^2 + 1/(H^2))\epsilon, (1+\epsilon) r_0^2 l(\langle v^2 \rangle - (D^2))\}.$$

Then the following hold.

(1) *If $d > N$, then for any μ-semi-stable sheaf F_1 with*

$$v(F_1) = a_1 w_0 + l_1 \varrho_Y - (d_1 \widehat{H} + \widehat{D}_1 + (d_1 \widehat{H} + \widehat{D}_1, \widetilde{\xi}_0)\varrho_Y / r_0),$$
$$0 < d_1 \le d,\ d_1/a_1 \le d/a,\ D_1 \in \operatorname{NS}(X) \otimes \mathbb{Q} \cap H^\perp,$$

we have $l_1 \le l d_1 / d$.

(2) *If $d > N$, then for any μ-semi-stable sheaf E_1 with*

$$v(E_1) = l_1 v_0^\vee + a_1 \varrho_X + (d_1 H + D_1 - (d_1 H + D_1, \xi_0)\varrho_X / r_0),$$
$$0 < d_1 \le d,\ d_1/l_1 < d/l,\ D_1 \in \operatorname{NS}(X) \otimes \mathbb{Q} \cap H^\perp,$$

we have $a_1 < a d_1 / d$.

Proof. It is sufficient to prove the claims (1) and (2) for μ-stable sheaves. Indeed, for a μ-semi-stable sheaf F_1, we take a Jordan–Hölder filtration with respect to μ-stability. We assume that F_1 is S-equivalent to $\bigoplus_i F_{1,i}$, where $F_{1,i}$ are μ-stable torsion-free sheaves with

$$v(F_{1,i}) = a_{1,i} w_0 + l_{1,i} \varrho_Y - (d_{1,i} \widehat{H} + \widehat{D}_{1,i} + (d_{1,i} \widehat{H} + \widehat{D}_{1,i}, \widetilde{\xi}_0)\varrho_Y / r_0),$$
$$d_{1,i}/a_{1,i} = d_1/a_1,\ D_{1,i} \in \operatorname{NS}(X) \otimes \mathbb{Q} \cap H^\perp.$$

Then since $d_{1,i}/a_{1,i} = d_1/a_1 \le d/a$ and $d_{1,i} \le d_1 \le d$, if the claim holds for μ-stable sheaves, then we have $l_{1,i} \le l d_{1,i}/d$ for all i. Thus $l_1 \le \sum_i l d_{1,i}/d = l d_1/d$. Hence we may assume that F_1 is μ-stable. In the same way, we may assume that E_1 is μ-stable. For μ-stable sheaves, the claims follow from the following numerical statement. □

Lemma 2.9. *Assume that $l, a > 0$ as above.*

(1) *If $d > N$, then for any Mukai vector $w_1 \in H^{ev}(Y, \mathbb{Z})$ such that $\langle w_1^2 \rangle \ge -2\epsilon$ and*

$$\begin{aligned} w_1 &= a_1 w_0 + l_1 \varrho_Y - (d_1 \widehat{H} + \widehat{D}_1 + (d_1 \widehat{H} + \widehat{D}_1, \widetilde{\xi}_0)\varrho_Y/r_0), \\ &\quad 0 < d_1 \le d, d_1/a_1 \le d/a, D_1 \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp}, \end{aligned} \tag{2.7}$$

we have $l_1 \le l d_1/d$.

(2) *If $d > N$, then for any Mukai vector $v_1 \in H^{ev}(X, \mathbb{Z})$ such that $\langle v_1^2 \rangle \ge -2\epsilon$ and*

$$\begin{aligned} v_1 &= l_1 v_0^{\vee} + a_1 \varrho_X + (d_1 H + D_1 - (d_1 H + D_1, \xi_0)\varrho_X/r_0), \\ &\quad 0 < d_1 \le d, d_1/l_1 < d/l, D_1 \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp}, \end{aligned}$$

we have $a_1 < a d_1/d$.

Proof. We set $(H^2) := 2n$ and $s := \langle v^2 \rangle/2 = -r_0 l a + d^2 n + (D^2)/2$. By our assumption, $d^2 n - s + (D^2)/2 = r_0 l a > 0$. We shall first prove part (1).

Case I. We assume that $d_1 < d$. By (1.7), we get that $1/(2nr_0) \le d_1 \le d - 1/(2nr_0)$. We note that

$$\begin{aligned} \langle w_1^2 \rangle &= 2nd_1^2 - 2l_1 a_1 r_0 + (D_1^2) \\ &\le 2nd_1^2 - 2l_1 r_0 d_1 a/d \\ &= 2nd_1^2 - 2l_1 r_0 \frac{d_1}{d} \frac{d^2 n - s + (D^2)/2}{r_0 l} \\ &= 2nd_1^2 - 2l_1 d_1 \frac{d^2 n - s + (D^2)/2}{dl}. \end{aligned} \tag{2.8}$$

We first show that $l_1 < l$ for $d > N$. Assume that $l_1 \ge l$. By (2.8), we see that

$$\begin{aligned} -2\epsilon &\le \langle w_1^2 \rangle \\ &\le 2d_1^2 n - 2(d^2 n - s + (D^2)/2) d_1/d \\ &= 2nd_1 \left(d_1 - d + \frac{s - (D^2)/2}{dn} \right). \end{aligned} \tag{2.9}$$

We set $n_1 := \max\{(4r_0 + 1/(2nr_0))\epsilon, (1+\epsilon) r_0 (\langle v^2 \rangle - (D^2))\}$. Since $N \ge n_1$, by Lemma 2.10 below, we get a contradiction. Therefore, we have $l_1 < l$ for $d > N$.

We next show that $l_1 \le l d_1/d$. By (2.8), we get that

$$-2\epsilon \le \langle w_1^2 \rangle \le 2nd_1 \left(\left(d_1 - \frac{l_1}{l} d \right) + \frac{l_1}{dnl} (s - (D^2)/2) \right).$$

We note that

$$2nd_1 \left(\left(d_1 - \frac{l_1}{l} d \right) + \frac{l_1}{dnl} (s - (D^2)/2) \right) < -2\epsilon \tag{2.10}$$

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if and only if $(l_1/(dnl))(s-(D^2)/2) < l_1 d/l - (d_1 + \epsilon/(nd_1))$. We shall show that $(l_1/l)d - d_1 \le 0$, if $d > N$. Assume that $(l_1/l)d - d_1 > 0$. If $\epsilon = 0$ and $d > N$, then it is easy to see that (2.10) holds. Hence we assume that $\epsilon = 1$. Since $1/(2nr_0) \le d_1 \le (l_1/l)d - 1/(2nlr_0^2)$, we get that

$$l_1 d/l - (d_1 + 1/(nd_1)) \ge \min\left\{\frac{l_1 d}{l} - \frac{1}{2nr_0} - 2r_0, \frac{1}{2nlr_0^2} - \frac{1}{n(l_1 d/l - 1/(2nlr_0^2))}\right\}.$$

We set $n_2 := 4l^2 r_0^3 + 1/(2n)$. Then we see that $n(l_1 d/l - 1/(2nlr_0^2)) > 4nlr_0^2$ for $d > n_2$. We set $n_3 := 2r_0^2 l + l/(2n) + 1/(4nr_0)$. Then we get that $l_1 d/l - 1/(2nr_0) - 2r_0 \ge 1/(4nlr_0^2)$. Hence $l_1 d/l - (d_1 + 1/(nd_1)) \ge 1/(4nlr_0^2)$ for $d \ge \max\{n_2, n_3\}$. So if $d > \max\{n_1, n_2, n_3, 4r_0^2 l(s-(D^2)/2)\} = N$, then $\langle w_1^2\rangle < -2$, which is a contradiction. Therefore, $(l_1/l)d - d_1 \le 0$ for $d > N$.

Case II. We next assume that $d_1 = d$. If $l_1 \ge l + 1/r_0$, then we get

$$\begin{aligned}\langle w_1^2\rangle &\le \langle v^2\rangle - (D^2) - 2a \\ &= \frac{-(2nd^2) + (lr_0+1)(\langle v^2\rangle - (D^2))}{(lr_0)}.\end{aligned}$$

Since $d > 4r_0^3 l^2$, we get $nd^2 > 4nr_0^3 l^2 d$. Then we see that $nd^2 > 4nr_0^3 l^2 d > (lr_0+1)(\langle v^2\rangle - (D^2))$ and $nd^2 > n(4r_0^3 l^2)^2 > 2lr_0$, and hence $\langle w_1^2\rangle < -2$. Therefore, we get our claim.

We next prove part (2). Since $\mathcal{F}_{\mathcal{E}}(v_1) = w_1$, the claim follows from part (1). □

LEMMA 2.10. *Assume that* $1/(2nr_0) \le d_1 \le d - 1/(2nr_0)$. *Then*

$$2nd_1\left(d_1 - d + \frac{s-(D^2)/2}{dn}\right) < -2\epsilon \tag{2.11}$$

for $d > n_1$.

Proof. It is easy to see that (2.11) follows from the following inequality:

$$\begin{aligned} d - \frac{s-(D^2)/2}{dn} &> \max\left\{d_1 + \frac{\epsilon}{nd_1}\,\middle|\, \frac{1}{2nr_0} \le d_1 \le d - \frac{1}{2nr_0}\right\} \\ &= \max\left\{d_1 + \frac{\epsilon}{nd_1}\,\middle|\, d_1 = \frac{1}{2nr_0}, d - \frac{1}{2nr_0}\right\} \end{aligned} \tag{2.12}$$

for all $d > n_1$. Hence we shall show (2.12). If $\epsilon = 0$ and $d > r_0(\langle v^2\rangle - (D^2)) = n_1$, then (2.12) holds. Hence we assume that $\epsilon = 1$. For $d > n_1$, we have $n(d - 1/(2nr_0)) > 4nr_0$ and $(s - (D^2)/2)/(dn) < 1/(4nr_0)$. Hence

$$d - \frac{1}{2nr_0} + \frac{1}{n(d-1/(2nr_0))} < d - \frac{1}{2nr_0} + \frac{1}{4nr_0} = d - \frac{1}{4nr_0} < d - \frac{s-(D^2)/2}{dn}.$$

We also get that

$$\frac{1}{2nr_0} + 2r_0 \le -\frac{1}{4nr_0} + 1 + 2r_0 < -\frac{s-(D^2)/2}{dn} + n_1 < -\frac{s-(D^2)/2}{dn} + d.$$

Therefore, (2.12) holds. □

PROPOSITION 2.11. *Assume that* $l = 0$ *and* $d > 0$. *We set*

$$N := \max\{(\langle v^2\rangle - (D^2))/2 + \epsilon, (2r_0+1)\epsilon\}.$$

Then the following hold.

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(1) *If $a > N$, then for any μ-semi-stable sheaf F_1 with*

$$v(F_1) = a_1 w_0 + l_1 \varrho_Y - (d_1 \widehat{H} + \widehat{D}_1 + (d_1 \widehat{H} + \widehat{D}_1, \widetilde{\xi}_0)\varrho_Y / r_0),$$
$$0 < d_1 \leq d, d_1/a_1 \leq d/a, D_1 \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp},$$

we have $l_1 \leq 0$.

(2) *If $a > N$, then for any μ-semi-stable sheaf E_1 with*

$$v(E_1) = l_1 v_0^{\vee} + a_1 \varrho_X + (d_1 \widehat{H} + \widehat{D}_1 - (d_1 H + D_1, \xi_0)\varrho_X / r_0),$$
$$0 < d_1 \leq d, l_1 > 0, D_1 \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp},$$

we have $a_1/d_1 < a/d$.

In the same way as in the proof of Proposition 2.8, we may assume that E_1 and F_1 are μ-stable. Then the claims follow from the following two lemmas.

Lemma 2.12. *Assume that $l = 0$ and $d > 0$. We set $N' := \max\{(\langle v^2 \rangle - (D^2))/2, (2r_0 + 1)\epsilon\}$. Then the following hold.*

(1) *If $a > N'$, then for any Mukai vector $w_1 \in H^{ev}(Y, \mathbb{Z})$ such that $\langle w_1^2 \rangle \geq -2\epsilon$ and*

$$w_1 = a_1 w_0 + l_1 \varrho_Y - (d_1 \widehat{H} + \widehat{D}_1 + (d_1 \widehat{H} + \widehat{D}_1, \widetilde{\xi}_0)\varrho_Y / r_0),$$
$$d_1 < d, 0 < d_1/a_1 \leq d/a, D_1 \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp},$$

we have $l_1 \leq 0$.

(2) *If $a > N'$, then for any Mukai vector $v_1 \in H^{ev}(X, \mathbb{Z})$ such that $\langle v_1^2 \rangle \geq -2\epsilon$ and*

$$v_1 = l_1 v_0^{\vee} + a_1 \varrho_X + (d_1 \widehat{H} + \widehat{D}_1 - (d_1 H + D_1, \xi_0)\varrho_X / r_0),$$
$$0 < d_1 < d, l_1 > 0, D_1 \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp},$$

we have $a_1/d_1 < a/d$.

Proof. We shall only prove part (1). We may assume that F_1 is μ-stable. Assume that $l_1 > 0$. Then $r_0 l_1 \geq 1$, and hence we see that

$$-2\epsilon \leq \langle w_1^2 \rangle \leq d_1^2 (H^2) - 2r_0 l_1 a_1 \leq d_1^2 (H^2) - 2a_1 \leq d_1^2 (H^2) - 2a d_1/d.$$

We set

$$\begin{aligned} n_1 &:= \max\left\{ d\left(\frac{(H^2)}{2} d_1 + \frac{\epsilon}{d_1} \right) \middle| \frac{1}{r_0 (H^2)} \leq d_1 \leq d - \frac{1}{r_0 (H^2)} \right\} \\ &= \max\left\{ d\left(\frac{(H^2)}{2} d_1 + \frac{\epsilon}{d_1} \right) \middle| d_1 = \frac{1}{r_0 (H^2)}, d - \frac{1}{r_0 (H^2)} \right\}. \end{aligned}$$

Then we have $d_1^2 (H^2) - 2a d_1/d < -2\epsilon$ for $a > n_1$. Therefore, $l_1 \leq 0$ for $a > n_1$. It is easy to see that $N' > n_1$. Hence part (1) holds. □

Lemma 2.13. *Assume that $l = 0$ and $d > 0$.*

(1) *If $a > (\langle v^2 \rangle - (D^2))/2 + \epsilon$, then for any Mukai vector $w_1 \in H^{ev}(Y, \mathbb{Z})$ such that $\langle w_1^2 \rangle \geq -2\epsilon$ and*

$$w_1 = a_1 w_0 + l_1 \varrho_Y - (d_1 \widehat{H} + \widehat{D}_1 + (d_1 \widehat{H} + \widehat{D}_1, \widetilde{\xi}_0)\varrho_Y / r_0),$$
$$d_1 = d, 0 < d_1/a_1 \leq d/a, D_1 \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp},$$

we have $l_1 \leq 0$.

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(2) *If* $a > (\langle v^2\rangle - (D^2))/2 + \epsilon$, *then for any Mukai vector* $v_1 \in H^{ev}(X, \mathbb{Z})$ *such that* $\langle v_1^2\rangle \geq -2\epsilon$ *and*

$$v_1 = l_1 v_0^{\vee} + a_1 \varrho_X + (d_1 H + D_1 - (d_1 H + D_1, \xi_0)\varrho_X / r_0),$$
$$d_1 = d, l_1 > 0, D_1 \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp},$$

we have $a_1/d_1 < a/d$.

Proof. We only prove part (2). If $a_1 \geq a$, then we see that

$$\langle v_1^2\rangle - (D_1^2) \leq d^2(H^2) - 2l_1 a r_0 \leq d^2(H^2) - 2a = \langle v^2\rangle - (D^2) - 2a < -2\epsilon.$$

Therefore, $a_1 < a$. □

Corollary 2.14. *Assume that* $d > N$ *if* $l > 0$ *and* $a > N$ *if* $l = 0$, $d > 0$. *Let* F *be a* μ*-semi-stable sheaf with* $v(F) = \mathcal{F}_{\mathcal{E}}(v)$. *Then* F *is* G_2*-twisted semi-stable. Moreover, if* F *is* G_2*-twisted stable, then it is* μ*-stable.*

Proof. Assume that F is not μ-stable. Let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F$$

be the Jordan–Hölder filtration of F with respect to the μ-stability. We set

$$v(F_i/F_{i-1}) = a_i w_0 + l_i \varrho_Y - (d_i \widehat{H} + \widehat{D}_i + (d_i \widehat{H} + \widehat{D}_i, \widetilde{\xi}_0)\varrho_Y / r_0), \quad D_i \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp}.$$

Applying Proposition 2.8, or 2.11 to each F_i/F_{i-1}, we get that $l_i \leq l d_i/d$. Then we see that $\sum_i l_i \leq \sum_i l d_i/d = l$. Since $\sum_i l_i = l$, we have $l_i = l d_i/d$ for all i. Since $d_i/a_i = d/a$, we get $l_i/a_i = l/a$, which implies that F is G_2-twisted semi-stable. By the same proof, we also see that F is μ-stable, provided that F is G_2-twisted stable. □

Remark 2.3. Assume that $d > N$ if $l > 0$ and $a > N$ if $l = 0$ and $d > 0$. Let F be a μ-semi-stable sheaf with $v(F) = \mathcal{F}_{\mathcal{E}}(v)$. Then F is locally free. Indeed we note that

$$v(F^{\vee\vee}) = a w_0 + l' \varrho_Y - (d\widehat{H} + \widehat{D} + (d\widehat{H} + \widehat{D}, \widetilde{\xi}_0)\varrho_Y / r_0), \quad l' \geq l.$$

By Proposition 2.8 or 2.11, we get $l' \leq l$. Hence $v(F^{\vee\vee}) = v(F)$, which implies that F is locally free.

Remark 2.4. Assume that $l > 0$. If $d > N$ and

$$\min\{-(D^2) | (D, H) = 0, D \in \mathrm{NS}(X) \setminus \{0\}\} > (r_0 l)^2(\langle w^2\rangle + 2(r_0 l)^2 \epsilon)/4, \tag{2.13}$$

then $\widehat{H}$ is a general polarization with respect to w.

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To prove the claim, for a μ-semi-stable sheaf F with $v(F) = w$, assume that there is an exact sequence

$$0 \to F_1 \to F \to F_2 \to 0$$

such that $F_i(\neq 0)$, $i = 1, 2$, are μ-semi-stable sheaves with

$$v(F_i) := a_i w_0 + l_i \varrho_Y - (\widehat{\xi_i} + (\widehat{\xi_i}, \widetilde{\xi_0})\varrho_Y/r_0), \quad \xi_i \in \mathrm{NS}(X) \otimes \mathbb{Q}$$

and $(\xi_1, H)/a_1 = (\xi_2, H)/a_2$. By the proof of Corollary 2.14, we see that $l_1/a_1 = l_2/a_2 = l/a$, and F_i, $i = 1, 2$, are G_2-twisted semi-stable sheaves. Assume that F_i are S-equivalent to a direct sum of G_2-twisted stable sheaves: $F_i \sim \bigoplus_{i=1}^{s_i} F_{ij}$. Since $\chi(F_{ij}, F_{ij'}) \le 2\epsilon$ and $l_i r_0 \ge s_i$, we see that $\langle v(F_i)^2 \rangle \ge -2s_i^2\epsilon \ge -2l_i^2 r_0^2 \epsilon$. By using [Yos03a, part I, (5.3)], we see that

$$\begin{aligned}\frac{\langle v(\widehat{\mathcal{F}}_{\mathcal{E}}(w))^2\rangle}{r_0 l_1 r_0 l_2} &= -\left(\left(\frac{\xi_1}{r_0 l_1} - \frac{\xi_2}{r_0 l_2}\right)^2\right) + \frac{r_0 l}{r_0^3 l_1^2 l_2}\langle v(\widehat{\mathcal{F}}_{\mathcal{E}}(F_1))^2\rangle + \frac{r_0 l}{r_0^3 l_1 l_2^2}\langle v(\widehat{\mathcal{F}}_{\mathcal{E}}(F_2))^2\rangle \\ &\ge -\left(\left(\frac{\xi_1}{r_0 l_1} - \frac{\xi_2}{r_0 l_2}\right)^2\right) - 2\left(\frac{l}{l_2} + \frac{l}{l_1}\right).\end{aligned}$$

Hence we get

$$r_0^2 l_1 l_2 (\langle w^2 \rangle + 2r_0^2 l^2 \epsilon) \ge -((r_0 l_2 \xi_1 - r_0 l_1 \xi_2)^2).$$

Since $r_0 l_2 \xi_1 - r_0 l_1 \xi_2 = (r_0 l_2) c_1(\widehat{\mathcal{F}}_{\mathcal{E}}(v_1)) - (r_0 l_1) c_1(\widehat{\mathcal{F}}_{\mathcal{E}}(v_2)) \in \mathrm{NS}(X)$, we get our claim.

3. Asymptotic stability theorem

3.1 Positive rank case

THEOREM 3.1. *Assume that H is general with respect to v_0 and there is a universal family $\mathcal{E}$. Let E be a G_1-twisted semi-stable sheaf with $v(E) = v := l v_0^\vee + a\varrho_X + (dH + D) - (dH + D, \xi_0)\varrho_X/r_0$, where $l, a > 0$ and $D \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^\perp$. If*

$$d > \max\{(4l^2 r_0^3 + 1/(H^2))\epsilon, (1+\epsilon) r_0^2 l (\langle v^2 \rangle - (D^2))\},$$

then WIT_0 holds for E with respect to $\mathcal{F}_{\mathcal{E}}$ and $\mathcal{F}_{\mathcal{E}}^0(E)$ is G_2-twisted semi-stable. Moreover, $\mathcal{F}_{\mathcal{E}}$ preserves the S-equivalence classes. Conversely for a G_2-twisted semi-stable sheaf F with $v(F) = v(\mathcal{F}_{\mathcal{E}}(v))$, WIT_2 holds for F with respect to $\widehat{\mathcal{F}}_{\mathcal{E}}$ and $\widehat{\mathcal{F}}_{\mathcal{E}}^2(F)$ is a G_1-twisted semi-stable sheaf. In particular, $\mathcal{F}_{\mathcal{E}}$ induces an isomorphism

$$\mathcal{M}_H^{G_1}(v)^{ss} \to \mathcal{M}_{\widehat{H}}^{G_2}(\mathcal{F}_{\mathcal{E}}(v))^{ss}$$

which preserves the S-equivalence classes.

Proof. We set $\mathfrak{A}_2^* := \mathfrak{A}_2$ or $\mathfrak{A}_2'$ according to whether $\mathcal{E}$ is locally free or not. For a G_1-twisted semi-stable sheaf E with $v(E) = v$, we assume that $\mathcal{F}_{\mathcal{E}}(E)$ is not μ-stable. That is, WIT_0 does not hold for E with respect to $\mathcal{F}_{\mathcal{E}}$, or WIT_0 holds for E but $\mathcal{F}_{\mathcal{E}}^0(E)$ is not μ-stable. We set

$$\begin{gathered} v(\mathcal{F}_{\mathcal{E}}^0(E)) := a' w_0 + l'\varrho_Y - (d'\widehat{H} + \widehat{D}' + (d'\widehat{H} + \widehat{D}', \widetilde{\xi_0})\varrho_Y/r_0), \\ a' > 0, D' \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^\perp, \\ v(\mathcal{F}_{\mathcal{E}}^1(E)) := a'' w_0 + l''\varrho_Y + (d''\widehat{H} + \widehat{D}'' + (d''\widehat{H} + \widehat{D}'', \widetilde{\xi_0})\varrho_Y/r_0), \\ a'' \ge 0, D'' \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^\perp. \end{gathered}$$

By Lemma 2.5(1) or Lemma 2.6(1), we get (i) $\mu_{\max,G_2}(\mathcal{F}_{\mathcal{E}}^0(E)) < 0$, (ii) $\mathcal{F}_{\mathcal{E}}^1(E) \in \mathfrak{A}_2^*$ and (iii) $\mathcal{F}_{\mathcal{E}}^i(E) = 0$, $i \neq 0, 1$. In particular, $d' > 0$ and $d'' \ge 0$. Let $F_1 \neq 0$ be a G_2-twisted

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stable subsheaf of $\mathcal{F}^0_{\mathcal{E}}(E)$ such that $\mu_{G_2}(F_1)=\mu_{\max,G_2}(\mathcal{F}^0_{\mathcal{E}}(E))$ and $\mathcal{F}^0_{\mathcal{E}}(E)/F_1$ satisfies $\mu_{G_2}(\mathcal{F}^0_{\mathcal{E}}(E)/F_1)\le\mu_{G_2}(F_1)$ or $\mathcal{F}^0_{\mathcal{E}}(E)/F_1=0$. Then

$$v(F_1)=a_1w_0+l_1\varrho_Y-(d_1\widehat{H}+\widehat{D}_1+(d_1\widehat{H}+\widehat{D}_1,\widetilde{\xi}_0)\varrho_Y/r_0),$$
$$a_1>0,\,0<d_1/a_1\le d'/a',\,D_1\in\mathrm{NS}(X)\otimes\mathbb{Q}\cap H^{\perp}.$$

Let $F_2^{\bullet}$ be an object in $\mathbf{D}(Y)$ fitting in an exact triangle

$$F_1\to\mathcal{F}_{\mathcal{E}}(E)\to F_2^{\bullet}\to F_1[1]. \tag{3.1}$$

We note that $H^0(F_2^{\bullet})=\mathcal{F}^0_{\mathcal{E}}(E)/F_1$, $H^1(F_2^{\bullet})=\mathcal{F}^1_{\mathcal{E}}(E)$ and $H^k(F_2^{\bullet})=0$, $k\ne0,1$. By our choice of F_1, we get $d_1>0$ and $F_1,F_2^{\bullet}\in\mathfrak{A}_2^*[-1]$. Applying $\widehat{\mathcal{F}}_{\mathcal{E}}$ to (3.1), we have an exact triangle

$$\widehat{\mathcal{F}}_{\mathcal{E}}(F_1)\to E[-2]\to\widehat{\mathcal{F}}_{\mathcal{E}}(F_2^{\bullet})\to\widehat{\mathcal{F}}_{\mathcal{E}}(F_1)[1].$$

By Theorem 2.1 or Proposition 2.7, we have $\widehat{\mathcal{F}}_{\mathcal{E}}(F_1),\widehat{\mathcal{F}}_{\mathcal{E}}(F_2^{\bullet})\in\mathfrak{A}_1[-2]$. Then we see that WIT_2 holds for F_1 and we have an exact sequence

$$0\to\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2^{\bullet})\to\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1)\xrightarrow{\phi}E\to\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_2^{\bullet})\to0.$$

Since E and $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2^{\bullet})$ are torsion free, $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1)$ is also torsion free. Therefore, $l_1>0$. Since $(l,-d,a)=(l'-l'',-d'-d'',a'-a'')$, we have $d'\le d$ and $d'/a'\le d/a$. Moreover, if $d'/a'=d/a$, then $d''=a''=0$. Since $0<d_1\le d'\le d$, applying Proposition 2.8 to the sheaf F_1, we get that $l_1\le ld_1/d$. If $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2^{\bullet})\ne0$, then $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2^{\bullet})\in\mathfrak{A}_1[-1]$ is a torsion-free sheaf with $\mu_{\max,G_1}(\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2^{\bullet}))\le0$, which implies that $\mu_{G_1}(\operatorname{im}\phi)>\mu_{G_1}(\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1))\ge\mu_{G_1}(E)$. This means that E is not μ-semi-stable. Therefore, $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2^{\bullet})=0$. Then we see that $l_1=ld_1/d$ and $a_1/l_1\le a/l$, which implies that $d_1/a_1=d'/a'=d/a$ and $\mathcal{F}^0_{\mathcal{E}}(E)$ is a μ-semi-stable sheaf. By Proposition 2.8, we have $l'\le l$. Since $a''=d''=0$, $\mathcal{F}^1_{\mathcal{E}}(E)$ is a zero-dimensional sheaf with $l''\ge0$. Hence we have $l''=l'-l=0$. Therefore, $\mathcal{F}^1_{\mathcal{E}}(E)=0$ and $\mathcal{F}^0_{\mathcal{E}}(E)$ is G_2-twisted semi-stable.

If E is G_1-twisted stable, then we also see that $\mathcal{F}_{\mathcal{E}}(E)$ is a G_2-twisted stable sheaf. Assume that E is S-equivalent to $E'=\bigoplus_i E_i$, where E_i are G_1-twisted stable sheaves. We shall prove that WIT_0 holds for all E_i, $\mathcal{F}^0_{\mathcal{E}}(E_i)$ are G_2-twisted stable and $\mathcal{F}_{\mathcal{E}}(E)=\mathcal{F}^0_{\mathcal{E}}(E)$ is S-equivalent to $\bigoplus_i\mathcal{F}^0_{\mathcal{E}}(E_i)$. Since $\mathcal{F}_{\mathcal{E}}(E')=\bigoplus_i\mathcal{F}_{\mathcal{E}}(E_i)$, WIT_0 holds for E_i and $\mathcal{F}_{\mathcal{E}}(E_i)=\mathcal{F}^0_{\mathcal{E}}(E_i)$ are G_2-twisted semi-stable sheaves. For every subsheaf $F_{i,2}\subset\mathcal{F}_{\mathcal{E}}(E_i)$ with $\mu(\mathcal{F}_{\mathcal{E}}(E_i))=\mu(F_{i,2})$, we regard $F_{i,2}$ as a subsheaf of $\mathcal{F}_{\mathcal{E}}(E)$ and apply the same argument as above. Then we see that $\widehat{\mathcal{F}}_{\mathcal{E}}(F_{i,2})$ makes E_i properly G_1-twisted semi-stable. Therefore, $\mathcal{F}_{\mathcal{E}}(E_i)$ are G_2-twisted stable. Then it is easy to see that $\mathcal{F}_{\mathcal{E}}(E)$ is S-equivalent to $\bigoplus_i\mathcal{F}_{\mathcal{E}}(E_i)$. Therefore, $\mathcal{F}_{\mathcal{E}}$ preserves the S-equivalence classes.

Conversely for a G_2-twisted semi-stable sheaf F with $v(F)=\mathcal{F}_{\mathcal{E}}(v)$, assume that $\widehat{\mathcal{F}}_{\mathcal{E}}(F)[2]$ is not a μ-stable sheaf. We set

$$v(\widehat{\mathcal{F}}^1_{\mathcal{E}}(F)):=l'v^{\vee}+a'\varrho_X-((d'H+D')-(d'H+D',\xi_0)\varrho_X/r_0),$$
$$l'\ge0,\,D'\in\mathrm{NS}(X)\otimes\mathbb{Q}\cap H^{\perp},$$
$$v(\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)):=l''v^{\vee}+a''\varrho_X+(d''H+D'')-(d''H+D'',\xi_0)\varrho_X/r_0,$$
$$l''>0,\,D''\in\mathrm{NS}(X)\otimes\mathbb{Q}\cap H^{\perp}.$$

By Theorem 2.1 or Proposition 2.7, $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F)\in\mathfrak{F}_1$ and $\mu_{\min,G_1}(\widehat{\mathcal{F}}^2_{\mathcal{E}}(F))>0$. In particular, $d'\ge0$ and $d''>0$. Let $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)\to E_2$ be a quotient such that E_2 is a G_1-twisted stable torsion-free sheaf with $\mu_{G_1}(E_2)=\mu_{\min,G_1}(\widehat{\mathcal{F}}^2_{\mathcal{E}}(F))$. We set

$$v(E_2)=l_2v^{\vee}+a_2\varrho_X+(d_2H+D_2)-(d_2H+D_2,\xi_0)\varrho_X/r_0,$$
$$0<d_2/l_2\le d''/l'',\,D_2\in\mathrm{NS}(X)\otimes\mathbb{Q}\cap H^{\perp}.$$

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Since $(l, d, a) = (l'' - l', d'' + d', a'' - a')$, $d_2/l_2 \leq d''/l'' \leq d/l$ and $d_2/l_2 < d/l$ unless $l' = d' = 0$. Since $l_2 \leq l''$, we also have $d_2 \leq d'' \leq d$. Let $E_1^{\bullet}$ be an object in $\mathbf{D}(X)$ fitting in an exact triangle

$$E_1^{\bullet} \to \widehat{\mathcal{F}}_{\mathcal{E}}(F)[2] \to E_2 \to E_1^{\bullet}[1]. \tag{3.2}$$

We note that $H^{-1}(E_1^{\bullet}) = \widehat{\mathcal{F}}^1_{\mathcal{E}}(F)$, $H^0(E_1^{\bullet}) = \ker(\widehat{\mathcal{F}}^2_{\mathcal{E}}(F) \to E_2)$ and $H^i(E_1^{\bullet}) = 0$, $i \neq -1, 0$. In particular, $E_1^{\bullet}, E_2 \in \mathfrak{A}_1$. We apply $\mathcal{F}_{\mathcal{E}}$ to (3.2). Then by using Lemma 2.5 or Lemma 2.6, we have an exact sequence

$$0 \to \mathcal{F}^0_{\mathcal{E}}(E_1^{\bullet}) \to F \xrightarrow{\phi} \mathcal{F}^0_{\mathcal{E}}(E_2) \to \mathcal{F}^1_{\mathcal{E}}(E_1^{\bullet}) \to 0$$

with $\mathcal{F}^0_{\mathcal{E}}(E_1^{\bullet}) \in \mathfrak{F}_2$ and $\mu_{\min,G_2}(\mathcal{F}^1_{\mathcal{E}}(E_1^{\bullet})) \geq 0$ or $\mathcal{F}^1_{\mathcal{E}}(E_1^{\bullet}) = 0$. Assume that $d_2/l_2 < d/l$. Since $0 < d_2 \leq d$, we can apply Proposition 2.8 to conclude $a_2/d_2 < a/d$. Then $\phi(F)$ gives a destabilizing quotient sheaf. Therefore, $d_2/l_2 = d''/l'' = d/l$, $l' = d' = 0$ and $\mathcal{F}^1_{\mathcal{E}}(E_1^{\bullet})$ is a zero-dimensional sheaf. In particular, $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F) = 0$. If $a_2/l_2 \leq a/l$, then $d/a \leq d_2/a_2$, which also means that $\phi(F)$ gives a destabilizing quotient sheaf, unless $d/a = d_2/a_2$ and ϕ is surjective. If $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)$ contains a zero-dimensional sheaf, then F must contain $\mathcal{E}_{|\{x\} \times Y}$, $x \in X$. Therefore, $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)$ is a G_1-twisted semi-stable sheaf. It is easy to see that $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)$ is G_1-twisted stable if F is G_2-twisted stable. It is also easy to see the preservation of the S-equivalence classes. □

3.2 Rank zero case

Proposition 3.2. *Assume that H is general with respect to v_0 and there is a universal family $\mathcal{E}$. Let E be a G_1-twisted stable sheaf with $v(E) := v = a\varrho_X + (dH + D) - (dH + D, \xi_0)\varrho_X/r_0$, where $d > 0$ and $D \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp}$. If*

$$a > \max\{(2r_0 + 1)\epsilon, (\langle v^2 \rangle - (D^2))/2 + \epsilon\},$$

then WIT_0 holds for E with respect to $\mathcal{F}_{\mathcal{E}}$ and $\mathcal{F}^0_{\mathcal{E}}(E)$ is G_2-twisted semi-stable. Conversely for a G_2-twisted semi-stable sheaf F on Y with $v(F) = \mathcal{F}_{\mathcal{E}}(v)$, WIT_2 holds with respect to $\widehat{\mathcal{F}}_{\mathcal{E}}$ and $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)$ is a G_1-twisted semi-stable sheaf. In particular, $\mathcal{F}_{\mathcal{E}}$ induces an isomorphism

$$\mathcal{M}_H^{G_1}(v)^{ss} \to \mathcal{M}_{\widehat{H}}^{G_2}(\mathcal{F}_{\mathcal{E}}(v))^{ss}$$

which preserves the S-equivalence classes.

Proof. Assume that $\mathcal{F}_{\mathcal{E}}(E)$ is not μ-stable. We set

$$v(\mathcal{F}^0_{\mathcal{E}}(E)) := a'w_0 + l'\varrho_Y - (d'\widehat{H} + \widehat{D}' + (d'\widehat{H} + \widehat{D}', \widetilde{\xi}_0)\varrho_Y/r_0), \quad D' \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp},$$
$$v(\mathcal{F}^1_{\mathcal{E}}(E)) := a''w_0 + l''\varrho_Y + (d''\widehat{H} + \widehat{D}'' + (d''\widehat{H} + \widehat{D}'', \widetilde{\xi}_0)\varrho_Y/r_0), \quad D'' \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp}.$$

In the same way as in the proof of Theorem 3.1, by using Lemma 2.5(1) or Lemma 2.6(1), we see that $d' > 0$, $d'' \geq 0$ and we have an exact triangle

$$F_1 \to \mathcal{F}_{\mathcal{E}}(E) \to F_2^{\bullet} \to F_1[1] \tag{3.3}$$

such that $F_1 (\neq 0)$ is a G_2-twisted stable sheaf with

$$v(F_1) = a_1 w_0 + l_1\varrho_Y - (d_1\widehat{H} + \widehat{D}_1 + (d_1\widehat{H} + \widehat{D}_1, \widetilde{\xi}_0)\varrho_Y/r_0),$$
$$0 < d_1/a_1 \leq d'/a', D_1 \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp}$$

and $F_2^{\bullet}$ is a complex such that $\mu_{\max,G_2}(H^0(F_2^{\bullet})) < 0$ or $H^0(F_2^{\bullet}) = 0$, and $H^1(F_2^{\bullet}) \in \mathfrak{T}_2$. Applying $\widehat{\mathcal{F}}_{\mathcal{E}}$ to (3.3), we see that WIT_2 holds for F_1, $\widehat{\mathcal{F}}^i_{\mathcal{E}}(F_2^{\bullet}) = 0$ for $i \neq 1, 2$ and there is an exact sequence

$$0 \to \widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2^{\bullet}) \to \widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1) \xrightarrow{\phi} E \to \widehat{\mathcal{F}}^2_{\mathcal{E}}(F_2^{\bullet}) \to 0.$$

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Since $(0,-d,a)=(l'-l'',-d'-d'',a'-a'')$, we have $d'\le d$ and $d'/a'\le d/a$. Moreover, if $d'/a'=d/a$, then $d''=a''=0$. Since $0<d_1\le d'\le d$, applying Proposition 2.11 to the sheaf F_1, we get that $l_1\le 0$. Hence $l_1=0$ and $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2^\bullet)=0$ by the torsion-freeness of $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F_2^\bullet)$. Since E is G_1-twisted semi-stable, $a_1/d_1\le a/d$. Hence $d_1/a_1=d'/a'=d/a$, $d''=a''=0$ and $\mathcal{F}^0_{\mathcal{E}}(E)$ is a μ-semi-stable sheaf. By Proposition 2.11, we have $l'\le 0$. Since $a''=d''=0$, $\mathcal{F}^1_{\mathcal{E}}(E)$ is a zero-dimensional sheaf with $l''\ge 0$. Then we have $l''=l'=0$. Therefore, $\mathcal{F}^1_{\mathcal{E}}(E)=0$ and $\mathcal{F}^0_{\mathcal{E}}(E)$ is G_2-twisted semi-stable.

If E is G_1-twisted stable, then we also see that $\mathcal{F}_{\mathcal{E}}(E)$ is G_2-twisted stable. Moreover, it preserves the S-equivalence classes.

Conversely for a G_2-twisted semi-stable sheaf F with $v(F)=\mathcal{F}_{\mathcal{E}}(v)$, assume that $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F)\neq 0$. Then $\operatorname{rk}(\widehat{\mathcal{F}}^1_{\mathcal{E}}(F))=\operatorname{rk}(\widehat{\mathcal{F}}^2_{\mathcal{E}}(F))>0$. Let $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)\to E_2$ be a quotient such that E_2 is a G_1-twisted stable torsion-free sheaf with $\mu(E_2)=\mu_{\min,G_1}(\widehat{\mathcal{F}}^2_{\mathcal{E}}(F))$. We set

$$v(E_2)=l_2v^\vee+a_2\varrho_X+(d_2H+D_2)-(d_2H+D_2,\xi_0)\varrho_X/r_0,\quad D_2\in \operatorname{NS}(X)\otimes\mathbb{Q}\cap H^\perp.$$

By Lemma 2.5(2) or Lemma 2.6(4), we have $d_2>0$. Let $E_1^\bullet$ be an object in $\mathbf{D}(X)$ which defines an exact triangle

$$E_1^\bullet\to\widehat{\mathcal{F}}_{\mathcal{E}}(F)[2]\to E_2\to E_1^\bullet[1]. \tag{3.4}$$

We note that $H^{-1}(E_1^\bullet)=\widehat{\mathcal{F}}^1_{\mathcal{E}}(F)$, $H^0(E_1^\bullet)=\ker(\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)\to E_2)$ and $H^i(E_1^\bullet)=0$, $i\neq -1,0$. Thus $E_1^\bullet, E_2\in\mathfrak{A}_1$. We apply $\mathcal{F}_{\mathcal{E}}$ to (3.4). Then, by using Lemma 2.5 or Lemma 2.6, we have an exact sequence

$$0\to\mathcal{F}^0_{\mathcal{E}}(E_1^\bullet)\to F\xrightarrow{\phi}\mathcal{F}^0_{\mathcal{E}}(E_2)\to\mathcal{F}^1_{\mathcal{E}}(E_1^\bullet)\to 0$$

with $\mu_{\max,G_2}(\mathcal{F}^0_{\mathcal{E}}(E_1^\bullet))\le 0$ and $\mu_{\min,G_2}(\mathcal{F}^1_{\mathcal{E}}(E_1^\bullet))\ge 0$. Then $d=d_2-\deg_{G_2}(\mathcal{F}_{\mathcal{E}}(E_1^\bullet))/r_0(H^2)\ge d_2$. Since $0<d_2\le d$, we can apply Lemmas 2.12 and 2.13 to conclude that $a_2/d_2<a/d$. Then the image of $\phi(F)$ gives a destabilizing quotient sheaf. Therefore, $\widehat{\mathcal{F}}^1_{\mathcal{E}}(F)=0$. If $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)$ contains a zero-dimensional sheaf, then F must contain $\mathcal{E}_{|\{x\}\times Y}$, $x\in X$. Thus $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)$ is purely one-dimensional. Let $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)\to E_2$ be a quotient such that E_2 is a G_1-twisted stable purely one-dimensional sheaf with

$$v(E_2)=a_2\varrho_X+(d_2H+D_2)-(d_2H+D_2,\xi_0)\varrho_X/r_0.$$

If $a_2/d_2\le a/d$, then by the same argument we see that $\phi(F)$ gives a destabilizing quotient sheaf, unless $d_2/a_2=d/a$ and ϕ is surjective. By Corollary 2.14, ϕ is surjective. Thus F is properly G_2-twisted semi-stable. Therefore, $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)$ is G_1-twisted semi-stable, and $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)$ is G_1-twisted stable if F is G_2-twisted stable. It is also easy to see the preservation of the S-equivalence classes. □

Definition 3.1. Let v be a Mukai vector with $\operatorname{rk} v=0$. A polarization H is general with respect to v and $G\in K(X)\otimes\mathbb{Q}$ if for a G-twisted semi-stable sheaf E with $v(E)=v$ and a non-trivial subsheaf F of E,

$$\frac{\chi_G(F)}{(c_1(F),H)}=\frac{\chi_G(E)}{(c_1(E),H)}\text{ if and only if } v(F)\in\mathbb{Q}v.$$

If $\langle v(G),v\rangle\neq 0$, then there is a general polarization.

Lemma 3.3. *For an effective divisor class $\xi\in\operatorname{NS}(X)$, we set*

$$D_\xi:=\{\xi_1\in\operatorname{NS}(X)|\xi_1 \textit{ and } \xi-\xi_1 \textit{ are represented as effective divisors}\}.$$

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(1) *D_ξ is a finite set.*

(2) *We set $\xi = c_1(v)$. Assume that $(\langle v(G), v\rangle\xi_1 - b\xi, H) \neq 0$ for all $\xi_1 \in D_\xi$ and $b \in \mathbb{Z}$ satisfying $0 \leq |b| < |\langle v(G), v\rangle|$ and $\langle v(G), v\rangle\xi_1 - b\xi \neq 0$. Then H is a general polarization with respect to v and G.*

Proof. (1) We prove the claim for any smooth projective surface X. We fix an ample divisor L on X such that $L \pm K_X$ are ample. We shall show that

$$\{D \in \mathrm{NS}(X) \mid \ D \text{ is represented by an effective divisor with } (L, D) = d\}$$

is a finite set. We set $D' := (L^2)D - (D, L)L$. Then $(D', L) = 0$ and $({D'}^2) = (L^2)^2(D^2) - (D, L)^2(L^2)$. We shall bound (D^2) in terms of (D, L). Then the claim follows from the Hodge index theorem. For an irreducible and reduced curve C, we have $(C, C + K_X) \geq -2$. Hence $(D, D + K_X) \geq -2(D, L)$. Since $|(D, K_X)| < (D, L)$, we have $(D^2) \geq -2(D, L) - (D, K_X) \geq -3(D, L)$. Therefore, (D^2) is bounded below.

(2) Let F be a proper subsheaf of a coherent sheaf E with $v(E) = v$. Then $c_1(F) \in D_\xi$. If

$$\frac{\chi_G(F)}{(c_1(F), H)} = \frac{\chi_G(E)}{(c_1(E), H)},$$

then $|\chi_G(F)|$ is an integer which is smaller than $\chi_G(E)$. Hence the claim holds. □

Remark 3.1. Assume that H satisfies the assumption in Lemma 3.3(2) for v and G_1. Then H also satisfies this condition for $v\exp(mH)$ and G_1.

LEMMA 3.4. *Let $v = a\varrho_X + (dH + D) - (dH + D, \xi_0)\varrho_X/r_0$ be the Mukai vector in Proposition 3.2. Let H be a general polarization with respect to v and G_1. Assume that*

$$a > \max\{(2r_0 + 1)\epsilon, (\langle v^2\rangle - (D^2))/2 + \epsilon\}.$$

Then $\widehat{H}$ is a general polarization with respect to $\mathcal{F}_{\mathcal{E}}(v)$ (cf. Definition 1.4).

Proof. For a μ-semi-stable sheaf F with $v(F) = \mathcal{F}_{\mathcal{E}}(v)$, assume that there is an exact sequence

$$0 \to F_1 \to F \to F_2 \to 0$$

such that F_1 and F_2 are μ-semi-stable sheaves with $\mu_{G_2}(F_1) = \mu_{G_2}(F_2)$. By Corollary 2.14, F is G_2-twisted semi-stable and F is S-equivalent to $F_1 \oplus F_2$ with respect to G_2-twisted stability. By Proposition 3.2, $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F)$ is a G_1-twisted semi-stable sheaf which is S-equivalent to $\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1) \oplus \widehat{\mathcal{F}}^2_{\mathcal{E}}(F_2)$. Since H is a general polarization, we have $v(\widehat{\mathcal{F}}^2_{\mathcal{E}}(F_1)) \in \mathbb{Q}v$, which implies that $v(F_1) \in \mathbb{Q}\mathcal{F}_{\mathcal{E}}(v)$. Therefore, $\widehat{H}$ is a general polarization. □

The following corollary is a supplement to [Yos01a, Theorem 8.1] and [Yos03a].

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Corollary 3.5. *Let X be a $K3$ surface or an abelian surface. We set $v := \xi + a\varrho_X$, $(\xi^2) \geq -2\epsilon$, $(\xi, H) > 0$. Then $\overline{M}_H(v)$ is a normal variety, if H is general with respect to v. Moreover, if X is a $K3$ surface and v is primitive, then $\overline{M}_H(v)$ is an irreducible symplectic manifold which is deformation equivalent to $\mathrm{Hilb}_X^{\langle v^2\rangle/2+1}$. In particular, $M_H(v) \neq \emptyset$.*

Proof. If X is an abelian surface, we assume that $\mathcal{E}$ is the Poincaré line bundle on $X \times \widehat{X}$ and if X is a $K3$ surface, we assume that $\mathcal{E} = I_\Delta$, where $\Delta \subset X \times X$ is the diagonal. We shall show that $\overline{M}_H(\xi + a\varrho_X)$ is isomorphic to a moduli space of stable torsion-free sheaves. Then the claim follows from [Yos03a].

Replacing v by $v\,\mathrm{ch}(H^{\otimes m})$, we may assume that $a \gg d = (\xi, H)/(H^2)$. By Proposition 3.2, we have a desired isomorphism $\overline{M}_H(\xi + a\varrho_X) \to \overline{M}_{\widehat{H}}(a - \widehat{\xi})$. □

3.3 A special case

Let (X, H) be a polarized abelian surface with $\mathrm{NS}(X) = \mathbb{Z}H$. Let $\mathcal{P}$ be the Poincaré line bundle on $X \times \widehat{X}$. Then a Mukai vector $v \in v(\mathbf{D}(X))$ is written as

$$v = r + dH + a\varrho_X$$

and

$$\mathcal{F}_{\mathcal{P}}(v) = a - d\widehat{H} + r\varrho_{\widehat{X}}.$$

In this special case, we get more precise results. We set

$$n := (H^2)/2, \quad s := \langle v^2\rangle/2 = d^2 n - ra.$$

We first treat positive rank cases.

3.3.1 *Positive rank cases.*

Proposition 3.6. *We set $k := \gcd(r, d) > 0$. Assume that $r > 0$. We take a pair of integers (r', d') such that $rd' - r'd = -k$ and $0 \leq r' < r$. If*

$$dn > \max\left\{\frac{1}{k}\left(r' + \frac{(k-1)}{k}r\right)s, s\right\}, \tag{3.5}$$

then the following assertions hold.

(1) *For any μ-semi-stable sheaf F_1 with*

$$v(F_1) = a_1 \pm d_1\widehat{H} + r_1\varrho_{\widehat{X}}, \quad 0 < d_1 < d, d_1/a_1 \leq d/a,$$

we have $r_1 \leq rd_1/d$.

(2) *For any μ-semi-stable sheaf E_1 with*

$$v(E_1) = r_1 + d_1 H + a_1\varrho_X, \quad 0 < d_1 < d, d_1/r_1 < d/r,$$

we have $a_1 < ad_1/d$.

Proof. We shall prove claim (1). We may assume that F_1 is μ-stable. We set $v(F_1) := a_1 \pm d_1\widehat{H} + r_1\varrho_{\widehat{X}}$, $0 < d_1 < d$ and $d_1/a_1 \leq d/a$. If $r_1 \leq 0$, then obviously our claim holds. If $r_1 > 0$, then we see that

$$0 \leq \langle v(F_1)^2\rangle \leq \frac{2d_1}{rd}(nd(rd_1 - r_1 d) + r_1 s).$$

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If $r_1 \geq r$, then we see that

$$nd(rd_1 - r_1 d) + r_1 s \leq (nd(d_1 - d) + s)r_1 \leq (-nd + s)r_1 < 0, \tag{3.6}$$

which is a contradiction. Assume that $r_1 < r$. If $rd_1 - r_1 d < 0$, then there is a positive integer m such that $rd_1 - r_1 d = -km$. Then $r_1 - r'm$ is divisible by r/k and $r_1 - r'm < r$. Hence we get $r_1 - r'm \leq r - r/k$, which implies that

$$\begin{aligned} nd(rd_1 - r_1 d) + r_1 s &= -mknd + r_1 s \\ &\leq -mknd + rs - rs/k + r'ms < 0. \end{aligned}$$

This is a contradiction. Therefore, $rd_1 - r_1 d \geq 0$. □

Remark 3.2. Assume that $d = rt + 1, t > 0$ and $s = dn$. If $v(F_1) \notin \mathbb{Q}(t^2 n + t\widehat{H} + \varrho_{\widehat{X}})$ and $v(E_1) \notin \mathbb{Q}(1 + tH + t^2 n\varrho_X) = \mathbb{Q}e^{tH}$, then the same assertions (1) and (2) of Proposition 3.6 hold.

In the proof of Proposition 3.6(1), if $r_1 \geq r$, then we have

$$0 \leq nd(rd_1 - r_1 d) + r_1 s \leq (nd(d_1 - d) + s)r_1 \leq (-nd + s)r_1 = 0.$$

Hence $\langle v(F_1)^2 \rangle = 0$, $d_1 = d - 1 = rt$ and $r_1 = r$. Then $v(F_1) = r(t^2 n + t\widehat{H} + \varrho_{\widehat{X}})$. Assume that $r_1 < r$. Then we see that $0 \leq -mnd + r_1 s = (r_1 - m)nd \leq 0$, which implies that $\langle v(F_1)^2 \rangle = 0$, $m = r_1, d_1 = r_1 t$. Hence $v(F_1) = m(t^2 n + t\widehat{H} + \varrho_{\widehat{X}})$.

Theorem 3.7. *Under the condition (3.5), $\mathcal{G}_{\mathcal{P}}$ induces an isomorphism*

$$\mathcal{M}_H(r + dH + a\varrho_X)^{ss} \to \mathcal{M}_{\widehat{H}}(a + d\widehat{H} + r\varrho_{\widehat{X}})^{ss}.$$

In particular, for $E \in \mathcal{M}_H(r + dH + a\varrho_X)^{ss}$, WIT_2 with respect to $\mathcal{G}_{\mathcal{P}}$ holds for E and $\mathcal{G}^2_{\mathcal{P}}(E)$ is semi-stable.

Proof. We only prove one direction, that is, we show that $\mathcal{G}_{\mathcal{P}}$ preserves semi-stability. Let E be a semi-stable sheaf with $v(E) = r + dH + a\varrho_X$. We first note that the claim of Proposition 3.6 is slightly weaker than that in Proposition 2.8, since $d_1 < d$. Hence WIT_0 does not hold with respect to $\mathcal{F}_{\mathcal{P}}$ in general. However, by the same argument as in the proof of Theorem 3.1, we see that $\mathcal{F}^0_{\mathcal{P}}(E)$ is a μ-semi-stable sheaf and $\mathcal{F}^1_{\mathcal{P}}(E)$ is a zero-dimensional sheaf. Then $\mathcal{D}_{\widehat{X}}(\mathcal{F}_{\mathcal{P}}(E)) = \mathbf{R}\mathcal{H}om(\mathcal{F}_{\mathcal{P}}(E), \mathcal{O}_{\widehat{X}})$ is a μ-semi-stable sheaf. In particular, WIT_2 holds for E with respect to $\mathcal{G}_{\mathcal{P}}$. Assume that $\mathcal{G}^2_{\mathcal{P}}(E)$ is not semi-stable. Then we have an exact sequence

$$0 \to G_1 \to \mathcal{G}^2_{\mathcal{P}}(E) \to G_2 \to 0$$

where G_1 is a μ-semi-stable sheaf with $v(G_1) = a_1 + d_1\widehat{H} + r_1\varrho_{\widehat{X}}$, $d_1/a_1 = d/a$ and G_2 is a stable sheaf such that $v(G_2) = a_2 + d_2\widehat{H} + r_2\varrho_{\widehat{X}}$, $d_2/a_2 = d/a$ and $r_2/a_2 < r/a$. Then $\mathcal{D}_{\widehat{X}}(G_i)[1] \in \mathfrak{A}_2$, $i = 1, 2$, and we have an exact sequence in $\mathfrak{A}_2$:

$$0 \to \mathcal{D}_{\widehat{X}}(G_2)[1] \to \mathcal{F}_{\mathcal{P}}(E)[1] \to \mathcal{D}_{\widehat{X}}(G_1)[1] \to 0.$$

By using Theorem 2.1, we see that $\widehat{\mathcal{G}}^i_{\mathcal{P}}(G_2) = 0$, $i \neq 2$, $\widehat{\mathcal{G}}_{\mathcal{P}}(G_1)[2] \in \mathfrak{A}_1$ and we have an exact sequence

$$0 \to \widehat{\mathcal{G}}^1_{\mathcal{P}}(G_1) \to \widehat{\mathcal{G}}^2_{\mathcal{P}}(G_2) \to E \to \widehat{\mathcal{G}}^2_{\mathcal{P}}(G_1) \to 0.$$

From this description, we get a contradiction. Therefore, $\mathcal{G}^2_{\mathcal{P}}(E)$ is semi-stable. □

Remark 3.3. If $d = rt + 1$ and $t \geq 0$, then the condition is $dn > s$.

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Remark 3.4. If $r=1$ and $d\geq 2$, then IT_0 holds with respect to $\mathcal{F}_{\mathcal{P}}$ under the assumption $2(d-1)n>s$ (cf. [Ter98, Theorem 1.1]).

If $dn\leq s$, then $\mathcal{G}_{\mathcal{P}}$ does not always preserve the stability.

Lemma 3.8. *Assume that $r>0$, $d=tr+1$, $t\geq 0$ and $dn<s\leq (d^2-(d-1)^2/r)n$. Then there is a μ-stable sheaf E with $v(E)=r+dH+((d^2n-s)/r)\varrho_X$ such that E satisfies WIT_2 with respect to $\mathcal{G}_{\mathcal{P}}$, but $\mathcal{G}^2_{\mathcal{P}}(E)$ is not μ-semi-stable.*

Proof. We set $v:=r+dH+((d^2n-s)/r)\varrho_X$. Since $\gcd(r,d)=1$, it is sufficient to find a member $E\in\mathcal{M}_H(v)^{ss}$ such that E satisfies WIT_2 with respect to $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{G}^2_{\mathcal{P}}(E)$ is not stable. We first note that $t>0$. Indeed if $t=0$, then the second condition does not hold.

Claim 3.1. There is a μ-stable sheaf E with $v(E)=v$ such that $H^0(X,E(-tH))\neq 0$ and WIT_2 holds with respect to $\mathcal{G}_{\mathcal{P}}$.

We first assume this claim and show that $\mathcal{G}^2_{\mathcal{P}}(E)$ is not μ-semi-stable. We set

$$F:=\operatorname{coker}(\mathcal{O}_X\to E(-tH)).$$

Then we have an exact sequence

$$0\to\mathcal{G}^2_{\mathcal{P}}(F(tH))\to\mathcal{G}^2_{\mathcal{P}}(E)\to\mathcal{G}^2_{\mathcal{P}}(\mathcal{O}_X(tH))\to 0.$$

Since $v(\mathcal{G}^2_{\mathcal{P}}(\mathcal{O}_X(tH)))=nt^2+t\widehat{H}+\varrho_X$, we get that

$$\begin{aligned}\frac{\deg(\mathcal{G}^2_{\mathcal{P}}(\mathcal{O}_X(tH)))}{\mathrm{rk}(\mathcal{G}^2_{\mathcal{P}}(\mathcal{O}_X(tH)))}-\frac{\deg(\mathcal{G}^2_{\mathcal{P}}(E(tH)))}{\mathrm{rk}(\mathcal{G}^2_{\mathcal{P}}(E(tH)))}&=\frac{t(H^2)}{t^2n}-\frac{rd(H^2)}{d^2n-s}\\&=\frac{-2(s-dn)}{t(d^2n-s)}<0.\end{aligned}$$

Thus $\mathcal{G}^2_{\mathcal{P}}(E)$ is not μ-semi-stable. Therefore, we get our lemma.

Proof of Claim 3.1. We note that $s\geq n$. Let F be a stable sheaf with

$$v(F)=(r-1)+H-\{(s-n)/r\}\varrho_X.$$

Then since $\chi(F)\leq 0$,

$$\mathrm{Ext}^1(F\otimes\mathcal{P}_{|X\times\{y\}},\mathcal{O}_X)=H^1(X,F\otimes\mathcal{P}_{|X\times\{y\}})^{\vee}\neq 0$$

for some $y\in\widehat{X}$. Let E be a sheaf on X such that $E(-tH)$ is defined as a non-trivial extension

$$0\to\mathcal{O}_X\to E(-tH)\to F\otimes\mathcal{P}_{|X\times\{y\}}\to 0.$$

Then E is μ-stable (see [Yos99a, Lemma 2.1]). Moreover, since

$$\begin{aligned}\chi(F(tH))&=(d^2n-s)/r-nt^2\\&=((d^2-(d-1)^2/r)n-s)/r\geq 0,\end{aligned}$$

Theorem 3.14 in § 3.3.3 implies that WIT_2 holds for a general F with respect to $\mathcal{G}_{\mathcal{P}}$. Since WIT_2 holds for $\mathcal{O}_X(tH)$, we get our claim. □

Proposition 3.9. *Assume that $r>0$ and $d=tr+1$, $t\geq 0$. Then $\mathcal{G}_{\mathcal{P}}$ induces an isomorphism*

$$\mathcal{M}_H(r+dH+a\varrho_X)^{ss}\to\mathcal{M}_{\widehat{H}}(a+d\widehat{H}+r\varrho_{\widehat{X}})^{ss}\tag{3.7}$$

if and only if $dn\geq s$.

Proof. We first prove that $\mathcal{G}_{\mathcal{P}}$ induces an isomorphism (3.7) for $dn \geq s$. If $dn > s$, then Theorem 3.7 and Remark 3.3 imply the claim. If $dn = s$, then we have $v = r + dH + dtn\varrho_X$. Assume that $t > 0$. Let E be a semi-stable sheaf on X with $v(E) = v$. For a coherent sheaf F_1 on $\widehat{X}$ with $v(F_1) = t^2 n + t\widehat{H} + \varrho_{\widehat{X}}$, we have

$$\frac{\chi(\mathcal{G}_{\mathcal{P}}(E)(k\widehat{H}))}{\operatorname{rk} \mathcal{G}_{\mathcal{P}}(E)} < \frac{\chi(F_1(k\widehat{H}))}{\operatorname{rk} F_1}, \quad k \gg 0.$$

Then, by using Remark 3.2, we can show that WIT_2 holds for E and $\mathcal{G}^2_{\mathcal{P}}(E)$ is a semi-stable sheaf. Conversely we also see that WIT_2 holds for $F \in \mathcal{M}_{\widehat{H}}(a + d\widehat{H} + r\varrho_{\widehat{X}})^{ss}$ and $\widehat{\mathcal{G}}^2_{\mathcal{P}}(F)$ is a semi-stable sheaf. Thus the claim holds. If $t = 0$, then $d = 1$. In this case, [Yos01a, Remark 3.1] implies the claim.

We next assume that $dn < s$ and prove that there is a semi-stable sheaf $E \in \mathcal{M}_H(r + dH + a\varrho_X)^{ss}$ such that WIT_i does not hold with respect to $\mathcal{G}_{\mathcal{P}}$ or WIT_i holds but $\mathcal{G}^i_{\mathcal{P}}(E)$ is not semi-stable. We note that

$$(d^2 - (d-1)^2/r)n - dn = (t^2(r^2 - r) + tr)n \geq 0.$$

If $dn < s \leq (d^2 - (d-1)^2/r)n$, then Lemma 3.8 implies the claim. Assume that $(d^2 - (d-1)^2/r)$ $n - s < 0$. Then for the sheaf F in the proof of Claim 3.1 we have $\chi(F(tH)) = (d^2 - (d-1)^2/r)$ $n - s < 0$. By using Theorem 3.14, we have $\mathcal{G}^1_{\mathcal{P}}(E) \cong \mathcal{G}^1_{\mathcal{P}}(F(tH)) \neq 0$ and $\mathcal{G}^2_{\mathcal{P}}(\mathcal{O}_X(tH)) \neq 0$ is a quotient sheaf of $\mathcal{G}^2_{\mathcal{P}}(E)$. Thus WIT_i does not hold for E with respect to $\mathcal{G}_{\mathcal{P}}$. Therefore, the claim holds. □

3.3.2 *Rank zero case.* For the rank zero case, we have the following results, the proofs of which are left to the reader.

LEMMA 3.10. *Assume that $r = 0$ and $d > 0$. If $a > d(d-1)n$, then:*

(1) *for any μ-semi-stable sheaf F_1 with $v(F_1) = a_1 \pm d_1\widehat{H} + r_1\varrho_{\widehat{X}}$, $0 < d_1 < d$ and $d_1/a_1 \leq d/a$, we have $r_1 \leq 0$; and*

(2) *for any μ-semi-stable sheaf E_1 with $v(E_1) = r_1 + d_1 H + a_1\varrho_X$, $0 < d_1 < d$ and $r_1 > 0$, we have $a_1 < ad_1/d$.*

PROPOSITION 3.11. *Assume that $d > 0$. $\mathcal{G}_{\mathcal{P}}$ induces an isomorphism $\mathcal{M}_H(dH + a\varrho_X)^{ss} \to \mathcal{M}_{\widehat{H}}(a + d\widehat{H})^{ss}$, if $a > d(d-1)n$. Moreover, $\mathcal{F}_{\mathcal{P}}$ induces an isomorphism $\mathcal{M}_H(dH + a\varrho_X)^{ss} \to \mathcal{M}_{\widehat{H}}(a - d\widehat{H})^{ss}$, if $a > d^2 n$.*

3.3.3 *Birational maps.* By Proposition 3.9, $\mathcal{G}_{\mathcal{P}}$ does not preserve the stability in general. On the other hand, we can show that $\mathcal{G}_{\mathcal{P}}$ or $\mathcal{F}_{\mathcal{P}}$ always preserves the stability for a general member of the moduli spaces (Theorem 3.14).

PROPOSITION 3.12. *Assume that $r, d > 0$. If $\langle v^2 \rangle < 2r$, then WIT_2 holds with respect to $\mathcal{G}_{\mathcal{P}}$ for all μ-semi-stable sheaves E with $v(E) = v$.*

Proof. We shall prove our claim by induction on $\langle v^2 \rangle$. Obviously our claim holds for semi-homogeneous sheaves. Let E be a μ-semi-stable sheaf with $v(E) = v$. Assume that E is S-equivalent to $\bigoplus_{i=1}^m E_i$, where E_i, $1 \leq i \leq m$, are μ-stable sheaves. Then

$$\sum_i \frac{\langle v(E_i)^2 \rangle}{\operatorname{rk} E_i} = \frac{\langle v^2 \rangle}{r} < 2.$$

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Since $\langle v(E_j)^2\rangle \geq 0$ for all j, we get $\langle v(E_i)^2\rangle/\operatorname{rk} E_i \leq \langle v^2\rangle/r < 2$. Therefore, we shall prove our claim for μ-stable sheaves.

We first note that $a = nd^2/r - \langle v^2\rangle/2r > nd^2/r - 1 \geq -1$. If $a = 0$, then $2nd^2 < 2r$. Hence the claim follows from Proposition 3.11. We assume that $a > 0$. Assume that $\operatorname{Ext}^1(E, \mathcal{P}_{|X\times\{y\}}) \neq 0$, $y \in \widehat{X}$. We take a non-trivial extension

$$0 \to \mathcal{P}_{|X\times\{y\}} \to G \to E \to 0.$$

Assume that G is not μ-semi-stable. Let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_l = G$$

be the Harder–Narasimhan filtration of G with respect to the μ-semi-stability. We set $v(F_i/F_{i-1}) := r_i + d_iH + a_i\varrho_X$. Then $0 < d_l/r_l < \cdots < d_2/r_2 < d_1/r_1 < d/r$ and $r_i \leq r$. We see that

$$\begin{aligned}\sum_i \frac{\langle v(F_i/F_{i-1})^2\rangle}{r_i} &< \sum_i 2\left(n\frac{d}{r}d_i - a_i\right)\\ &= 2n\frac{d^2}{r} - 2a = \frac{\langle v^2\rangle}{r} < 2.\end{aligned}$$

Since $\langle v(F_j/F_{j-1})^2\rangle \geq 0$ for all j, we get

$$\frac{\langle v(F_i/F_{i-1})^2\rangle}{r_i} < \frac{\langle v^2\rangle}{r} < 2.$$

Since $r_i \leq r$, we get $\langle v(F_i/F_{i-1})^2\rangle < \langle v^2\rangle$. By the induction hypothesis, our claim holds for F_i/F_{i-1}. Hence G satisfies WIT_2 with respect to $\mathcal{G}_{\mathcal{P}}$. Since $\mathcal{P}_{|X\times\{y\}}$ also satisfies WIT_2 with respect to $\mathcal{G}_{\mathcal{P}}$, E satisfies WIT_2 with respect to $\mathcal{G}_{\mathcal{P}}$.

Assume that G is μ-semi-stable. Since $\langle v(G)^2\rangle = \langle v^2\rangle - 2a$, by the induction hypothesis, our claim holds for G. Therefore, E satisfies WIT_2 with respect to $\mathcal{G}_{\mathcal{P}}$. □

Lemma 3.13. *Assume that* $r, d > 0$.

(1) *If* $a \geq 0$, *then there is a stable sheaf* E *with* $v(E) = v$ *such that* $\operatorname{Ext}^1(E, \mathcal{O}_X) = H^1(X, E)^\vee = 0$. *In particular,* WIT_2 *holds for* E *with respect to* $\mathcal{G}_{\mathcal{P}}$.

(2) *If* $a \leq 0$, *then there is a stable sheaf* E *with* $v(E) = v$ *such that* $H^0(X, E) = 0$. *In particular,* WIT_1 *holds for* E *with respect to* $\mathcal{F}_{\mathcal{P}}$.

Remark 3.5. Let E be a stable sheaf with $v(E) = v$. Since $\mathcal{F}^0_{\mathcal{P}}(E)$ is torsion free, $H^0(X, E) = 0$ implies that $\mathcal{F}^0_{\mathcal{P}}(E) = 0$. Since $\operatorname{Hom}(E \otimes \mathcal{P}_{|X\times\{y\}}, \mathcal{O}_X) = 0$ for all $y \in \widehat{X}$, $\mathcal{G}^1_{\mathcal{P}}(E)$ is also torsion free. Thus $H^1(X, E) = 0$ implies $\mathcal{G}^1_{\mathcal{P}}(E) = 0$.

Proof of Lemma 3.13. If $\langle v^2\rangle = 0$, then obviously the claim holds. Hence we assume that $\langle v^2\rangle > 0$. We take an integer b such that

$$\begin{aligned}0 &\leq \langle (r + dH + (a+b)\varrho_X)^2\rangle\\ &= \langle v^2\rangle - 2rb < 2r.\end{aligned}$$

We note that $b \geq 0$. Let F be a stable sheaf with $v(F) = r + dH + (a+b)\varrho_X$ such that $H^1(X, F) = 0$. We consider a surjective homomorphism $\phi : F \to \bigoplus_{i=1}^b \mathbb{C}_{x_i}$, where

$x_1, x_2, \ldots, x_b \in X$. If we choose a sufficiently general ϕ, then

$$\begin{cases} H^1(X, \ker \phi) = 0, & \text{if } a \geq 0, \\ H^0(X, \ker \phi) = 0, & \text{if } a \leq 0. \end{cases}$$

Let $\mathcal{M}_H(v)^{\mu ss}$ be the moduli stack of μ-semi-stable sheaves E with $v(E) = v$. Then the usual deformation theory says that $\dim \mathcal{M}_H(v)^{\mu ss} \geq \langle v^2 \rangle + 1$. On the other hand, [Yos99b, Lemma 2.3(2)] implies that $\dim(\mathcal{M}_H(v)^{\mu ss} \setminus \dim \mathcal{M}_H(v)^s) \leq \langle v^2 \rangle$. Hence $\ker \phi \in \mathcal{M}_H(v)^{\mu ss}$ deforms to a stable sheaf. Therefore, we get our claim. □

By [Yos01a, Corollary 4.15], we get the following theorem which was conjectured in [Yos01a, Conjecture 4.16].

THEOREM 3.14. *Assume that $r, d > 0$.*

(1) *If $a \geq 0$, then $\mathcal{G}_{\mathcal{P}}$ induces a birational map*

$$\overline{M}_H(r + dH + a\varrho_X) \cdots \to \overline{M}_{\widehat{H}}(a + d\widehat{H} + r\varrho_{\widehat{X}}).$$

(2) *If $a \leq 0$, then $\mathcal{F}_{\mathcal{P}}$ induces a birational map*

$$\overline{M}_H(r + dH + a\varrho_X) \cdots \to \overline{M}_{\widehat{H}}(-a + d\widehat{H} - r\varrho_{\widehat{X}}).$$

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Appendix A.

Assume that H is a primitive ample divisor on X. Then we can write the $\mathbb{Q}$-ample divisor $\widehat{H}$ on Y as $\widehat{H} = \lambda \widetilde{H}$, where $\widetilde{H}$ is a primitive element of $H^2(Y, \mathbb{Z})$ and $\lambda \in \mathbb{Q}_{>0}$. We shall study this λ. Since we also want to treat the case of the twisted Fourier–Mukai transform, ξ_0 may not belong to $\mathrm{NS}(X)$ and the universal family $\mathcal{E}$ is a suitable twisted sheaf in general. We consider primitive sublattices

$$L_1 := (\mathbb{Q}v_0^{\vee} + \mathbb{Q}H + \mathbb{Q}\varrho_X) \cap H^{ev}(X, \mathbb{Z}),$$
$$L_2 := (\mathbb{Q}w_0 + \mathbb{Q}\widehat{H} + \mathbb{Q}\varrho_Y) \cap H^{ev}(Y, \mathbb{Z}).$$

We set

$$\begin{aligned} l_1 &:= \min\{\langle \varrho_X, x \rangle > 0 \mid x \in L_1\}, \\ l_2 &:= \min\{\langle \varrho_Y, y \rangle > 0 \mid y \in L_2\} \\ &= \min\{\langle v_0^{\vee}, x \rangle > 0 \mid x \in L_1\}. \end{aligned}$$

Then $L_1 = \mathbb{Z}u_1 + \mathbb{Z}H + \mathbb{Z}\varrho_X$, where $u_1 \in L_1$ satisfies $\langle u_1, \varrho_X \rangle = l_1$. Hence the determinant of the intersection matrix of L_1 is $l_1^2(H^2)$.

LEMMA APPENDIX A.1. *Let $\widetilde{H}$ be a primitive ample divisor on Y such that $r_0\widehat{H} \in \mathbb{Z}\widetilde{H}$. Then $l_1^2(H^2)$
$= l_2^2(\widetilde{H}^2)$. In particular, $\widehat{H} = l_2/l_1 \widetilde{H}$.*

Proof. $\mathcal{F}_{\mathcal{E}}$ induces an isometry $L_1 \to L_2$. Since the determinant of the intersection matrix of L_1 (respectively L_2) is $l_1^2(H^2)$ (respectively $l_2^2(\widetilde{H}^2)$), we have the assertion. □

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Example Appendix A.1. Assume that $v_0 = r_0 + d_0 H + a_0 \varrho_X$ with $d_0^2(H^2) = 2r_0 a_0$. We set $l := \gcd(r_0, a_0)$. Then $L_1 = \mathbb{Z}1 + \mathbb{Z}H + \mathbb{Z}\varrho_X$ and $l_1 = 1$. It is easy to see that $l | d_0(H^2)$, and hence, $l_2 = \gcd(r_0, d_0(H^2), a_0) = l$. Therefore, $\widehat{H} = l\widetilde{H}$.

Example Appendix A.2. Assume that $(r_0, \xi_0 \mod H)$ is primitive in $\mathbb{Z} \times H^2(X, \mathbb{Z})/\mathbb{Z}H$. Then $L_1 = \mathbb{Z}v_0^\vee + \mathbb{Z}H + \mathbb{Z}\varrho_X$. In this case, $l_1 = r_0$ and $l_2 = \gcd(r_0, (\xi_0, H))$. Therefore, $\widetilde{H} = (r_0/l_2)\widehat{H}$. In particular, $\widehat{H} \notin \mathrm{NS}(Y)$ if $l_2 \neq r_0$.

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