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# TOWARD HILBERT–KUNZ DENSITY FUNCTIONS IN CHARACTERISTIC 0

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**Abstract.** For a pair $(R, I)$, where $R$ is a standard graded domain of dimension $d$ over an algebraically closed field of characteristic 0, and $I$ is a graded ideal of finite colength, we prove that the existence of $\lim_{p\to\infty} e_{HK}(R_p, I_p)$ is equivalent, for any fixed $m \geqslant d-1$, to the existence of $\lim_{p\to\infty} \ell(R_p/I_p^{[p^m]})/p^{md}$. This we get as a consequence of Theorem 1.1: as $p \longrightarrow \infty$, the convergence of the Hilbert–Kunz (HK) density function $f(R_p, I_p)$ is equivalent to the convergence of the truncated HK density functions $f_m(R_p, I_p)$ (in $L^\infty$ norm) of the *mod p reductions* $(R_p, I_p)$, for any fixed $m \geqslant d-1$. In particular, to define the HK density function $f^\infty_{R,I}$ in char 0, it is enough to prove the existence of $\lim_{p\to\infty} f_m(R_p, I_p)$, for any fixed $m \geqslant d-1$. This allows us to prove the existence of $e^\infty_{HK}(R, I)$ in many new cases, for example, when Proj R is a Segre product of curves.

## CONTENTS



## §1. Introduction

Let $R$ be a Noetherian ring of prime characteristic $p > 0$ and of dimension $d$, and let $I \subseteq R$ be an ideal of finite colength. Then, we recall that the *Hilbert–Kunz multiplicity* of $R$ with respect to $I$ is defined as

$$e_{HK}(R, I) = \lim_{n\to\infty} \frac{\ell(R/I^{[p^n]})}{p^{nd}},$$

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where $I^{[p^n]}$ = the $n$th Frobenius power of $I$ = the ideal generated by $p^n$th power of elements of $I$. This is an ideal of finite colength, and $\ell(R/I^{[p^n]})$ denotes the length of the $R$-module $R/I^{[p^n]}$. This invariant was introduced by E. Kunz and existence of the limit was proved by Monsky (see Theorem 1.8 in [Mo1]). It carries information about char $p$ related properties of the ring, but at the same time is difficult to compute (even in the graded case), as various standard techniques, used for studying multiplicities, are not applicable for the invariant $e_{HK}$.

It is natural to ask whether the notion of this invariant can be extended to the "char 0" case by studying the behavior of *mod p reductions*.

A natural way to attempt this, for *a pair* $(R, I)$ (from now onwards, unless stated otherwise, by a pair $(R, I)$, we mean that $R$ is a standard graded ring and $I \subset R$ is a graded ideal of finite colength), could be as follows. Suppose that $R$ is a finitely generated algebra and a domain over a field $k$ of characteristic 0, and $I \subseteq R$ is an ideal of finite colength. Let $(A, R_A, I_A)$ be a *spread* of the pair $(R, I)$ (see Definition 3.2), where $A \subset k$ is a finitely generated algebra over $\mathbb{Z}$. Then, we may define

$$e_{HK}^{\infty}(R, I) := \lim_{p_s \to \infty} e_{HK}(R_s, I_s),$$

where $R_s = R_A \otimes_A \bar{k}(s)$ and $I_s = I_A \otimes_A \bar{k}(s)$, with $\bar{k}(s)$ as the algebraic closure of $k(s)$ with char $k(s) = p_s$, and $s$ is a closed point of Spec$(A)$ (the definition is tentative, since the existence of this limit is not known in general). Or consider a simpler situation: $R$ is a finitely generated $\mathbb{Z}$-algebra and a domain, $I \subset R$, such that $R/I$ is an abelian group of finite rank. Then, let

$$e_{HK}^{\infty}(R, I) := \lim_{p \to \infty} e_{HK}(R_p, I_p), \quad \text{where } R_p = R \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{p\mathbb{Z}} \text{ and } I_p = I \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{p\mathbb{Z}}.$$

In case of dimension $R = 1$, we know that the Hilbert–Kunz multiplicity coincides with the Hilbert–Samuel multiplicity; hence, it is independent of $p$, for large $p$.

For homogeneous coordinate rings of plane curves with respect to the maximal graded ideal (in [T1], [Mo3]), nonsingular curves with respect to a graded ideal $I$ (in [T2]), and diagonal hypersurfaces (in [GM] and [HM]), it has been shown that $e_{HK}(R_p, I_p)$ varies with $p$, and the limit exists as $p \to \infty$. Then, there are other cases where $e_{HK}(R_p, I_p)$ is independent of $p$:

plane cubics (by [BC], [Mo2] and [P]), certain monomial ideals (by [Br], [C], [E], [W]), two-dimensional invariant rings for finite group actions (by [WY2]), and full flag varieties and elliptic curves (by [FT]). Therefore, the limit exists in all of these cases.

Since

$$e_{HK}^{\infty}(R,I) := \lim_{p\to\infty}\lim_{n\to\infty}\frac{\ell(R_p/I_p^{[p^n]})}{(p^n)^d},$$

it seems harder to compute as such, as the inner limit $\lim_{n\to\infty}\ell(R_p/I_p^{[p^n]})/(p^n)^d$ itself does not seem easily computable (even in the graded case). In the special situation considered by Gessel and Monsky (see [GM]), the existence of $e_{HK}^{\infty}$ is proved by reducing the problem to the existence of $\lim_{p\to\infty}(\ell(R_p/I_p^{[p]})/p^d)$. To make this invariant more approachable in a general graded case, the following question was posed in [BLM] (see the introduction).

QUESTION. Supposing that $e_{HK}^{\infty}(R,I)$ exists, is it true that for any fixed $n\geqslant 1$,

$$e_{HK}^{\infty}(R,I) = \lim_{p\to\infty}\frac{\ell(R_p/I_p^{[p^n]})}{(p^n)^d}?$$

The main result of their paper was to give an affirmative answer in the case of a two-dimensional standard graded normal domain $R$ with respect to a homogeneous ideal $I$ of finite colength. Note that the existence of $e_{HK}^{\infty}(R,I)$, in this case, was proved earlier in [T2].

Recall that for a vector bundle $V$ on a smooth (projective and polarized) variety, we have the well defined *Harder–Narasimhan (HN) data*, namely $\{r_i(V),\mu_i(V)\}_i$, where $r_i(V)=\mathrm{rank}(E_i/E_{i-1})$, $\mu_i(V)=$ slope of $E_i/E_{i-1}$, and

$$0\subset E_1\subset E_2\subset\cdots\subset E_l\subset V$$

is the HN filtration of $V$.

Let $X_p=\mathrm{Proj}\ R_p$, which is a nonsingular projective curve, and let $I_p$ be generated by homogeneous elements of degrees $d_1,\ldots,d_\mu$; then, we have the vector bundle $V_p$ on $X_p$ given by the following canonical exact sequence of $\mathcal{O}_{X_p}$-modules:

$$0\longrightarrow V_p\longrightarrow \oplus_i\mathcal{O}_{X_p}(1-d_i)\longrightarrow \mathcal{O}_{X_p}(1)\longrightarrow 0.$$

Then, by [T2, Proposition 1.16], there is a constant $C$ determined by genus of $X_p$ and rank $V_p$ (hence independent of $p$), such that for $s \geqslant 1$,

$$\left| \sum_j r_j(F^{s*}V_p)\mu_j(F^{s*}V_p)^2 - \sum_i r_i(V_p)\mu_i(V_p)^2 \right| \leqslant C/p. \tag{1}$$

(Here, $F$ is the absolute Frobenius morphism, and $F^s$ is the $s$-fold iterate.) Note that the HN filtration and hence the HN data of $V_p$ stabilize for $p \gg 0$ (see [Mar]).

Thus, here,

(1) one relates $\ell(R_p/I_p^{[p^s]})$ with the HN data of $F^{s*}V_p$, for $s \geqslant 1$ (see [B] and [T1]);
(2) the HN data of $F^{s*}V_p$ are related to the HN data of $V_p$ (see [T2]);
(3) the restriction of the relative HN filtration of $V_A$ on $X_A$ (where $V_A$ is a spread of $V_0$ in char 0) remains the HN filtration of $V_p$ for large $p$ (see [Mar]).

In particular, for a pair $(R, I)$, where char $R = p > 0$, with the associated syzygy bundle $V$ (as above), the proof uses the comparison of $\ell(R/I^{[p^s]})$ with the HN data of the syzygy bundle $V$ and the other well behaved invariants of $(R, I)$ (which have well defined notion in all characteristics and are well behaved vis-a-vis reduction mod $p$).

However, note that (3) is valid for $\dim R \geqslant 2$, and (2) also holds for $\dim R \geqslant 3$ (proved relatively recently in [T3]). However, (1) does not seem to hold in higher dimension, due to the existence of cohomologies other than $H^0(-)$ and $H^1(-)$ (therefore, one cannot use anymore the semistability property of a vector bundle to compute $h^0$ of almost all of its twists, by powers of a very ample line bundle).

In this paper, we approach the problem by a completely different method (see Corollary 2.12), comparing directly $(1/(p^n)^d)\ell(R/I^{[p^n]})$ and $(1/(p^{n+1})^d)\ell(R/I^{[p^{n+1}]})$, for $n \geqslant 1$, taking into account that both are graded.

For this, we phrase the problem in a more general setting: by the theory of the Hilbert–Kunz (HK) *density function* (which was introduced and developed in [T4]), for a pair $(R_p, I_p)$, where $R_p$ is a domain of char $p > 0$, there exists a sequence of functions $\{f_n(R_p, I_p) : [0, \infty) \longrightarrow \mathbb{R}\}_n$ such that

$$\frac{1}{(p^n)^d}\ell\left(\frac{R_p}{I_p^{[p^n]}}\right) = \int_0^\infty f_n(R_p, I_p)(x)\, dx$$

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and

$$\lim_{n\to\infty}\frac{1}{(p^n)^d}\ell\left(\frac{R_p}{I_p^{[p^n]}}\right)=\int_0^\infty f_{R_p,I_p}(x)\,dx,$$

where the map

$$f_{R_p,I_p}:[0,\infty)\to\mathbb{R}\quad\text{is given by } f_{R_p,I_p}(x)=\lim_{n\to\infty}f_n(R_p,I_p)(x)$$

is called the HK density function of $(R_p,I_p)$ (the existence and properties of the limit defining $f_{R_p,I_p}$ are proved in [T4]). We show here that, for each $x\in[1,\infty)$,

$$\begin{aligned}f^\infty_{R,I}(x):=\lim_{p\to\infty}\lim_{n\to\infty}f_n(R_p,I_p)(x)\quad&\text{exists}\\ \iff\quad\lim_{p\to\infty}f_m(R_p,I_p)(x)\quad&\text{exists},\end{aligned}$$

for any fixed $m\geqslant d-1$, where $d-1=\dim\operatorname{Proj}\,R$. Moreover, if it exists, then

$$f^\infty_{R,I}(x)=\lim_{p\to\infty}f_m(R_p,I_p)(x),\quad\text{for any } m\geqslant d-1.$$

The main point (Proposition 2.11) is to give a bound on the difference $\|f_n(R_p,I_p)-f_{n+1}(R_p,I_p)\|$, in terms of a power of $p$ and invariants that are well behaved under reduction mod $p$, where $\|g\|:=\sup\{g(x)\mid x\in[1,\infty)\}$ is the $L^\infty$ norm. Since the union of the support of all $f_n$ is contained in a compact interval, a similar bound (Corollary 2.12) holds for the difference $|\ell(R/I^{[p^n]})/(p^n)^d-\ell(R/I^{[p^{n+1}]})/(p^{n+1})^d|$. More precisely, we prove the following theorem.

THEOREM 1.1. *Let $R$ be a standard graded domain of dimension $d\geqslant 2$, over an algebraically closed field $k$ of characteristic $0$. Let $I\subset R$ be a homogeneous ideal of finite colength. Let $(A,R_A,I_A)$ be a spread (see Definition 3.2 and Notations 3.3). Then, for a closed point $s\in\operatorname{Spec}(A)$, let the function $f_n(R_s,I_s)(x):[1,\infty)\longrightarrow[0,\infty)$ be given by*

$$f_n(R_s,I_s)(x)=\frac{1}{q^{d-1}}\ell\left(\frac{R_s}{I_s^{[q]}}\right)_{\lfloor xq\rfloor},$$

*where $q=p_s^n$, for $p_s=\operatorname{char}k(s)$, and $\lfloor y\rfloor$ denotes the largest integer $\leqslant y$ and $\ell(R_s/I_s^{[q]})_m$ denotes the length of the $m$th graded piece of the ring $R_s/I_s^{[q]}$.*

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*Let the HK density function of* $(R_s, I_s)$ *be given by*

$$f_{R_s,I_s}(x) = \lim_{n\to\infty} f_n(R_s, I_s)(x).$$

*Let* $s_0 \in \mathrm{Spec}\ Q(A)$ *denote the generic point of* $\mathrm{Spec}(A)$*. Then, we have the following.*

(1) *There exist a constant* $C$ *(given in terms of invariants of* $(R_{s_0}, I_{s_0})$ *of the generic fiber) and an open dense subset* $\mathrm{Spec}(A')$ *of* $\mathrm{Spec}(A)$ *such that for every closed point* $s \in \mathrm{Spec}(A')$ *and* $n \geqslant 1$*,*

$$\|f_n(R_s, I_s) - f_{n+1}(R_s, I_s)\| < C/p_s^{n-d+2},$$

*where* $p_s = \mathrm{char}\ k(s)$*. In particular, for any* $m \geqslant d-1$*,*

$$\lim_{p_s\to\infty} \|f_m(R_s, I_s) - f_{R_s,I_s}\| = 0.$$

(2) *There exist a constant* $C_1$ *(given in terms of invariants of* $(R_{s_0}, I_{s_0})$*) and an open dense subset* $\mathrm{Spec}(A')$ *of* $\mathrm{Spec}(A)$*, such that for every closed point* $s \in \mathrm{Spec}(A')$ *and* $n \geqslant 1$*, we have*

$$\left|\frac{1}{p_s^{nd}}\ell\left(\frac{R_s}{I_s^{[p_s^n]}}\right) - \frac{1}{p_s^{(n+1)d}}\ell\left(\frac{R_s}{I_s^{[p_s^{n+1}]}}\right)\right| \leqslant \frac{C_1}{p_s^{n-d+2}}.$$

(3) *For any* $m \geqslant d-1$*,*

$$\lim_{p_s\to\infty}\left[\frac{1}{p_s^{md}}\ell\left(\frac{R_s}{I_s^{[p_s^m]}}\right) - e_{HK}(R_s, I_s)\right] = 0.$$

As a result, we have the following corollary.

COROLLARY 1.2. *Let* $R$ *be a standard graded domain and a finitely generated* $\mathbb{Z}$*-algebra of characteristic* 0*, and let* $I \subset R$ *be a homogeneous ideal of finite colength, such that for almost all* $p$*, the fiber over* $p$*,* $R_p := R \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$*, is a standard graded ring of dimension* $d$*, which is geometrically integral, and* $I_p \subset R_p$ *is a homogeneous ideal of finite colength. Then, we have the following.*

(1) *There exists a constant* $C_1$ *given in terms of invariants of* $R$ *and* $I$ *such that, for* $n \geqslant 1$*, we have*

$$\left|\frac{1}{p^{nd}}\ell\left(\frac{R_p}{I_p^{[p^n]}}\right) - \frac{1}{p^{(n+1)d}}\ell\left(\frac{R_p}{I_p^{[p^{n+1}]}}\right)\right| \leqslant \frac{C_1}{p^{n-d+2}}.$$

(2) *For any fixed $m \geqslant d-1$,*

$$\lim_{p\to\infty}\left[e_{HK}(R_p, I_p) - \frac{1}{p^{md}}\ell\left(\frac{R_p}{I_p^{[p^m]}}\right)\right] = 0.$$

*In particular, for any fixed $m \geqslant d-1$,*

$$e_{HK}^{\infty}(R, I) := \lim_{p\to\infty} e_{HK}(R_p, I_p) \quad \textit{exists}$$
$$\iff \quad \lim_{p\to\infty} \frac{1}{p^{md}}\ell\left(\frac{R_p}{I_p^{[p^m]}}\right) \quad \textit{exists.}$$

In particular, the last assertion of the above corollary answers the abovementioned question of [BLM] affirmatively, for all $(R, I)$, where $R$ is a standard graded domain and $I \subset R$ is a graded ideal of finite colength.

Moreover, the proof, even in the case of dimension 2 (unlike the proof in [BLM]), does not rely on earlier results of [B], [T1], and [T2]. In particular, since we do not use HN filtrations, we do not need a normality hypothesis on the ring $R$.

REMARK 1.3. If $e_{HK}^{\infty}(R, I)$ exists for a pair $(R, I)$, whenever $R$ is a standard graded domain, defined over an algebraically closed field of characteristic 0, then one can check that $e_{HK}^{\infty}(R, I)$ exists for any pair $(R, I)$, where $R$ is a standard graded ring over a field $k$ of characteristic 0. Let $\bar{R} = R \otimes_k \bar{k}$. Let $\{q_1, \ldots, q_r\} = \{q \in \mathrm{Ass}(\bar{R}) \mid \dim \bar{R}/q = \dim R\}$; then, we have a spread $(A, \bar{R}_A, \bar{I}_A)$ of $(\bar{R}, \bar{I})$ such that $\{q_{1s}, \ldots, q_{rs}\} = \{q_s \in \mathrm{Ass}(\bar{R}_s) \mid \dim \bar{R}_s/q_s = \dim \bar{R}_s\}$ (here, $q_{is} = q_i \otimes_k \bar{k(s)} \subset \bar{R}$). Moreover, for each $i$, $\ell((\bar{R}_s)q_{is}) = l_i$, a constant independent of $s$. This implies that

$$e_{HK}(\bar{R}_s, \bar{I}_s) = \sum_{i=1}^{r} l_i e_{HK}\left(\frac{\bar{R}_s}{q_{is}}, \frac{\bar{I}_s + q_{is}}{q_{is}}\right),$$

which implies

$$\lim_{p_s\to\infty} e_{HK}(\bar{R}_s, \bar{I}_s) = \sum_{i=1}^{r} l_i \lim_{p_s\to\infty} e_{HK}\left(\frac{\bar{R}_s}{q_{is}}, \frac{\bar{I}_s + q_{is}}{q_{is}}\right)$$
$$= \sum_{i=1}^{r} l_i e_{HK}^{\infty}\left(\frac{\bar{R}}{q_i}, \frac{\bar{I} + q_i}{q_i}\right).$$

Hence, in this situation, one can define

$$e_{HK}^{\infty}(R, I) := e_{HK}^{\infty}(\bar{R}, \bar{I}) = \sum_{i=1}^{r} l_i e_{HK}^{\infty}\left(\frac{\bar{R}}{q_i}, \frac{\bar{I}+q_i}{q_i}\right).$$

In Section 4, we study some properties of $f_{R,I}^{\infty}$ (when it exists), and prove that $f_{R,I}^{\infty}$ behaves well under Segre product (Propositions 4.3 and 4.4). In the case of nonsingular projective curves (Theorem 4.6), the function $f_{R_s,I_s} - f_{R,I}^{\infty}$ is nonnegative, continuous, and characterizes the behavior of the HN filtration of the syzygy bundle of the curve, reduction mod char $k(s)$. Hence, $f_{(S_1\#\cdots\#S_n)_p} - f_{S_1\#\cdots\#S_n}^{\infty} = 0$ if and only if the HN filtrations of the syzygy bundles of Proj $S_i$ are the strong HN filtrations reduction mod $p$, for all $i$.

We give an example to show that, for an arbitrary Segre product of plane trinomial curves, the function $f_{(S_1\#\cdots\#S_n)_p} = f_{S_1\#\cdots\#S_n}^{\infty}$, for a Zariski dense set of primes. Moreover, the function $f_{(S_1\#\cdots\#S_n)_p} \neq f_{S_1\#\cdots\#S_n}^{\infty}$, for a Zariski dense set of primes, if one of the trinomials is symmetric. It is easy to check that if $f_{R,I}^{\infty}$ exists (in $L^{\infty}$ norm), then $e_{HK}^{\infty}(R, I)$ exists. One can ask the converse, that is, the following question.

QUESTION. Does the existence of $e_{HK}^{\infty}(R, I)$ imply the existence of $f_{R,I}^{\infty}$?

By Proposition 4.3, an affirmative answer to this question will imply the existence of the $e_{HK}^{\infty}$ for Segre products of the rings for which $e_{HK}^{\infty}$ exist.

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## §2. A key proposition

Throughout this section, $R$ is a Noetherian standard graded integral domain of dimension $d \geqslant 2$ over an algebraically closed field $k$ of char $p > 0$, $I$ is a homogeneous ideal of $R$ such that $\ell(R/I) < \infty$. Let $h_1, \ldots, h_\mu$ be a set of homogeneous generators of $I$ of degrees $d_1, \ldots, d_\mu$, respectively. Moreover, $\mathbf{m}$ denotes the graded maximal ideal of $R$.

Let $X = \text{Proj } R$; then, we have an associated canonical short exact sequence of locally free sheaves of $\mathcal{O}_X$-modules (moreover, the sequence is locally split exact),

$$0 \longrightarrow V \longrightarrow \oplus_i \mathcal{O}_X(1-d_i) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0, \tag{2}$$

where $\mathcal{O}_X(1-d_i) \longrightarrow \mathcal{O}_X(1)$ is given by the multiplication by the element $h_i$.

For a coherent sheaf $\mathcal{Q}$ of $\mathcal{O}_X$-modules, the sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow F^{n*}V \otimes \mathcal{Q}(m) \longrightarrow \oplus_i \mathcal{Q}(q - qd_i + m) \longrightarrow \mathcal{Q}(q+m) \longrightarrow 0$$

is exact as the short exact sequence (2) is locally split as $\mathcal{O}_X$-modules (as usual, $q = p^n$, and $F^n$ is the $n$th iterate of the absolute Frobenius morphism). Therefore, we have a long exact sequence of cohomologies

$$\begin{aligned} &0 \longrightarrow H^0(X, F^{n*}V \otimes \mathcal{Q}(m)) \\ &\longrightarrow \oplus_i H^0(X, \mathcal{Q}(q - qd_i + m)) \stackrel{\phi_{m,q}(\mathcal{Q})}{\to} H^0(X, \mathcal{Q}(q+m)) \\ &\longrightarrow H^1(X, F^{n*}V \otimes \mathcal{Q}(m)) \longrightarrow \cdots, \end{aligned} \tag{3}$$

for $m \geqslant 0$ and $q = p^n$.

We recall the definition of (Castelnuovo–Mumford) regularity.

DEFINITION 2.1. Let $\mathcal{Q}$ be a coherent sheaf of $\mathcal{O}_X$-modules, and let $\mathcal{O}_X(1)$ be a very ample line bundle on $X$. For $\tilde{m} \in \mathbb{N}$, we say that $\mathcal{Q}$ is $\tilde{m}$-regular (or $\tilde{m}$ is a regularity number of $\mathcal{Q}$) with respect to $\mathcal{O}_X(1)$ if, for all $m \geqslant \tilde{m}$,

(1) the canonical multiplication map $H^0(X, \mathcal{Q}(m)) \otimes H^0(X, \mathcal{O}_X(1)) \longrightarrow H^0(X, \mathcal{Q}(m+1))$ is surjective, and
(2) $H^i(X, \mathcal{Q}(m-i)) = 0$, for $i \geqslant 1$.

NOTATIONS 2.2.

(1) Let

$$P_{(R,\mathbf{m})}(m) = \tilde{e}_0 \binom{m+d-1}{d} - \tilde{e}_1 \binom{m+d-2}{d-1} + \cdots + (-1)^d \tilde{e}_d$$

be the Hilbert–Samuel polynomial of $R$ with respect to the graded maximal ideal $\mathbf{m}$. Therefore,

$$\begin{aligned} \chi(X, \mathcal{O}_X(m)) = \tilde{e}_0 \binom{m+d-1}{d-1} - \tilde{e}_1 \binom{m+d-2}{d-2} \\ + \cdots + (-1)^{d-1} \tilde{e}_{d-1}. \end{aligned}$$

(2) Let $\bar{m}$ be a positive integer such that

(a) $\bar{m}$ is a regularity number for $(X, \mathcal{O}_X(1))$, and

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(b) $R_m = h^0(X, \mathcal{O}_X(m))$, for all $m \geqslant \bar{m}$. In particular, $\ell(R/\mathbf{m}^m) = P_{(R,\mathbf{m})}(m)$, for all $m \geqslant \bar{m}$.

(3) Let $l_1 = h^0(X, \mathcal{O}_X(1))$.
(4) Let $n_0 \geqslant 1$ be an integer such that $R_{n_0} \subseteq I$, where $R = \oplus_{n\geqslant 0} R_n$.
(5) We denote $\dim_k$ Coker $\phi_{m,q}(\mathcal{Q})$ by coker $\phi_{m,q}(\mathcal{Q})$.

REMARK 2.3.

(1) The canonical map $\oplus_m R_m \longrightarrow \oplus_m H^0(X, \mathcal{O}_X(m))$ is injective, as $R$ is an integral domain.
(2) For $m + q \geqslant m_R(q) = \bar{m} + n_0(\sum_i d_i)q$, we have coker $\phi_{m,q}(\mathcal{O}_X) = \ell(R/I^{[q]})_{m+q} = 0$, because $m_R(q) = \bar{m} + n_0\mu q + n_0(\sum_i (d_i - 1))q \Rightarrow q - qd_i + m \geqslant \bar{m}$, for all $i$. Hence, the map $\phi_{m,q}(\mathcal{O}_X)$ is the map $\oplus_i R_{q-qd_i+m} \longrightarrow R_{m+q}$, where the map $R_{q-qd_i+m} \to R_{m+q}$ is given by multiplication by the element $h_i^q$. Therefore, coker $\phi_{m,q}(\mathcal{O}_X) = \ell(R/I^{[q]})_{m+q}$. Moreover, by the proof of Lemma 2.10, we have $\ell(R/I^{[q]})_{m+q} = 0$, as $m + q \geqslant \bar{m} + n_0\mu q$.
(3) For $C_R = (\mu)h^0(X, \mathcal{O}_X(\bar{m}))$, we have

$$|\text{coker } \phi_{m,q}(\mathcal{O}_X) - \ell(R/I^{[q]})_{m+q}| \leqslant C_R, \tag{4}$$

for all $n, m \geqslant 0$ and $q = p^n$, because
if $m + q < \bar{m}$, then

$$\begin{aligned}|\text{coker } \phi_{m,q}(\mathcal{O}_X) - \ell(R/I^{[q]})_{m+q}| &\leqslant h^0(X, \mathcal{O}_X(m+q)) \\ &\leqslant h^0(X, \mathcal{O}_X(\bar{m})).\end{aligned}$$

If $m + q \geqslant \bar{m}$, then $h^0(X, \mathcal{O}_X(m+q)) = \ell(R_{m+q})$, and therefore

$$\begin{aligned}|\text{coker } \phi_{m,q}(\mathcal{O}_X) - \ell(R/I^{[q]})_{m+q}| \leqslant \sum_{i=1}^{\mu} &|h^0(X, \mathcal{O}_X(q - qd_i + m)) \\ &- \ell(R_{q-qd_i+m})|.\end{aligned}$$

Now, if $q - qd_i + m < \bar{m}$, then $\ell(R_{q-qd_i+m}) \leqslant h^0(X, \mathcal{O}_X(q - qd_i + m)) \leqslant h^0(X, \mathcal{O}_X(\bar{m}))$, and if $q - qd_i + m \geqslant \bar{m}$, then $R_{q-qd_i+m} = H^0(X, \mathcal{O}_X(q - qd_i + m))$.
Hence,

$$|\text{coker } \phi_{m,q}(\mathcal{O}_X) - \ell(R/I^{[q]})_{m+q}| \leqslant \mu h^0(X, \mathcal{O}_X(\bar{m})).$$

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Definition 2.4. Let $\mathcal{Q}$ be a coherent sheaf of $\mathcal{O}_X$-modules of dimension $\bar{d}$, and let $\tilde{m} \geqslant 1$ be the least integer that is a regularity number for $\mathcal{Q}$ with respect to $\mathcal{O}_X(1)$. Then, we define $C_0(\mathcal{Q})$ and $D_0(\mathcal{Q})$ as follows. Let $a_1, \ldots, a_{\bar{d}} \in H^0(X, \mathcal{O}_X(1))$ be such that we have a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{Q}_i(-1) \xrightarrow{a_i} \mathcal{Q}_i \to \mathcal{Q}_{i-1} \to 0, \quad \text{for } 0 < i \leqslant \bar{d},$$

where $\mathcal{Q}_d = \mathcal{Q}$ and $\mathcal{Q}_i = \mathcal{Q}/(a_{\bar{d}}, \ldots, a_{i+1})\mathcal{Q}$, for $0 \leqslant i < \bar{d}$, with dim $\mathcal{Q}_i = i$. We define

$$C_0(\mathcal{Q}) = \min\left\{ \sum_{i=0}^{\bar{d}} h^0(X, \mathcal{Q}_i) \mid a_1, \ldots, a_{\bar{d}} \text{ is a } \mathcal{Q}\text{-sequence as above} \right\},$$

$$D_0(\mathcal{Q}) = h^0(X, \mathcal{Q}(\tilde{m})) + 2(\bar{d}+1)\left(\max\{q_0, q_1, \ldots, q_{\bar{d}}\}\right),$$

where

$$\chi(X, \mathcal{Q}(m)) = q_0\binom{m+\bar{d}}{\bar{d}} - q_1\binom{m+\bar{d}-1}{\bar{d}-1} + \cdots + (-1)^{\bar{d}} q_{\bar{d}}$$

is the Hilbert polynomial of $\mathcal{Q}$.

A more general version of the following lemma has been stated and proved in [T4, Lemma 2.6]. Here, we state and prove a relevant part of it, for a self-contained treatment (avoiding additional complications).

Lemma 2.5. *Let $\mathcal{Q}$ be a coherent sheaf of $\mathcal{O}_X$-modules of dimension $\bar{d}$. Let $P$ be a locally free sheaf of $\mathcal{O}_X$-modules that fits into a short exact sequence of locally free sheaves of $\mathcal{O}_X$-modules*

(5) $$0 \longrightarrow P \longrightarrow \oplus_i \mathcal{O}_X(-b_i) \longrightarrow P'' \longrightarrow 0, \quad \text{where } b_i \geqslant 0.$$

*Then, for $\tilde{\mu} = \mathrm{rank}(P) + \mathrm{rank}(P'')$ and for all $n, m \geqslant 0$, we have*

$$h^0(X, \mathcal{Q}(m+q)) \leqslant D_0(\mathcal{Q})(m+q)^{\bar{d}}$$

*and*

$$h^0(F^{n*}P \otimes \mathcal{Q}(m)) \leqslant (\tilde{\mu}) C_0(\mathcal{Q})(m^{\bar{d}} + 1).$$

*Proof.* Let $\tilde{m}$ be a regularity number for $\mathcal{Q}$; then, by Definition 2.4, we have

$$h^0(X, \mathcal{Q}(m+q)) \leqslant D_0(\mathcal{Q})(m+q)^{\bar{d}} \quad \text{for all } n, m \geqslant 0.$$

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Let $\mathcal{Q}_{\bar{d}} = \mathcal{Q}$. Let $a_{\bar{d}}, \ldots, a_1 \in H^0(X, \mathcal{O}_X(1))$, with the exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{Q}_i(-1) \xrightarrow{a_i} \mathcal{Q}_i \longrightarrow \mathcal{Q}_{i-1} \longrightarrow 0,$$

where $\mathcal{Q}_i = \mathcal{Q}_{\bar{d}}/(a_{\bar{d}}, \ldots, a_{i+1})\mathcal{Q}_{\bar{d}}$, for $0 \leqslant i \leqslant \bar{d}$, and realizing the minimal value $C_0(\mathcal{Q})$. Now, by the exact sequence (5), we have the following short exact sequence of $\mathcal{O}_X$-sheaves:

$$0 \longrightarrow F^{n*}P \otimes \mathcal{Q}_i \longrightarrow \oplus_j \mathcal{Q}_i(-qb_j) \longrightarrow F^{n*}P'' \otimes \mathcal{Q}_i \longrightarrow 0.$$

This implies $H^0(X, F^{n*}P \otimes \mathcal{Q}_i) \hookrightarrow \oplus_j H^0(X, \mathcal{Q}_i(-qb_j))$. Therefore,

$$h^0(X, F^{n*}P \otimes \mathcal{Q}_i) \leqslant \sum_j h^0(X, \mathcal{Q}_i(-qb_j)) \leqslant (\tilde{\mu})h^0(X, \mathcal{Q}_i), \tag{6}$$

as $-b_j \leqslant 0$. Since $F^{n*}P$ is a locally free sheaf of $\mathcal{O}_X$-modules, we have

$$0 \longrightarrow F^{n*}P \otimes \mathcal{Q}_i(m-1) \xrightarrow{a_i} F^{n*}P \otimes \mathcal{Q}_i(m) \longrightarrow F^{n*}P \otimes \mathcal{Q}_{i-1}(m) \longrightarrow 0,$$

which is a short exact sequence of $\mathcal{O}_X$-sheaves. Now, by induction on $i$, we prove that, for $m \geqslant 1$,

$$h^0(X, F^{n*}P \otimes \mathcal{Q}_i(m)) \leqslant (\tilde{\mu}) \left[h^0(X, \mathcal{Q}_i) + \cdots + h^0(X, \mathcal{Q}_0)\right] (m^i).$$

For $i = 0$, the inequality holds as $h^0(X, F^{n*}P \otimes \mathcal{Q}_0(m)) \leqslant (\tilde{\mu})h^0(X, \mathcal{Q}_0)$ (as $\dim \mathcal{Q}_0 = 0$).

Now, for $m \geqslant 1$, by the inequality (6) and by induction on $i$, we have

$$\begin{aligned} h^0(X, F^{n*}P \otimes \mathcal{Q}_i(m)) &\leqslant h^0(X, F^{n*}P \otimes \mathcal{Q}_i) + h^0(X, F^{n*}P \otimes \mathcal{Q}_{i-1}(1)) \\ &\quad + \cdots + h^0(X, F^{n*}P \otimes \mathcal{Q}_{i-1}(m)) \\ &\leqslant (\tilde{\mu})h^0(X, \mathcal{Q}_i) + \tilde{\mu}[h^0(X, \mathcal{Q}_{i-1}) + \cdots + h^0(X, \mathcal{Q}_0)] \\ &\quad \times (1 + 2^{i-1} + \cdots + m^{i-1}) \\ &\leqslant (\tilde{\mu})[h^0(X, \mathcal{Q}_i) + \cdots + h^0(X, \mathcal{Q}_0)]m^i. \end{aligned}$$

This implies

$$h^0(X, F^{n*}P \otimes \mathcal{Q}(m)) = h^0(X, F^{n*}P \otimes \mathcal{Q}_{\bar{d}}(m)) \leqslant \tilde{\mu}C_0(\mathcal{Q})m^{\bar{d}},$$

for all $m \geqslant 1$. Therefore, for all $0 \leqslant i \leqslant \bar{d}$ and for all $m \geqslant 0$, we have $h^0(X, F^{n*}P \otimes \mathcal{Q}(m)) \leqslant \tilde{\mu}C_0(\mathcal{Q})(m^{\bar{d}} + 1)$. This proves the lemma. ∎

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LEMMA 2.6. *There exists a short exact sequence of coherent sheaves of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow F_* \mathcal{O}_X \longrightarrow \mathcal{Q} \longrightarrow 0,$$

*where $\mathcal{Q}$ is a coherent sheaf of $\mathcal{O}_X$-modules such that* $\dim \operatorname{supp}(\mathcal{Q}) < d-1$.

*Proof.* Note that $X = \operatorname{Proj} R$, where $R = \oplus_{n \geqslant 0} R_n$, is a standard graded domain such that $R_0$ is an algebraically closed field. Therefore, there exists a Noether normalization

$$k[X_0, \ldots, X_{d-1}] \longrightarrow R,$$

which is an injective, finite separable graded map of degree 0 (as $k$ is an algebraically closed field). This induces a finite separable affine map $\pi : X \longrightarrow \mathbb{P}_k^{d-1} = S$.

Note that there is also an isomorphism

$$\eta : \mathcal{O}_S^{\oplus n_0} \oplus \mathcal{O}_S(-1)^{\oplus n_1} \oplus \cdots \oplus \mathcal{O}_S(-d+1)^{\oplus n_{d-1}} \longrightarrow F_* \mathcal{O}_S$$

of $\mathcal{O}_S$-modules, where $\sum n_i = p^{d-1}$.

Now, the isomorphism of $\eta$ implies that the map

$$\pi^*(\eta) : \oplus_{i=0}^{d-1} \mathcal{O}_X(-i)^{\oplus n_i} \longrightarrow \pi^* F_* \mathcal{O}_s$$

is an isomorphism of $\mathcal{O}_X$-sheaves. Since $0 \leqslant i \leqslant d-1$, we also have an injective and generically isomorphic map of $\mathcal{O}_X$-sheaves

$$\oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow \oplus_{i=0}^{d-1} \mathcal{O}_X(-i)^{\oplus n_i}.$$

Composing this map with $\pi^*(\eta)$ gives an injective and generically isomorphic map of $\mathcal{O}_X$-sheaves

$$\alpha : \oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow \pi^* F_* \mathcal{O}_S.$$

Since $\pi$ is separable, there is a canonical map $\beta : \pi^* F_* \mathcal{O}_S \longrightarrow F_* \mathcal{O}_X$, of sheaves of $\mathcal{O}_X$-modules, which is generically isomorphic.

Now, we have the composite map

$$\beta \circ \alpha : \oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow \pi^* F_* \mathcal{O}_S \rightarrow F_* \mathcal{O}_X,$$

which is generically an isomorphism. Hence, $\dim \operatorname{Coker}(\beta \circ \alpha) < \dim X = d-1$, and the map $\beta \circ \alpha : \oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow F_* \mathcal{O}_X$ is injective, as $X$ is an integral scheme. This proves the lemma. □

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LEMMA 2.7. *Let*

$$0 \longrightarrow \oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow F_*\mathcal{O}_X \longrightarrow \mathcal{Q} \longrightarrow 0,$$

*as in Lemma 2.6. Then, we have the following.*

(1) $\mathcal{Q}$ *is* $\tilde{m}$*-regular, where* $\tilde{m} = \max\{\bar{m} + d, l_1 - 1\}$*, where* $\bar{m}$ *and* $l_1$ *are as given in Notations 2.2.*

(2) *For a given* $d$*, there exists a universal polynomial function, with rational coefficients,* $\bar{P}_1^d(X_0, \ldots, X_{d-1}, Y)$ *(and hence independent of* $p$*), such that*

$$2C_0(\mathcal{Q}) + D_0(\mathcal{Q}) \leqslant p^{d-1}\bar{P}_1^d(\tilde{e}_0, \tilde{e}_1, \ldots, \tilde{e}_{d-1}, \bar{m}),$$

*where we define* $(\dim \operatorname{supp}(\mathcal{Q}) = \bar{d} < d-1)$

$$C_0(\mathcal{Q}) = \min\left\{\sum_{i=0}^{\bar{d}} h^0(X, \mathcal{Q}_i) \mid a_1, \ldots, a_{\bar{d}} \text{ is a } \mathcal{Q}\text{-sequence and } \mathcal{Q}_i = \mathcal{Q}/(a_{\bar{d}}, \ldots, a_{i+1})\mathcal{Q}\right\}$$

*and*

$$D_0(\mathcal{Q}) = h^0(X, \mathcal{Q}(\tilde{m})) + 2(\bar{d}+1)(\max\{q_0, q_1, \ldots, q_{\bar{d}}\}),$$

*where* $q_0, \ldots, q_{\bar{d}}$ *are the coefficients of the Hilbert polynomial* $\chi(X, \mathcal{Q}(m))$*.*

*Proof.* (1) The above short exact sequence of $\mathcal{O}_X$-sheaves gives a long exact sequence of cohomologies

$$\cdots \longrightarrow \oplus^{p^{d-1}} H^i(X, \mathcal{O}_X(m-d)) \longrightarrow H^i(X, \mathcal{O}_X(mp)) \longrightarrow H^i(X, \mathcal{Q}(m)) \longrightarrow \oplus^{p^{d-1}} H^{i+1}(X, \mathcal{O}_X(m-d)) \longrightarrow \cdots .$$

However, $h^i(X, \mathcal{O}_X(m-d-i)) = 0$, for all $m \geqslant \bar{m} + d$ and $i \geqslant 1$, which implies that if $m \geqslant \bar{m} + d$, then $h^i(X, \mathcal{Q}(m-i)) = 0$, for $i \geqslant 1$, and the canonical map

$$f_{1,m} : H^0(X, (F_*\mathcal{O}_X)(m)) \longrightarrow H^0(X, \mathcal{Q}(m))$$

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is surjective. Moreover, the canonical map

$$H^0(X, (F_*\mathcal{O}_X)(m)) \otimes H^0(X, \mathcal{O}_X(1)) \longrightarrow H^0(X, (F_*\mathcal{O}_X)(m+1))$$

is the same as the canonical map

$$f_{2,m} : H^0(X, \mathcal{O}_X(mp)) \otimes H^0(X, \mathcal{O}_X(1))^{[p]} \longrightarrow H^0(X, \mathcal{O}_X(mp+p)),$$

which is surjective for $m \geqslant \tilde{m}$. The map $f_{2m}$ fits into the following canonical diagram:

$$\begin{array}{ccc} R_{mp} \otimes R_1^{[p]} & \longrightarrow & R_{mp+p} \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{O}_X(mp)) \otimes H^0(X, \mathcal{O}_X(1))^{[p]} & \xrightarrow{f_{2,m}} & H^0(X, \mathcal{O}_X(mp+p)), \end{array}$$

where $H^0(X, \mathcal{O}_X(1))^{[p]}$, as a set, is $H^0(X, \mathcal{O}_X(1))$, and the upper symbol $^{[p]}$ indicates that the $k$-space structure is through the $p$th power map of $k$, where the top horizontal map is surjective for $m \geqslant l_1 - 1$, and the right vertical map is identity as $mp + p \geqslant \bar{m}$. Now, the commutative diagram of canonical maps

$$\begin{array}{ccc} H^0(X, (F_*\mathcal{O}_X)(m)) \otimes H^0(X, \mathcal{O}_X(1)) & \longrightarrow & H^0(X, \mathcal{Q}(m)) \otimes H^0(X, \mathcal{O}_X(1)) \\ \downarrow f_{2,m} & & \downarrow \\ H^0(X, (F_*\mathcal{O}_X)(m+1)) & \xrightarrow{f_{1,m+1}} & H^0(X, \mathcal{Q}(m+1)) \end{array}$$

implies that the second vertical map is surjective, for $m \geqslant \tilde{m}$, as the maps $f_{2,m}$ and $f_{1,m+1}$ are surjective. This proves that $\mathcal{Q}$ is $\tilde{m}$-regular. Hence, the assertion (1).

(2) If

$$\text{(7)}\quad \chi(X, \mathcal{Q}(m)) = q_0 \binom{m+d-2}{d-2} - q_1 \binom{m+d-3}{d-3} + \cdots + (-1)^{d-2} q_{d-2},$$

then, by Lemma A.1(1) (in the appendix, below),

$$|q_i| \leqslant p^{d-1} P_i^d(\tilde{e}_0, \ldots, \tilde{e}_{i+1}),$$

where $P_i^d(X_0, \ldots, X_{i+1})$ is a universal polynomial function with rational coefficients.

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Now, $\mathcal{Q}$ is $\tilde{m}$-regular implies that, for $0 \leqslant i < d$, $\mathcal{Q}_i := \mathcal{Q}/(a_{\bar{d}}, \ldots, a_{i+1})\mathcal{Q}$ is $\tilde{m}$-regular, for any $\mathcal{Q}$-sequence $a_1, \ldots, a_{\bar{d}} \in H^0(X, \mathcal{O}_X(1))$. Therefore, the short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{Q}_{i+1}(-1) \longrightarrow \mathcal{Q}_{i+1} \longrightarrow \mathcal{Q}_i \longrightarrow 0$$

gives the short exact sequence of $k$-vector spaces

$$0 \longrightarrow H^0(X, \mathcal{Q}_{i+1}(\tilde{m}-1)) \longrightarrow H^0(X, \mathcal{Q}_{i+1}(\tilde{m})) \longrightarrow H^0(X, \mathcal{Q}_i(\tilde{m})) \longrightarrow 0,$$

for $m \geqslant \tilde{m}$. Hence,

$$\begin{aligned} h^0(X, \mathcal{Q}_i) &\leqslant h^0(X, \mathcal{Q}_i(\tilde{m})) \leqslant \cdots \leqslant h^0(X, \mathcal{Q}(\tilde{m})) = \chi(X, \mathcal{Q}(\tilde{m})) \\ &\leqslant |q_0|\binom{\tilde{m}+d-2}{d-2} + |q_1|\binom{\tilde{m}+d-3}{d-3} + \cdots + |q_{d-2}|. \end{aligned}$$

This implies that $h^0(X, \mathcal{Q}_i) \leqslant p^{d-1}P^d(\tilde{e}_0, \ldots, \tilde{e}_{d-1}, \tilde{m})$, where $P^d(X_0, \ldots, X_{d-1}, Y)$ is a universal polynomial function with rational coefficients. Therefore,

$$C_0(\mathcal{Q}) \leqslant (d-1)p^{d-1}P^d(\tilde{e}_0, \ldots, \tilde{e}_{d-1}, \tilde{m}).$$

The inequality for $D_0(\mathcal{Q})$ follows similarly. This proves the assertion (2) and hence the lemma. □

LEMMA 2.8. *Given $d \geqslant 2$, there exist universal polynomials $\bar{P}_2^d$, $\bar{P}_3^d \in \mathbb{Q}[X_0, \ldots, X_{d-1}, Y]$ such that, if $X$ is an integral projective variety of dimension $d-1 \geqslant 1$ with Hilbert polynomial and $\bar{m}$ as in Notations 2.2, and if there are short exact sequences of $\mathcal{O}_X$-modules*

$$\begin{gathered} 0 \longrightarrow \mathcal{O}_X(-m_0) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{M}_1 \longrightarrow 0 \quad \text{and} \\ 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(n_2) \longrightarrow \mathcal{M}_2 \longrightarrow 0, \end{gathered}$$

*then*

$$\begin{aligned} 2C_0(\mathcal{M}_1) + D_0(\mathcal{M}_1) &\leqslant m_0^{d-1}\bar{P}_2^d(\tilde{e}_0, \ldots, \tilde{e}_{d-1}, \bar{m}), \\ 2C_0(\mathcal{M}_2) + D_0(\mathcal{M}_2) &\leqslant n_2^{d-1}\bar{P}_3^d(\tilde{e}_0, \ldots, \tilde{e}_{d-1}, \bar{m}), \end{aligned}$$

*where $m_0 \geqslant 0$ and $n_2 \geqslant 0$ are two integers.*

*Proof.* Without loss of generality, one can assume that $m_0 \geqslant 1$ and $n_2 \geqslant 1$. Since $\mathcal{O}_X$ is $\bar{m}$-regular, the sheaf $\mathcal{M}_1$ is an $\bar{m}+m_0$-regular sheaf of $\mathcal{O}_X$-modules of dimension $d-2$. Therefore, for any $\mathcal{M}_1$-sequence $a_1, \ldots, a_{d-2}$, the sheaf of $\mathcal{O}_X$-modules $\mathcal{M}_{1i} := \mathcal{M}_1/(a_{d-2}, \ldots, a_{i+1})\mathcal{M}_1$ is $\bar{m}+m_0$-regular. This implies

$$\begin{aligned} h^0(X, \mathcal{M}_{1i}) &\leqslant h^0(X, \mathcal{M}_{1i}(\bar{m}+m_0)) \\ &\leqslant h^0(X, \mathcal{M}_1(\bar{m}+m_0)) \leqslant h^0(X, \mathcal{O}_X(\bar{m}+m_0)) \\ &= \tilde{e}_0\binom{\bar{m}+m_0+d-1}{d-1} - \tilde{e}_1\binom{\bar{m}+m_0+d-2}{d-2} \\ &\quad + \cdots + (-1)^{d-1}\tilde{e}_{d-1} \\ &\leqslant \tilde{e}_0^2\binom{\bar{m}+d+d-1}{d-1}m_0^{d-1} + \tilde{e}_1^2\binom{\bar{m}+d+d-2}{d-2}m_0^{d-2} \\ &\quad + \cdots + \tilde{e}_{d-1}^2 \\ &\leqslant {m_0}^{d-1}\tilde{P}^d(\tilde{e}_0, \ldots, \tilde{e}_{d-1}, \bar{m}), \end{aligned}$$

where the second last inequality follows as $\tilde{e}_i \leqslant \tilde{e}_i^2$ and $\bar{m}+m_0+k \leqslant (\bar{m}+d+k)m_0$, for any $k \geqslant 0$, and $\tilde{P}^d(X_0, \ldots, X_{d-1}, Y)$ is a universal polynomial function with rational coefficients.

Let $e_i(\mathcal{M}_1)$ denote the $i$th coefficient of the Hilbert polynomial of $\mathcal{M}_1$ with respect to the line bundle $\mathcal{O}_X(1)$. Then, by Lemma A.1, we have $e_i(\mathcal{M}_1) \leqslant m_0^{i+1}P_i^d(\tilde{e}_0, \ldots, \tilde{e}_i)$, where $P_i^d(X_0, \ldots, X_i)$ is a universal polynomial with rational coefficients.

Now, the bound for $2C_0(\mathcal{M}_1) + D_0(\mathcal{M}_1)$ follows. The identical proof follows for $\mathcal{M}_2$. □

NOTATIONS 2.9. For a pair $(R, I)$, where $R$ is a standard graded ring of char $p > 0$ and of dimension $d \geqslant 2$, and $I \subset R$ is a graded ideal of finite colength, we define (similarly to the sequence of functions that had been defined in [T4]) a sequence of functions $\{f_n : [1, \infty) \to [0, \infty)\}_n$ as follows.

Fix $n \in \mathbb{N}$ and denote $q = p^n$. Let $x \in [1, \infty)$; then, there exists a unique nonnegative integer $m$ such that $(m+q)/q \leqslant x < (m+q+1)/q$. Then,

$$f_n(x) := f_n(R, I)(x) = \frac{\ell(R/I^{[q]})_{m+q}}{q^{d-1}}.$$

LEMMA 2.10. *Each $f_n : [1, \infty) \longrightarrow [0, \infty)$, defined as in Notations 2.9, is a compactly supported function such that $\cup_{n\geqslant 1}\mathrm{Supp}\ f_n \subseteq [1, n_0\mu]$, where $R_{n_0} \subseteq I$ and $\mu \geqslant \mu(I)$.*

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*Proof.* Since $R$ is a standard graded ring, for $m \geqslant n_0\mu q$, we have $R_m \subseteq (R_{n_0})^{\mu q} \subseteq I^{\mu q} \subseteq I^{[q]}$. This implies that $\ell(R/I^{[q]})_m = 0$, if $m \geqslant n_0\mu q$. Therefore, support $f_n \subseteq [1, n_0\mu]$, for every $n \geqslant 0$. This proves the lemma. ∎

PROPOSITION 2.11. *For $f_n$ as given in Notations 2.9, we have*

(1) $|f_n(x) - f_{n+1}(x)| \leqslant C/p^{n-d+2}$, *for every $x \geqslant 1$ and for all $n \geqslant 0$.*
(2) *In particular, $\|f_n - f_{n+1}\| \leqslant C/p^{n-d+2}$ and $\|f_{d-1} - f_d\| \leqslant C/p$, where*

$$C = 2C_R + \mu\left(\bar{m} + n_0\left(\sum_{i=1}^{\mu} d_i\right) + 1\right)^{d-2}(\bar{P}_1^d + d^{d-1}\bar{P}_2^d + \bar{P}_3^d), \tag{8}$$

*the integers $\bar{m}$ and $n_0$ are given as in Notations 2.2, and $d_1, \ldots, d_\mu$ are degrees of a chosen generator of $I$. Moreover, $C_R = \mu h^0(X, \mathcal{O}_X(\bar{m}))$, for $X = \operatorname{Proj} R$, and $\bar{P}_1^d$, $\bar{P}_2^d$, and $\bar{P}_3^d$ are given as in Lemmas 2.7 and 2.8.*

*Proof.* Fix $x \in [1, \infty)$. Therefore, for given $q = p^n$, there exists a unique integer $m \geqslant 0$, such that $(m+q)/q \leqslant x < (m+q+1)/q$ and

$$\frac{(m+q)p + n_2}{qp} \leqslant x < \frac{(m+q)p + n_2 + 1}{qp}, \quad \text{for some } 0 \leqslant n_2 < p.$$

Hence,

$$f_n(x) = \frac{1}{q^{d-1}}\ell\left(\frac{R}{I^{[q]}}\right)_{m+q}$$

and

$$f_{n+1}(x) = \frac{1}{(qp)^{d-1}}\ell\left(\frac{R}{I^{[qp]}}\right)_{mp+qp+n_2}.$$

Now, by Equation (4) in Remark 2.3, we have

$$\left|f_n(x) - \frac{\operatorname{coker}\,\phi_{m,q}(\mathcal{O}_X)}{q^{d-1}}\right| < \frac{C_R}{q^{d-1}} \quad \text{and} \tag{9}$$

$$\left|f_{n+1}(x) - \frac{\operatorname{coker}\,\phi_{mp+n_2,qp}(\mathcal{O}_X)}{(qp)^{d-1}}\right| < \frac{C_R}{(qp)^{d-1}}.$$

Consider the short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{M}_1 \longrightarrow 0.$$

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Then, for any locally free sheaf $P$ of $\mathcal{O}_X$-modules and for $m \geqslant 0$, we have the following short exact sequence of $\mathcal{O}_X$-modules:

$$0 \longrightarrow F^{n*}P \otimes \mathcal{O}_X(-d+m) \longrightarrow F^{n*}P \otimes \mathcal{O}_X(m) \longrightarrow F^{n*}P \otimes \mathcal{M}_1(m) \longrightarrow 0.$$

Since

$$\begin{aligned}\text{coker } \phi_{m,q}(\mathcal{O}_X) = h^0(X, \mathcal{O}_X(m+q)) - \sum_i h^0(X, \mathcal{O}_X(m+q-qd_i))\\ + h^0(X, (F^{n*}V)(m)),\end{aligned}$$

we have (by taking $P = V$ and $= \sum \mathcal{O}_X(1-d_i)$, respectively)

$$\begin{aligned}&|\text{coker } \phi_{m,q}(\mathcal{O}_X) - \text{coker } \phi_{m-d,q}(\mathcal{O}_X)|\\ &\quad \leqslant |h^0(X, \mathcal{O}_X(m+q)) - h^0(X, \mathcal{O}_X(m-d+q))|\\ &\qquad + \sum_i |h^0(X, \mathcal{O}_X(m+q-qd_i))\\ &\qquad - h^0(X, \mathcal{O}_X(m-d+q-qd_i))| + |h^0(X, (F^{n*}V)(m))\\ &\qquad - h^0(X, (F^{n*}V)(m-d))|\\ &\quad \leqslant h^0(X, \mathcal{M}_1(m+q)) + h^0\left(X, \sum_i \mathcal{O}_X(q-qd_i) \otimes \mathcal{M}_1(m)\right)\\ &\qquad + h^0(X, F^{n*}V \otimes \mathcal{M}_1(m)).\end{aligned}$$

Let $d-2=0$. Then, $\mathcal{M}_1$ is a zero-dimensional sheaf, which implies that $h^0(X, \mathcal{M}_1(m)) = h^0(X, \mathcal{M}_1)$, for every $m \in \mathbb{Z}$. Moreover, if $P$ is a locally free sheaf of $\mathcal{O}_X$-modules, then $h^0(X, P \otimes \mathcal{M}_1) = (\text{rank } P) h^0(X, \mathcal{M}_1)$. Therefore, we get

$$\begin{aligned}|\text{coker } \phi_{m,q}(\mathcal{O}_X) - \text{coker } \phi_{m-d,q}(\mathcal{O}_X)| &\leqslant [1 + \mu + (\mu - 1)] h^0(X, \mathcal{M}_1)\\ &= 2\mu C_0(\mathcal{M}_1).\end{aligned}$$

If $d-2>0$ then, by Lemma 2.5,

$$\begin{aligned}&|\text{coker } \phi_{m,q}(\mathcal{O}_X) - \text{coker } \phi_{m-d,q}(\mathcal{O}_X)|\\ &\quad \leqslant D_0(\mathcal{M}_1)(m+q)^{d-2} + 2(\mu) C_0(\mathcal{M}_1)(m^{d-2}+1)\\ &\quad \leqslant (\mu)[2C_0(\mathcal{M}_1) + D_0(\mathcal{M}_1)](m+q)^{d-2}.\end{aligned}$$

Therefore, for $d-1 \geqslant 1$, we have

$$\begin{aligned} &|p^{d-1}\text{coker } \phi_{m,q}(\mathcal{O}_X) - p^{d-1}\text{coker } \phi_{m-d,q}(\mathcal{O}_X)| \\ &\quad \leqslant (\mu)[2C_0(\mathcal{M}_1) + D_0(\mathcal{M}_1)](m+q)^{d-2}p^{d-1}. \end{aligned} \tag{10}$$

Since, for a locally free sheaf $P$, we have

$$h^0(X, F^{n*}P \otimes (F_*\mathcal{O}_X)(m)) = h^0(X, F^{(n+1)*}P \otimes \mathcal{O}_X(mp)),$$

the short exact sequence

$$0 \longrightarrow \oplus^{p^{d-1}}\mathcal{O}_X(-d) \longrightarrow F_*\mathcal{O}_X \longrightarrow \mathcal{Q} \longrightarrow 0,$$

as given in the statement of Lemma 2.7, gives a canonical long exact sequence

$$\begin{aligned} 0 &\longrightarrow \oplus H^0(X, (F^{n*}P)(m-d)) \longrightarrow H^0(X, (F^{(n+1)*}P)(mp)) \\ &\longrightarrow H^0(X, F^{n*}P \otimes \mathcal{Q}(m)), \end{aligned}$$

which implies (by taking $P = V$ and $V = \sum \mathcal{O}_X(1-d_i)$, respectively)

$$\begin{aligned} \text{coker } \phi_{mp,qp}(\mathcal{O}_X) &= h^0(X, (F_*\mathcal{O}_X)(m+q)) \\ &\quad - \sum_i h^0(X, (F_*\mathcal{O}_X)(q - qd_i + m)) \\ &\quad + h^0(X, F^{n*}V \otimes (F_*\mathcal{O}_X)(m+q)). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &|p^{d-1}\text{coker } \phi_{(m-d),q}(\mathcal{O}_X) - \text{coker } \phi_{mp,qp}(\mathcal{O}_X)| \\ &\quad \leqslant (\mu)[2C_0(\mathcal{Q}) + D_0(\mathcal{Q})](m+q)^{d-2}. \end{aligned} \tag{11}$$

The short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(n_2) \longrightarrow \mathcal{M}_2 \longrightarrow 0$$

gives

$$\begin{aligned} 0 &\longrightarrow H^0(X, (F^{(n+1)*}P)(mp)) \longrightarrow H^0(X, (F^{(n+1)*}P)(mp+n_2)) \\ &\longrightarrow H^0(X, (F^{(n+1)*}P) \otimes \mathcal{M}_2(mp)) \longrightarrow \cdots, \end{aligned}$$

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which gives

$$|\text{coker } \phi_{mp,qp}(\mathcal{O}_X) - \text{coker } \phi_{mp+n_2,qp}(\mathcal{O}_X)|$$
$$\leqslant \left[h^0(X, F^{(n+1)*}V \otimes \mathcal{M}_2(mp)) + h^0\left(X, \sum_i \mathcal{O}_X(qp - qpd_i) \otimes \mathcal{M}_2(mp)\right)\right.$$
$$\left. + h^0(X, \mathcal{M}_2(mp + qp))\right] \leqslant 2(\mu)C_0(\mathcal{M}_2)((mp)^{d-2} + 1)$$
$$+ (\mu)D_0(\mathcal{M}_2)(mp + qp)^{d-2}.$$

Therefore,

$$|\text{coker } \phi_{mp,qp}(\mathcal{O}_X) - \text{coker } \phi_{mp+n_2,qp}(\mathcal{O}_X)| \leqslant (\mu)[2C_0(\mathcal{M}_2) + D_0(\mathcal{M}_2)](mp + qp)^{d-2}. \tag{12}$$

Combining Equations (10)–(12), we get

$$(A) := |p^{d-1}\text{coker } \phi_{m,q}(\mathcal{O}_X) - \text{coker } \phi_{mp+n_2,qp}(\mathcal{O}_X)|$$
$$\leqslant (\mu)[2C_0(\mathcal{M}_1) + D_0(\mathcal{M}_1)](m + q)^{d-2}p^{d-1}$$
$$+ (\mu)[2C_0(\mathcal{Q}) + D_0(\mathcal{Q})](m + q)^{d-2}$$
$$+ (\mu)[2C_0(\mathcal{M}_2) + D_0(\mathcal{M}_2)](mp + qp)^{d-2}.$$

Therefore,

$$(A) \leqslant (\mu)(m + q)^{d-2}\,[(2C_0(\mathcal{M}_1) + D_0(\mathcal{M}_1))p^{d-1} + (2C_0(\mathcal{Q}) + D_0(\mathcal{Q}))$$
$$+ (2C_0(\mathcal{M}_2) + D_0(\mathcal{M}_2))p^{d-2}].$$

Now, if we denote $\bar{P}_i^d = \bar{P}_i^d(\tilde{e}_0, \ldots, \tilde{e}_{d-1}, \tilde{m})$, for $i = 1, 2$, and 3, then, by Lemma 2.7,

$$2C_0(\mathcal{Q}) + D_0(\mathcal{Q}) \leqslant p^{d-1}\tilde{P}_1^d,$$

and, by Lemma 2.8, we have

$$2C_0(\mathcal{M}_1) + D_0(\mathcal{M}_1) \leqslant d^{d-1}\tilde{P}_2^d$$

and

$$2C_0(\mathcal{M}_2) + D_0(\mathcal{M}_2) \leqslant n_2^{d-1}\tilde{P}_3^d \leqslant p^{d-1}\tilde{P}_3^d,$$

where the last inequality follows as $n_2 < p$. Therefore, we have

$$(A) \leqslant (\mu)(m+q)^{d-2}[p^{d-1}d^{d-1}\bar{P}_2^d + p^{d-1}\bar{P}_1^d + p^{d-1}p^{d-2}\bar{P}_3^d].$$

Now, multiplying the above inequality by $1/(qp)^{d-1}$, we get

$$\left|\frac{\text{coker }\phi_{m,q}(\mathcal{O}_X)}{q^{d-1}} - \frac{\text{coker }\phi_{mp+n_2,qp}(\mathcal{O}_X)}{(qp)^{d-1}}\right| \leqslant (\mu)\frac{(m+q)^{d-2}}{q^{d-1}}[d^{d-1}\bar{P}_2^d + \bar{P}_1^d + p^{d-2}\bar{P}_3^d].$$

Moreover, by Remark 2.3,

$$m+q \geqslant \bar{m} + n_0\left(\sum_i d_i\right)q + q \Rightarrow \text{coker }\phi_{m,q}(\mathcal{O}_X) = \text{coker }\phi_{mp+n_2,qp}(\mathcal{O}_X) = 0.$$

Furthermore, $m+q \leqslant \bar{m} + n_0(\sum_i d_i)q + q, \Rightarrow (m+q)^{d-2} \leqslant L_0 q^{d-2}$, where

$$L_0 = \left(\bar{m} + n_0\left(\sum_i d_i\right) + 1\right)^{d-2}.$$

Therefore, for every $m \geqslant 0$ and $n \geqslant 1$, where $q = p^n$, we have

$$\begin{aligned}
&|\text{coker }\phi_{m,q}(\mathcal{O}_X)/q^{d-1} - \text{coker }\phi_{mp+n_2,qp}(\mathcal{O}_X)/(qp)^{d-1}| \\
&\quad \leqslant ((\mu)L_0 q^{d-2}[d^{d-1}\bar{P}_2^d + \bar{P}_1^d + p^{d-2}\bar{P}_3^d])/q^{d-1} \\
&\quad \leqslant ((\mu)L_0[d^{d-1}\bar{P}_2^d + \bar{P}_1^d + \bar{P}_3^d]p^{d-2}q^{d-2})/q^{d-1}.
\end{aligned}$$

Now, by Equation (9), we have

$$\begin{aligned}
|f_n(x) - f_{n+1}(x)| &\leqslant \frac{C_R}{q^{d-1}} + \frac{C_R}{(qp)^{d-1}} + (\mu)L_0[d^{d-1}\bar{P}_2^d + \bar{P}_1^d + \bar{P}_3^d]\frac{p^{d-2}}{q} \\
&\leqslant C\frac{p^{d-2}}{q},
\end{aligned}$$

where $C = 2C_R + (\mu)L_0(d^{d-1}\bar{P}_2^d + \bar{P}_1^d + \bar{P}_3^d)$, which proves the proposition. □

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Corollary 2.12. *There exists a constant $C_1 = P_4^d(\tilde{e}_0, \tilde{e}_1, \ldots, \tilde{e}_d, \bar{m}) + (n_0\mu - 1)C$, where $C$ is as in Proposition 2.11, and $P_4^d(X_0, \ldots, X_d, Y)$ is a universal polynomial function with rational coefficients such that, for $n \geqslant 1$,*

$$\left|\frac{1}{(p^n)^d}\ell\left(\frac{R}{I^{[p^n]}}\right) - \frac{1}{(p^{n+1})^d}\ell\left(\frac{R}{I^{[p^{n+1}]}}\right)\right| \leqslant \frac{C_1}{p^{n-d+2}}.$$

*Proof.* Note that

$$\sum_{m=0}^{\infty} \frac{1}{p^{nd}}\ell\left(\frac{R}{I^{[p^n]}}\right)_{m+q} = \int_1^{\infty} f_n(x)\,dx = \int_1^{n_0\mu} f_n(x)\,dx,$$

where the last equality follows from Lemma 2.10. Hence,

$$\begin{aligned}
&\left|\frac{1}{(p^n)^d}\ell\left(\frac{R}{I^{[p^n]}}\right) - \frac{1}{(p^{n+1})^d}\ell\left(\frac{R}{I^{[p^{n+1}]}}\right)\right| \\
&\quad \leqslant \left|\frac{1}{p^{nd}}\ell\left(\frac{R}{\mathbf{m}^{p^n}}\right) - \frac{1}{p^{(n+1)d}}\ell\left(\frac{R}{\mathbf{m}^{p^{n+1}}}\right)\right| \\
&\qquad + \left|\int_1^{n_0\mu} f_n(x)\,dx - \int_1^{n_0\mu} f_{n+1}(x)\,dx\right|.
\end{aligned}$$

If $p^n \leqslant \bar{m}$, then

$$\begin{aligned}
\left|\frac{1}{p^{nd}}\ell\left(\frac{R}{\mathbf{m}^{p^n}}\right) - \frac{1}{p^{(n+1)d}}\ell\left(\frac{R}{\mathbf{m}^{p^{n+1}}}\right)\right| &\leqslant \left|\frac{P_{(R,\mathbf{m})}(\bar{m})}{p^{nd}} + \frac{P_{(R,\mathbf{m})}(\bar{m}^2)}{p^{(n+1)d}}\right| \\
&\leqslant \frac{P_{(R,\mathbf{m})}(\bar{m}^2)}{p^n},
\end{aligned}$$

where $P_{(R,\mathbf{m})}$ is the Hilbert polynomial of $R$ with respect to $\mathbf{m}$. If $p^n > \bar{m}$, then there exists a universal polynomial function $P_6^d(X_0, \ldots, X_d)$ with rational coefficients such that

$$\text{L.H.S.} \leqslant \left|\frac{P_{(R,\mathbf{m})}(p^n)}{p^{nd}} - \frac{P_{(R,\mathbf{m})}(p^{n+1})}{p^{(n+1)d}}\right| \leqslant \frac{P_6^d(\tilde{e}_0, \tilde{e}_1, \ldots, \tilde{e}_d)}{p^n}.$$

Therefore, combining this with Proposition 2.11, part (1), we get a universal polynomial function $P_4^d(X_0, \ldots, X_d, Y)$ with rational coefficients such that

$$\left|\frac{1}{(p^n)^d}\ell\left(\frac{R}{I^{[p^n]}}\right) - \frac{1}{(p^{n+1})^d}\ell\left(\frac{R}{I^{[p^{n+1}]}}\right)\right| \leqslant \frac{P_4^d(\tilde{e}_0, \tilde{e}_1, \ldots, \tilde{e}_d)}{p^n} + \frac{(n_0\mu - 1)C}{p^{n-d+2}}.$$

Since $d \geqslant 2$, the corollary follows. ∎

## §3. Hilbert–Kunz density function and reduction mod $p$

REMARK 3.1. Let $R$ be a standard graded integral domain of dimension $d \geqslant 2$, with $R_0 = k$, where $k$ is an algebraically closed field. Let $N = \ell(R_1) - 1$; then, we have a surjective graded map $k[X_0, \ldots, X_N] \longrightarrow R$ of degree 0, given by $X_i$ mapping to generators of $R_1$. This gives a closed immersion $X = \text{Proj } R \longrightarrow \mathbb{P}_k^N$ such that $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}_k^N}(1)\,|_X$. Therefore, if

$$P_{R,\mathbf{m}}(m) = \tilde{e}_0\binom{m+d-1}{d} - \tilde{e}_1\binom{m+d-2}{d-1} + \cdots + (-1)^d\tilde{e}_d$$

is the Hilbert–Samuel polynomial of $R$, with respect to $\mathbf{m}$, then, the Hilbert polynomial for the pair $(X, \mathcal{O}_X(1))$ is

$$\chi(X, \mathcal{O}_X(m)) = \tilde{e}_0\binom{m+d-1}{d-1} - \tilde{e}_1\binom{m+d-2}{d-2} + \cdots + (-1)^{d-1}\tilde{e}_{d-1}.$$

Since $R$ is a domain, the canonical graded map $R = \oplus_m R_m \longrightarrow \oplus_m H^0(X, \mathcal{O}_X(m))$ is injective.

Let $\mathcal{I}_X$ be the ideal sheaf of $X$ in $\mathbb{P}_k^N$; then, we have the canonical short exact sequence of $\mathcal{O}_{\mathbb{P}_k^N}$-modules

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}_k^N} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

and the image of the induced map $f_m : H^0(\mathbb{P}_k^N, \mathcal{O}_{\mathbb{P}_k^N}(m)) \longrightarrow H^0(X, \mathcal{O}_X(m))$ is $R_m$. Now, by Exp XIII, (6.2) (in [SGA 6]), there exists a universal polynomial $P_5^d(t_0, \ldots, t_{d-1})$ with rational coefficients such that the sheaf $\mathcal{I}_X$ is $\bar{m} = P_5^d(\tilde{e}_0, \ldots, \tilde{e}_{d-1})$-regular. Therefore, the map $f_m$ is surjective for $m \geqslant \bar{m}$.

In particular, we have

(1) $R_m = H^0(X, \mathcal{O}_X(m))$, for all $m \geqslant \bar{m}$, and
(2) the sheaf $\mathcal{O}_X$ is $\bar{m}$-regular with respect to $\mathcal{O}_X(1)$.

Next, we recall a notion of *spread*.

DEFINITION 3.2. Consider the pair $(R, I)$, where $R$ is a finitely generated $\mathbb{Z}_{\geqslant 0}$-graded $d$-dimensional domain such that $R_0$ is an algebraically closed field $k$ of characteristic 0, and $I \subset R$ is a homogeneous ideal of finite colength. For such a pair, there exist a finitely generated $\mathbb{Z}$-algebra $A \subseteq k$, a finitely generated $\mathbb{N}$-graded algebra $R_A$ over $A$, and a homogeneous ideal

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$I_A \subset R_A$ such that $R_A \otimes_A k = R$ and $I = \mathrm{Im}(I_A \otimes_A k)$. We call $(A, R_A, I_A)$ a *spread* of the pair $(R, I)$.

Moreover, if, for the pair $(R, I)$, we have a spread $(A, R_A, I_A)$ as above and $A \subset A' \subset k$, for some finitely generated $\mathbb{Z}$-algebra $A'$, then $(A', R_{A'}, I_{A'})$ satisfy the same properties as $(A, R_A, I_A)$. Hence, we may always assume that the spread $(A, R_A, I_A)$ as above is chosen such that $A$ contains a given finitely generated $\mathbb{Z}$-algebra $A_0 \subseteq k$.

NOTATIONS 3.3. Given a spread $(A, R_A, I_A)$ as above, for a closed point $s \in \mathrm{Spec}(A)$, we define $R_s = R_A \otimes_A \bar{k}(s)$ and the ideal $I_s = \mathrm{Im}(I_A \otimes_A \bar{k}(s)) \subset R_s$. Similarly, for $X_A := \mathrm{Proj}\ R_A$, we define $X_s := X \otimes \bar{k}(s) = \mathrm{Proj}\ R_s$, and, for a coherent sheaf $V_A$ on $X_A$, we define $V_s = V_A \otimes \bar{k}(s)$.

REMARK 3.4. Note that for a spread $(A, R_A, I_A)$ of $(R, I)$ as above, the induced map $\tilde{\pi} : X_A := \mathrm{Proj}\ R_A \longrightarrow \mathrm{Spec}(A)$ is a proper map; hence, by generic flatness, there is an open set (in fact nonempty as $A$ is an integral domain) $U \subset \mathrm{Spec}(A)$ such that $\tilde{\pi}\mid_{\tilde{\pi}^{-1}(U)}: \tilde{\pi}^{-1}(U) \longrightarrow U$ is a proper flat map. Therefore (see [EGA IV, 12.2.1]), the set

$$\{s \in \mathrm{Spec}(A) \mid X \otimes_{\mathrm{Spec}(A)} \mathrm{Spec}(k(s)) \text{ is geometrically integral}\}$$

is a nonempty open set of $\mathrm{Spec}(A)$. Hence, by replacing $A$ by a finitely generated $\mathbb{Z}$-algebra $A'$ such that $A \subset A' \subset k$ (if necessary), we can assume that $\tilde{\pi}$ is a flat map such that for every $s \in \mathrm{Spec}(A)$, the fiber over $s$ is geometrically integral.

Therefore, for any closed point $s \in \mathrm{Spec}(A)$ (i.e., a maximal ideal of $A$), the ring $R_s$ is a standard graded $d$-dimensional ring such that the ideal $I_s \subset R_s$ is a homogeneous ideal of finite colength. Moreover, $X_s$ is an integral scheme over $\bar{k}(s)$.

*Proof of Theorem 1.1.* For given $(R, I)$, and a given spread $(A, R_A, I_A)$, we can choose a spread $(A', R_{A'}, I_{A'})$, where $A \subset A'$, such that the induced projective morphism of Noetherian schemes $\tilde{\pi} : X_{A'} \longrightarrow A'$ is flat, and, for every $s \in \mathrm{Spec}(A')$, the scheme $X_s$ is an integral scheme over $\bar{k}(s)$ of dimension $= d - 1$. Let $R_{A'} = \oplus_{i \geqslant 0}(R_{A'})_i$, and let $(R_{A'})_1$ be generated by $N$ elements as an $A'$-module. Then, the canonical graded surjective map

$$A'[X_0, \ldots, X_N] \longrightarrow R_{A'}$$

gives a closed immersion $X_{A'} = \mathrm{Proj}\ R_{A'} \longrightarrow \mathbb{P}^N_{A'}$ such that $\mathcal{O}_{X_{A'}}(1) = \mathcal{O}_{\mathbb{P}^N_{A'}}(1)\mid_{X_{A'}}$. Let $X_s = X_{A'} \otimes \bar{k}(s)$. Then, $X_s = \mathrm{Proj}\ R_s$, and $\mathcal{O}_{X_s}(1)$ is the canonical line bundle induced by $\mathcal{O}_{X_{A'}}(1)$. Let $s_0 = \mathrm{Spec} Q(A) = \mathrm{Spec} Q(A')$ be the generic point of $\mathrm{Spec}(A')$. We now have the following.

(1) The Hilbert polynomial for the pair $(X_s, \mathcal{O}_{X_s}(1))$ is

$$\chi(X_s, \mathcal{O}_{X_s}(m)) = \tilde{e}_0\binom{m+d-1}{d-1} - \tilde{e}_1\binom{m+d-2}{d-2} + \cdots + (-1)^{d-1}\tilde{e}_{d-1},$$

where the coefficients $\tilde{e}_i$ are as above for $(X, \mathcal{O}_X(1))$ (from char 0). In particular, dim $X_s = d-1$ and we have the following.

(2) By Remark 3.1, there exists $\bar{m} = P_5^d(\tilde{e}_0, \ldots, \tilde{e}_{d-1})$ such that $(R_s)_m = H^0(X_s, \mathcal{O}_{X_s}(m))$ for all $m \geqslant \bar{m}$, and $(X_s, \mathcal{O}_{X_s}(1))$ is $\bar{m}$-regular.

(3) Moreover, by the semicontinuity theorem (Chapter III, Theorem 12.8 in [H]), by shrinking $\mathrm{Spec}(A')$ further, we have $h^i(X_s, \mathcal{O}_{X_s}(\bar{m}))$, and $h^0(X_s, \mathcal{O}_{X_s})$ is independent of $s$, for all $i \geqslant 0$.

(4) Again, by shrinking $\mathrm{Spec}(A')$ (if necessary), can choose $n_0 \in \mathbb{N}$ such that $(R_{A'})_1^{n_0} \subseteq I_{A'}$. This implies that $(R_s)_1^{n_0} \subseteq I_s$, for all $s \in \mathrm{Spec}(A')$.

Let $s \in \mathrm{Spec}(A')$, and let $p_s = \mathrm{char}\ k(s)$. We sketch the proof of the existence of the map $f_{R_s,I_s} : [1, \infty) \to \mathbb{R}$ and its relation to $e_{HK}(R, I)$ (note that we have proved this in a more general setting in [T4]). By Proposition 2.11, for any given $s$, the sequence $\{f_n^s\}_n$ of functions is uniformly convergent. Let $f_{R_s,I_s}(x) = \lim_{n\to\infty} f_n(R_s, I_s)(x)$. This implies that $\lim_{n\to\infty} \int_1^\infty f_n(R_s, I_s)(x) = \int_1^\infty f_{R_s,I_s}(x)$, as, by Lemma 2.10, there is a compact set containing $\cup_n \mathrm{supp}\ f_n(R_s, I_s)$. On the other hand,

$$\begin{aligned} e_{HK}(R_s, I_s) &= \lim_{n\to\infty} \frac{1}{p_s^{nd}} \ell\left(\frac{R_s}{I_s^{[p_s^n]}}\right) \\ &= \lim_{n\to\infty} \frac{1}{p_s^{nd}} \ell\left(\frac{R_s}{\mathbf{m}_s^{p_s^n}}\right) + \lim_{n\to\infty} \frac{1}{p_s^{nd}} \sum_{m\geqslant 0} \ell\left(\frac{R}{I_s^{[p_s^n]}}\right)_{m+p_s^n} \\ &= \frac{e_0(R_s)}{d!} + \lim_{n\to\infty} \int_1^\infty f_n(R_s, I_s)(x)\,dx \\ &= \frac{e_0(R_s)}{d!} + \int_1^\infty f_{R_s,I_s}(x)\,dx, \end{aligned}$$

where $\mathbf{m}_s$ is the graded maximal ideal of $R_s$ and $e_0(R_s)$ denotes the Hilbert–Samuel multiplicity of $R_s$ with respect to $\mathbf{m}_s$.

Now, by Proposition 2.11, there exists a constant

$$C = 2C_{R_s} + \mu \left( \bar{m} + n_0 \left( \sum_{i=1}^{\mu} d_i \right) + 1 \right)^{d-2} (\bar{P}_1^d + d^{d-1}\bar{P}_2^d + \bar{P}_3^d),$$

which is independent of the choice of $s$ in $\mathrm{Spec}(A')$ (as $C_{R_s} = \mu h^0(X_s, \mathcal{O}_{X_s}(1))$), such that

(13) $$\|f_n(R_s, I_s) - f_{n+1}(R_s, I_s)\| \leqslant C/p_s^{n-d+2}, \quad \text{for all } n \geqslant 1.$$

In particular, for given $m \geqslant d-1$,

$$\|f_m(R_s, I_s) - f_{R_s,I_s}\| \leqslant \left[ \frac{C}{p_s} + \frac{C}{p_s^2} + \frac{C}{p_s^3} + \cdots \right] \frac{1}{p_s^{m-(d-1)}} \leqslant \frac{2C}{p_s^{m-d+2}}.$$

As $s \to s_0$, we have $p_s = \mathrm{char}\, k(s) \to \infty$, which implies

(14) $$\text{for any} \quad m \geqslant d-1, \quad \text{we have} \lim_{p_s \to \infty} \|f_m(R_s, I_s) - f_{R_s,I_s}\| = 0.$$

This completes the proof of Assertion (1) of the theorem.

It is easy to check that there exists a universal polynomial function $\bar{P}_9^d(X_0, \ldots, X_d)$ with rational coefficients such that for $C_2 := \bar{P}_9^d(\tilde{e}_0, \ldots, \tilde{e}_d)$ we have

$$\left| \frac{1}{p_s^{nd}} \ell \left( \frac{R_s}{\mathbf{m}_s^{p_s^n}} \right) - \frac{e_0(R_s)}{d!} \right| \leqslant \frac{C_2}{p_s^n}.$$

Moreover, for every $n \geqslant 1$, the function $f_n(R_s, I_s)$ has support in the compact interval $[1, n_0\mu]$. Therefore,

$$\begin{aligned}(A_1) &:= \left| \frac{1}{p_s^{nd}} \ell \left( \frac{R_s}{I_s^{[p_s^n]}} \right) - \frac{1}{p_s^{(n+1)d}} \ell \left( \frac{R_s}{I_s^{[p_s^{n+1}]}} \right) \right| \\ &\leqslant \left| \frac{1}{p_s^{nd}} \ell \left( \frac{R_s}{\mathbf{m}_s^{p_s^n}} \right) - \frac{e_0(R_s)}{d!} \right| \\ &\quad + \int_1^{n_0\mu} |f_n(R_s, I_s)(x) - f_{n+1}(R_s, I_s)(x)|\, dx.\end{aligned}$$

By (13), we have

$$(A_1) \leqslant \frac{C_2}{p_s^n} + \frac{(2C)(n_0\mu - 1)}{p_s^{n-d+2}}.$$

This proves Assertion (2). Now, similarly, Assertion (3) easily follows from (14). □

*Proof of Corollary 1.2.* This is easy to deduce from Theorem 1.1. □

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## §4. Some properties and examples

Throughout this section, $R$ is a standard graded integral domain of dimension $d \geqslant 2$, with $R_0 = k$, where $k$ is an algebraically closed field of characteristic 0, and $I \subset R$ is a homogeneous ideal of finite colength. Our choice of spread satisfies conditions as given in Remark 3.4.

DEFINITION 4.1. We denote $f^{\infty}_{R,I} = \lim_{p_s \to \infty} f_{R_s,I_s}$, if it exists, where for $(R, I)$, the pair $(R_s, I_s)$ is given as in Definition 3.2 and Notations 3.3.

DEFINITION 4.2. For a choice of spread $(A, R_A, I_A)$ of $(R, I)$, as in Remark 3.4, and a closed point $s \in \operatorname{Spec}(A)$, we define $F_{R_s} : [0, \infty) \longrightarrow [0, \infty)$ as

$$HSd(R_s)(x) = F_{R_s}(x) = \lim_{n\to\infty} F_n(R_s)(x),$$
$$\text{where} \quad F_n(R_s)(x) = \frac{1}{q^{d-1}}\ell(R_s)_{\lfloor xq\rfloor} \text{ and } q = p_s^n.$$

One can check that

$$F_{R_s} : \mathbb{R} \to \mathbb{R} \quad \text{is given by } F_{R_s}(x) = 0 \quad \text{for } x < 0,$$
$$\text{and} \quad F_{R_s}(x) = e_0(R)x^{d-1}/(d-1)! \quad \text{for } x \geqslant 0,$$

where $e_0(R)$ is the Hilbert–Samuel multiplicity of $R$ with respect to $\mathbf{m}$. Hence, we denote $F_{R_s}(x) = F_R(x)$. Moreover, for any $n \geqslant 1$, we have $\lim_{p_s\to\infty} F_n(R_s)(x) = F_{R_s}(x) = F_R(x)$.

PROPOSITION 4.3. *Let $R$ and $S$ be standard graded domains, where $R_0 = S_0 = k$, where $k$ is an algebraically closed field of characteristic 0 with $I \subset R$ and $J \subset S$ homogeneous ideals of finite colength, respectively. Then,*

$$f^{\infty}_{R,I}(x) \quad \textit{and} \quad f^{\infty}_{S,J}(x) \quad \textit{exist} \quad \Rightarrow \quad f^{\infty}_{R\#S,I\#J}(x) \quad \textit{exists},$$

*where $R\#S = \oplus_{m\geqslant 0} R_m \otimes_k S_m$. Moreover, in that case, we have*

$$f^{\infty}_{R\#S,I\#J}(x) = F_S(x)f^{\infty}_{R,I}(x) + F_R(x)f^{\infty}_{S,J}(x) - f^{\infty}_{R,I}(x)f^{\infty}_{S,J}(x).$$

*In particular, $f^{\infty}_{-,-}$ satisfies a multiplicative formula on Segre products.*

*Proof.* Let us denote $f^{\infty} = f^{\infty}_{R,I}$ and $g^{\infty} = f^{\infty}_{S,J}$. For $q = p_s^n$, where $p_s = \operatorname{char} k(s)$, we denote $f^s_n = f_n(R_s, I_s)$ and $g^s_n = f_n(S_s, J_s)$, where $s \in \operatorname{Spec}(A)$ denotes a closed point and $(A, R_A, I_A)$ and $(A, S_A, J_A)$ are spreads.

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For any $n \geqslant 1$, we have

$$f_n(R_s\#S_s, I_s\#J_s)(x) = F_n(R_s)(x)g_n^s(x) + F_n(S_s)(x)f_n^s(x) - f_n^s(x)g_n^s(x).$$

For a spread $(A, R_A, I_A)$, let $n_0$ and $\mu$ be positive integers such that ${(R_A)_1}^{n_0} \subseteq I_A$ and ${(S_A)_1}^{n_0} \subseteq J_A$, and also $\mu(I_A)$, $\mu(J_A) \leqslant \mu$. Then, by Lemma 2.10,

$$\bigcup_{n\geqslant 0, s\in \operatorname{Spec}(A)} \text{Support } (f_n^s) \bigcup \bigcup_{n\geqslant 0, s\in \operatorname{Spec}(A)} \text{Support } (g_n^s) \subseteq [1, n_0\mu]. \tag{15}$$

Moreover, there is a constant $C_1$ such that, for any $n \geqslant 1$ and every closed point $s \in \operatorname{Spec}(A)$, we have

$$f_n^s(x) \leqslant F_n(R_s)(x) \leqslant C_1 \quad \text{and} \quad g_n^s(x) \leqslant F_n(S_s)(x) \leqslant C_1,$$

for all $x \in [1, n_0\mu]$.

Since $f^\infty$ and $g^\infty$ exist, by Theorem 1.1(1), for given $n \geqslant d_1 + d_1 - 2$, we have

$$\lim_{p_s\to\infty} f_n^s = f^\infty \quad \text{and} \quad \lim_{p_s\to\infty} g_n^s = g^\infty.$$

Therefore, by (15), for given $n \geqslant d_1 + d_2 - 2$, we have

$$\begin{aligned} &\lim_{p_s\to\infty} F_n(R_s)(x)g_n^s(x) + F_n(S_s)(x)f_n^s(x) - f_n^s(x)g_n^s(x) \\ &\quad = F_R(x)g^\infty(x) + F_S(x)f^\infty(x) - f^\infty(x)g^\infty(x). \end{aligned}$$

Hence, for any $n \geqslant d_1 + d_2 - 2$,

$$\lim_{p_s\to\infty} f_n(R_s\#S_s, I_s\#J_s)(x) = F_R(x)g^\infty(x) + F_S(x)f^\infty(x) - f^\infty(x)g^\infty(x).$$

Now, by Theorem 1.1(1), the proposition follows. □

PROPOSITION 4.4. *Let the pairs $(R, I)$ and $(S, J)$ be as in Proposition 4.3. Let $(A, R_A, I_A)$, $(A, S_A, J_A)$ be spreads for $(R, I)$ and $(S, J)$, respectively, and let $s \in \operatorname{Spec}(A)$ be a closed point. Suppose that $f_{R_s,I_s} \geqslant f^\infty_{R,I}$ and $f_{S_s,J_s} \geqslant f^\infty_{S,J}$. Then,*

(1) $f_{R_s\#S_s, I_s\#J_s} \geqslant f^\infty_{R\#S, I\#J}$. *Moreover,*
(2) *if in addition $I_s \cap (R_s)_1 \neq 0$ and $J_s \cap (S_s)_1 \neq 0$, then*

$$f_{R_s,I_s} = f^\infty_{R,I} \quad \textit{and} \quad f_{S_s,J_s} = f^\infty_{S,J} \quad \Longleftrightarrow f_{R_s\#S_s, I_s\#J_s} = f^\infty_{R\#S, I\#J}.$$

*Proof.* (1) Let us denote $f^\infty = f^\infty_{R,I}$ and $g^\infty = g^\infty(S, J)$, and denote $f^s = f_{R_s,I_s}$ and $g^s = f_{S_s,J_s}$.

We know, by the multiplicative property of the HK density functions (see [T4, Proposition 2.18]), that

$$\begin{aligned} f_{R_s\#S_s,I_s\#J_s}(x) &= F_R(x)g^s(x) + F_S(x)f^s(x) - f^s(x)g^s(x) \\ &= (F_R(x) - f^s(x))g^s(x) + F_S(x)f^s(x) \\ &\geqslant (F_R(x) - f^s(x))g^\infty(x) + F_S(x)f^s(x) \\ &= F_R(x)g^\infty(x) + f^s(x)[F_S(x) - g^\infty(x)] \\ &\geqslant F_R(x)g^\infty(x) + f^\infty(x)[F_S(x) - g^\infty(x)] \\ &= f^\infty_{R\#S,I\#J}(x), \end{aligned}$$

where the third and fifth inequalities hold as $F_R(x) \geqslant f^s(x)$ and $F_S(x) \geqslant g^s(x)$, for every $s \in \operatorname{Spec} A$, and the last equality follows from Proposition 4.3.

(2) Suppose that $I$ and $J$ are the ideals of $R$ and $S$, respectively, and $s \in \operatorname{Spec}(A)$ is a closed point such that $I_s \cap (R_s)_1 \neq 0$ and $J_s \cap (S_s)_1 \neq 0$. Then, we make the following claim.

*Claim.* $F_R(x) > f^s(x)$ and $G_S(x) > g^s(x)$, for all $x \geqslant 1$.

*Proof of the Claim.* It is sufficient to prove that $F_R(x+1) > f^s(x+1)$, for $x > 0$. Choose an integer $n_0$ such that $x \geqslant 1/p_s^{n_0}$, where $p_s = \operatorname{char} k(s)$. Let $q = p_s^n$ for some $n$. For a given nonzero $y \in I_s \cap (R_s)_1$, we have an injective map of the $R_s$-linear map ($R$ is a domain) $\oplus_{m\geqslant 0}(R_s)_m \longrightarrow \oplus_{m\geqslant 0}(I_s^{[q]})_{m+q}$, of degree $q$, given by the multiplication by element $y^q$. Therefore, $\ell(I_s^{[q]})_{m+q} \geqslant \ell(R_s)_m$, for all $m \geqslant 0$. Since $\lfloor xq \rfloor = m$ if and only if $\lfloor (x+1)q \rfloor = m + q$, we have $\ell(I_s^{[q]})_{\lfloor (x+1)q\rfloor} \geqslant \ell(R_s)_{\lfloor xq \rfloor}$. Hence, $\ell(R_s/I_s^{[q]})_{\lfloor (x+1)q\rfloor} \leqslant \ell(R_s)_{\lfloor (x+1)q\rfloor} - \ell(R_s)_{\lfloor xq\rfloor}$.

Therefore, $f_n(R_s, I_s)(x+1) \leqslant F_n(R_s)(x+1) - F_n(R_s)(x)$.

However,

$$\lim_{n\to\infty} F_n(R_s)(x) = F_{R_s}(x) = \frac{e_0(R)x^{d-1}}{(d-1)!} \geqslant \frac{1}{(d-1)!}\frac{e_0(R)}{(p_s^{n_0})^{d-1}} > 0,$$

where $d = \dim R$. This implies that $f^s(x+1) = f_{R_s,I_s}(x+1) < F_R(x+1)$. This proves the claim.

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Now, retracing the above argument, we note that $f_{R_s\#S_s,I_s\#J_s} = f^{\infty}_{R\#S,I\#J}$ if and only if

$$[F_R(x) - f^s(x)]g^s(x) = [F_R(x) - f^s(x)]g^{\infty}(x)$$

and

$$[F_S(x) - g^{\infty}(x)]f^s(x) = [F_S(x) - g^{\infty}(x)]f^{\infty}(x).$$

Hence, by the above claim, we have $f^s(x) = f^{\infty}(x)$ and $g^s(x) = g^{\infty}(x)$ for all $x > 1$. For $x = 1$, we have $F_R(x) = f^s(x) = f^{\infty}(x)$ and $F_S(x) = g^s(x) = g^{\infty}(x)$. This proves the proposition. □

EXAMPLE 4.5. Let $R$ be a two-dimensional standard graded normal domain, where $R_0 = k$ is an algebraically closed field of char 0. Let $I = \mathbf{m} \subset R$ be the graded maximal ideal of $R$ generated by $h_1, \ldots, h_\mu$ of degree 1. Let $X = \text{Proj } R$ be the corresponding nonsingular projective curve, and let

$$0 \longrightarrow V \longrightarrow \oplus^{\mu}\mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

be the canonical short exact sequence of locally free sheaves of $\mathcal{O}_X$-modules. (Moreover, the sequence is locally split exact.)

Let $(A, R_A, I_A)$ and $(A, X_A, V_A)$ denote spreads for $(R, I)$ and $(X, V)$, respectively. Then, we have an associated canonical exact sequence of locally free sheaves of $\mathcal{O}_{X_A}$-modules

$$0 \longrightarrow V_A \longrightarrow \oplus^{\mu}\mathcal{O}_{X_A} \longrightarrow \mathcal{O}_{X_A}(1) \longrightarrow 0. \tag{16}$$

Restricting to the fiber $X_s$, we have the following exact sequence of locally free sheaves of $\mathcal{O}_{X_s}$-modules:

$$0 \longrightarrow V_s \longrightarrow \oplus^{\mu}\mathcal{O}_{X_s} \longrightarrow \mathcal{O}_{X_s}(1) \longrightarrow 0. \tag{17}$$

Moreover, we can choose a spread $(A, X_A, V_A)$ such that there is a filtration

$$0 = E_{0A} \subset E_{1A} \subset \cdots \subset E_{lA} \subset E_{l+1A} = V_A$$

of locally free sheaves of $\mathcal{O}_{X_A}$-modules such that

$$0 = E_{0s} \subset E_{1s} \subset \cdots \subset E_{ls} \subset E_{l+1s} = V_s$$

is the HN filtration of the vector bundles over $X_s$ for $s \in \text{Spec } A$.

THEOREM 4.6. *Let* $(R, I)$, $(A, R_A, I_A)$, *and* $(A, X_A, V_A)$ *be given as above. Then, for every closed point* $s \in \mathrm{Spec}(A)$, *we have*

(1) $f_{R_s,I_s} \geqslant f^{\infty}_{R,I}$ *and*
(2) $f_{R_s,I_s} = f^{\infty}_{R,I}$ *if and only if the filtration*

$$0 = E_{0s} \subset E_{1s} \subset \cdots \subset E_{ls} \subset E_{l+1s} = V_s$$

*is the strongly semistable HN filtration of* $V_s$ *on* $X_s$. *That is, for every* $n \geqslant 1$,

$$0 = F^{n*}E_{0s} \subset F^{n*}E_{1s} \subset \cdots \subset F^{n*}E_{ls} \subset F^{n*}E_{l+1s} = F^{n*}V_s$$

*is the HN filtration of* $F^{n*}V_s$.

*Proof.* We fix such an $s \in \mathrm{Spec}\ A$ and let $d = \deg X_s$ ($d$ independent of $s$), and let the HN filtration of $V_s$ be

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = V_s.$$

By [L, Theorem 2.7], there is $n_1 \geqslant 1$ such that $F^{n_1*}V_s$ has the strong HN filtration. (Note that $n_1$ may depend on $s$.)

Then, by [T2, Lemma 1.8], for $p_s = \mathrm{char}\ k(s) > 4(\mathrm{genus}(X_s))\mathrm{rank}(V_s)^3$, the HN filtration of $F^{n_1*}V_s$ is

$$\begin{aligned} 0 = E_{00} \subset E_{01} \subset \cdots \subset E_{0t_0} \subset E_{0,(t_0+1)} = F^{n_1*}E_1 = E_{10} \subset \cdots \subset \\ E_{i-1(t_{i-1}+1)} = F^{n_1*}E_i = E_{i0} \subset E_{i1} \subset \cdots \subset E_{it_i} \subset E_{i(t_i+1)} \\ E_{i(t_i+1)} = F^{n_1*}E_{i+1} = E_{i+1,0} \subset \cdots \subset F^{n_1*}V_s. \end{aligned}$$

Let, for $i \geqslant 0$ and $j \geqslant 1$,

$$a_{ij} = \frac{1}{p_s^{n_1}}\mu(E_{ij}/E_{i,j-1}) \quad \text{and} \quad r_{ij} = \mathrm{rank}(E_{ij}/E_{i,(j-1)}).$$

Let

$$\mu_0 = 1, \text{ and, for } i \geqslant 1, \text{ let } \mu_i = \mu(E_i/E_{i-1}) \quad \text{and} \quad r_i = \mathrm{rank}(E_i/E_{i-1}).$$

Note that, for any $i \geqslant 1$, the only possible inequalities are

$$a_{01} \geqslant \mu_1 \geqslant a_{0,(t_0+1)} > \cdots > a_{i0} \geqslant \mu_{i+1} \geqslant a_{i,(t_i+1)},$$

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and also note that $a_{ij} \leqslant 0$. By [T2, Lemma 1.14], for a given $i$,

$$a_{ij} = \mu_{i+1} + O(1/p_s), \tag{18}$$

where, by $O(1/p_s)$, we mean $O(1/p_s) = C/p_s$, where $|C|$ is bounded by a constant depending only on the degree of $X$ and rank of $V$ (and hence is independent of $p_s$).

*Claim.* If $1 - a_{ij_0}/d \leqslant x < 1 - a_{i(j_0+1)}/d$, for some $i \geqslant 0$ and $j_0 \geqslant 1$, then we have the following.

(1) $-[a_{ij}r_{ij} + d(x-1)r_{ij}] = -[\mu_{i+1}r_{ij} + d(x-1)r_{ij}] + O(1/p_s)$, for any $1 \leqslant k \leqslant t_i + 1$, and $-[a_{ij}r_{ij} + d(x-1)r_{ij}] \leqslant 0$ if $k \leqslant j_0$.
(2) $-\sum_{k \geqslant j_0+1}[a_{ik}r_{ik} + d(x-1)r_{ik}] \geqslant -[\mu_{i+1}r_{i+1} + d(x-1)r_{i+1}]$.

We skip the proof of the claim.

We also recall that, for $x$, as in the above claim, by [T4, Example 3.3], we have

$$f_{R_s,I_s}(x) = -\sum_{j \geqslant j_0+1}[a_{ij}r_{ij} + d(x-1)r_{ij}] - \sum_{k \geqslant i+1, j \geqslant 1}[a_{kj}r_{kj} + d(x-1)r_{kj}].$$

Let $x \geqslant 1$; then, $1 - \mu_i/d \leqslant x < 1 - \mu_{i+1}/d$, for some $i \geqslant 0$. (Note that $-\mu_i \geqslant 0$.) Now, there are three possibilities.

(1) $1 - \mu_i/d \leqslant x < 1 - a_{i-1,(t_{i-1}+1)}/d$. Then,

$$1 - \frac{a_{i-1,j_0}}{d} \leqslant x < 1 - \frac{a_{i-1,(j_0+1)}}{d}, \quad \text{for some } j_0 \geqslant 1,$$

and

$$\begin{aligned} f_{R_s,I_s}(x) = &-\sum_{j \geqslant j_0+1}[a_{i-1,j}r_{i-1,j} + d(x-1)r_{i-1,j}] \\ &-\sum_{k \geqslant i+2}[\mu_k r_k + d(x-1)r_k]. \end{aligned}$$

Therefore, by the above claim part (2),

$$f_{R_s,I_s}(x) \geqslant -\sum_{k \geqslant i+1}[\mu_k r_k + d(x-1)r_k].$$

(2) $1 - a_{i-1,(t_{i-1}+1)}/d \leqslant x < 1 - a_{i1}/d$. Then,

$$f_{R_s,I_s}(x) = -\sum_{k \geqslant i+1}[\mu_k r_k + d(x-1)r_k].$$

(3) $1 - a_{i1}/d \leqslant x < 1 - \mu_{i+1}/d$. Then

$$1 - \frac{a_{ij_0}}{d} \leqslant x < 1 - \frac{a_{i,(j_0+1)}}{d}, \quad \text{for some } j_0 \geqslant 1,$$

and

$$f_{R_s,I_s}(x) = -\sum_{j \geqslant j_0+1} [a_{ij}r_{ij} + d(x-1)r_{ij}] - \sum_{k \geqslant i+2} [\mu_k r_k + d(x-1)r_k].$$

Therefore, again by the above claim part (2),

$$f_{R_s,I_s}(x) \geqslant -\sum_{k \geqslant i+1} [\mu_k r_k + d(x-1)r_k].$$

By (18), we have that $f^{\infty}_{R,I} = \lim_{p_s \to \infty} f_{R_s,I_s}$ exists, and

$$1 \leqslant x < 1 - \mu_1/d \qquad \Rightarrow \quad f^{\infty}_{R,I}(x) = -\left[\sum_{i \geqslant 1} \mu_i r_i + d(x-1)r_i\right],$$

$$1 - \mu_i/d \leqslant x < 1 - \mu_{i+1}/d \quad \Rightarrow \quad f^{\infty}_{R,I}(x) = -\left[\sum_{k \geqslant i+1} \mu_k r_k + d(x-1)r_k\right].$$

This implies that $f_{R_s,I_s} \geqslant f^{\infty}_{R,I}$ for $1 \leqslant x < 1 - a_{l,(t_l+1)}/d$, and $f_{R_s,I_s} = f^{\infty}_{R,I} = 0$ otherwise. This proves part (1) of the theorem.

(2) If $V_s$ has strongly semistable HN filtration, then it is obvious that $f_{R_s,I_s} = f^{\infty}_{R,I}$. Let, as before, $n_1$ be such that $F^{n_1*}V$ has a strongly semistable HN filtration in the sense of [L, Theorem 2.7].

If the HN filtration of $V_s$ is not strongly semistable, then

$$0 = F^{n_1*}E_0 \subset F^{n_1*}E_1 \subset \cdots \subset F^{n_1*}E_l \subset F^{n_1*}E_{l+1} = F^{n_1*}V$$

is not the HN filtration of $F^{n_1*}V$. Therefore, there exists $i \geqslant 0$ such that

$$F^{n_1*}E_i = E_{i0} \subset E_{i1} \subset \cdots \subset F^{n_1*}E_{i+1},$$

where $E_{i1} \subset F^{n_1*}E_{i+1}$. Since $a_{i1} > \mu_{i+1}$, one can choose $1 - a_{i1}/d < x_0 \leqslant 1 - \mu_{i+1}/d \leqslant 1 - a_{i2}/d$. Now,

$$f_{R_s,I_s}(x) = -\sum_{j \geqslant 2} [a_{ij}r_{ij} + d(x-1)r_{ij}] - \sum_{k \geqslant i+2} [\mu_k r_k + d(x-1)r_k].$$

$$= [a_{i1}r_{i1} + d(x-1)r_{i1}] - \sum_{k \geqslant i+1} [\mu_k r_k + d(x-1)r_k] > f^{\infty}_{R,I}.$$

This proves the theorem. ∎

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Corollary 4.7. *Let $C_1 = \text{Proj } S_1, \ldots, C_n = \text{Proj } S_n$ be nonsingular projective curves, over a common field of characteristic 0. Suppose that each syzygy bundle $V_{C_i}$, given by*

$$0 \longrightarrow V_{C_i} \longrightarrow H^0(C_i, \mathcal{O}_{C_i}(1)) \otimes \mathcal{O}_{C_i} \longrightarrow \mathcal{O}_{C_i}(1) \longrightarrow 0,$$

*is semistable. (For example, if $\deg \mathcal{O}_{C_i}(1) > 2\text{genus}(C_i)$, then $V_{C_i}$ is semistable; see [PR] and [T6, Lemma 2.1].)*

*Then, there is $n_0$ such that for all $p \geqslant n_0$ we have*

(1) $f_{(S_1\#\cdots\#S_n)_p}(x) \geqslant f^{\infty}_{S_1\#\cdots\#S_n}(x)$ *and*
(2) $f_{(S_1\#\cdots\#S_n)_p}(x) = f^{\infty}_{S_1\#\cdots\#S_n}(x)$, *for all $x \in \mathbb{R}$, if and only if (mod $p$) reduction of the bundle $V_1 \boxtimes \cdots \boxtimes V_n$ is strongly semistable on $(C_1 \times \cdots \times C_n)_p$.*

*In particular,*

(1) $e^{\infty}_{HK}(S_1\# \cdots \#S_n)$ *exists and* $e_{HK}((S_1\# \cdots \#S_n)_p) \geqslant e^{\infty}_{HK}(S_1\# \cdots \#S_n)$, *and*
(2) $e_{HK}((S_1\# \cdots \#S_n)_p) = e^{\infty}_{HK}(S_1\# \cdots \#S_n)$ *if and only if (mod $p$) reduction of the bundle $V_1 \boxtimes \cdots \boxtimes V_n$ is strongly semistable on $(C_1 \times \cdots \times C_n)_p$,*

*where the HK density functions and HK multiplicities are considered with respect to the ideal $\mathbf{m}_1\# \cdots \#\mathbf{m}_n$ for the graded maximal ideals $\mathbf{m}_i \subset S_i$.*

*Proof.* The proof follows by Proposition 4.4 and Theorem 4.6. ∎

Remark 4.8. With the notations and assumptions as in the corollary above, one can easily compute $f^{\infty}_{S_1\#\cdots\#S_n}$, in terms of ranks of $V_i$ and degrees of $C_i$. In particular, if $d_1 = \deg C_1$ and $d_2 = \deg C_2$, with $r = \text{rank } V_1 \geqslant s = \text{rank } V_2$, then it follows that

$$e^{\infty}_{HK}(S_1\#S_2) = \frac{d_1 d_2}{3} + d_1 d_2 \left[\frac{1}{2s} + \frac{1}{6s^2} + \frac{1}{6r^2} + \frac{s}{6r^2}\right].$$

Notations 4.9. Let $R = k[x, y, z]/(h)$ be a plane trinomial curve of degree $d$. That is, $h = M_1 + M_2 + M_3$, where the $M_i$ are monomials of degree $d$. As given in [Mo2, Lemma 2.2], one can divide such an $h$ into two types.

(1) $h$ is irregular; that is, one of the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ of $\mathbb{P}^2$ has multiplicity $\geqslant d/2$ on the plane curve $h$. Here, we define $\lambda_R = 1$.

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(2) $h$ is regular and hence is one of the following types (up to a change of variables).

(a) $h = x^{a_1}y^{a_2} + y^{b_1}z^{b_2} + z^{c_1}x^{c_2}$, where $a_1, b_1, c_1 > d/2$. Here, we define $\alpha = a_1 + b_1 - d$, $\beta = a_1 + c_1 - d$, $\nu = b_1 + c_1 - d$, and $\lambda = a_1b_1 + a_2c_2 - b_1c_2$.

(b) $h = x^d + x^{a_1}y^{a_2}z^{a_3} + y^bz^c$, where $a_2, c > d/2$. Here, we define $\alpha = a_2$, $\beta = c$, $\nu = a_2 + c - d$, and $\lambda = a_2c - a_3b$.

We denote $\lambda_h = \lambda/a$, where $a = \text{g.c.d.}(\alpha, \beta, \nu, \lambda)$.

COROLLARY 4.10. *Let $S_1, \ldots, S_n$ be a set of irreducible plane trinomial curves given by trinomials $h_1, \ldots, h_n$ of degree $d_1, \ldots, d_n \geqslant 4$, respectively, over a field of characteristic 0. Then, there are spreads $\{(A_i, S_{iA}, \mathbf{m}_{iA})\}_i$ such that for every closed point $s \in \operatorname{Spec}(A)$,*

(1) *$f^s(S_1\# \cdots \#S_n)(x) = f^{\infty}_{S_1\#\cdots\#S_n}(x)$, for all $x \in \mathbb{R}$, if char $k(s) \equiv \pm 1 \pmod{l.c.m.(\lambda_{h_1}, \ldots, \lambda_{h_n})}$, where $\lambda_{h_i}$ is given as in Notations 4.9. In particular, there are infinitely many primes $p_s = \text{char } k(s)$ such that the function $f_{(S_1\#\cdots\#S_n)_{p_s}} - f^{\infty}_{S_1\#\cdots\#S_n} = 0$. Moreover,*

(2) *if, in addition, one of the curves, say $S_1$, is given by a symmetric trinomial $h_1 = x^{a_1}y^{a_2} + y^{a_1}z^{a_2} + z^{a_1}x^{a_2}$ such that $d \neq 5$, then*

$$f^s(S_1\# \cdots \#S_n)(x_0) > f^{\infty}_{S_1\#\cdots\#S_n}(x_0), \quad \textit{if}\, \text{char } k(s) \equiv \pm l \pmod{\lambda_{h_1}},$$

*for some $x_0 \in \mathbb{R}$ and for some $l \in (\mathbb{Z}/\lambda_{h_1}\mathbb{Z})^*$. In particular, there are infinitely many primes $p_s = \text{char } k(s)$ such that $f^s(S_1\# \cdots \#S_n) - f^{\infty}_{S_1\#\cdots\#S_n} \neq 0$.*

*Proof.* We can choose spreads $(A, S_{iA})$ with the property that char $k(s) > \max\{d_1, \ldots, d_n\}^2$, for every closed point $s \in \operatorname{Spec}(A)$. Now, for any irreducible plane curve given by $S = k[x, y, z]/(h)$, let $S \longrightarrow \tilde{S}$ be the normalization of $S$. Then, it is a finite graded map of degree 0 and $Q(S) = Q(\tilde{S})$ such that $\tilde{S}$ is a finitely generated $\mathbb{N}$-graded two-dimensional domain over $k$. Now, for pairs $(S, \mathbf{m})$ and $(\tilde{S}, \mathbf{m}\tilde{S})$, we can choose a spread $(A, S_A, \mathbf{m}_A)$ and $(A, \tilde{S}_A, \mathbf{m}\tilde{S}_A)$ such that for every closed point $s \in \operatorname{Spec}(A)$, the natural map $S_s = S_A \otimes k(s) \longrightarrow \tilde{S}_A \otimes k(s)$ is a finite graded map of degree 0. This implies, for every $x \geqslant 0$,

$$\lim_{q\to\infty} \frac{1}{q}\ell\left(\frac{S_s}{\mathbf{m}^{[q]}}\right)_{\lfloor xq\rfloor} = \lim_{q\to\infty} \frac{1}{q}\ell\left(\frac{\tilde{S}_s}{\mathbf{m}\tilde{S}_s^{[q]}}\right)_{\lfloor xq\rfloor},$$

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as kernel and cokernel of the map $S_s \longrightarrow \tilde{S}_s$ are zero-dimensional. Therefore, $f_{S_s,\mathbf{m}_s} = f_{\tilde{S}_s,\mathbf{m}\tilde{S}_s}$ and $f^{\infty}_{S,\mathbf{m}} = f^{\infty}_{\tilde{S},\mathbf{m}\tilde{S}}$. This also implies that $e_{HK}(S_s, \mathbf{m}_s) = e_{HK}(\tilde{S}_s, \mathbf{m}\tilde{S}_s)$. (This inequality about $e_{HK}$ can also be found in [Mo1, Lemma 1.3], [WY1, Theorem 2.7], and [BCP].) Let $\pi : \tilde{X}_s = \mathrm{Proj}\ \tilde{S}_s \longrightarrow X_s = \mathrm{Proj}\ S_s$ be the induced map. We also choose a spread $(A, X_A, V_A)$, where $V_A$ is given by

$$0 \longrightarrow V_A \longrightarrow \mathcal{O}_{X_A} \oplus \mathcal{O}_{X_A} \oplus \mathcal{O}_{X_A} \longrightarrow \mathcal{O}_{X_A}(1) \longrightarrow 0$$

and gives the syzygy bundle $V_s$ with its HN filtration as given in Example 4.5.

This gives a short exact sequence of sheaves of $\mathcal{O}_{\tilde{X}_s}$-modules

$$0 \longrightarrow \pi^* V_s \longrightarrow \mathcal{O}_{\tilde{X}_s} \oplus \mathcal{O}_{\tilde{X}_s} \oplus \mathcal{O}_{\tilde{X}_s} \longrightarrow \mathcal{O}_{\tilde{X}_s}(1) \longrightarrow 0.$$

Moreover, $\tilde{X}_s$ is a nonsingular curve. If $S$ is regular trinomial given by $h$, then, by [T5, Theorem 5.6], the bundle $\pi^*(V_s)$ is strongly semistable, provided that char $k(s) \equiv \pm 1 \pmod{2\lambda_{h_s}}$. Therefore, by Theorem 4.6, we have $f_{\tilde{S}_s,\mathbf{m}\tilde{S}_s} = f^{\infty}_{\tilde{S},\mathbf{m}\tilde{S}}$. This implies that $f_{S_s,\mathbf{m}_s} = f^{\infty}_{S,\mathbf{m}}$, for char $k(s) \equiv \pm 1 \pmod{2\lambda_{S_s}}$.

If $S$ is an irregular trinomial, then, by [T5, Theorem 1.1], $\pi^* V$ has an HN filtration $0 \subset \mathcal{L} \subset \pi^* V$. Therefore, $0 \subset \mathcal{L}_s \subset \pi^* V_s$ is the HN filtration and hence the strong HN filtration (as rank $V = 2$), for $\pi^* V_s$, for every closed point $s \in \mathrm{Spec}\ A$. In particular, by Theorem 4.6, $f_{S_s,\mathbf{m}_s} = f^{\infty}_{S,\mathbf{m}}$, for all such $s$. Now, assertion (1) follows by Proposition 4.4(2).

If $S_1 = k[x, y, z]/(h_1)$, where $h_1$ is as in statement (2) of the corollary, then $\pi^* V_s$ is semistable, but not strongly semistable, if char $k(s) \equiv \pm 2 \pmod{\lambda_{h_{1s}}}$. In particular, by Corollary 4.7, $f_{S_{1s},\mathbf{m}S_{1s}} > f^{\infty}_{S_{1s},\mathbf{m}S_{1s}}$, for such $s$. Therefore, the statement (2) follows from Proposition 4.4(2). □

## Appendix A

LEMMA A.1. *For an integer $d \geqslant 2$, there exist universal polynomials $P_i^d$, $P_i'^d$ in $\mathbb{Q}[X_0, \ldots, X_i]$, where $0 \leqslant i \leqslant d-2$, such that if, for a pair $(X, \mathcal{O}_X(1))$, we have $X$ an integral projective variety of char $p > 0$ and dimension $d-1$, and $\mathcal{Q}$ a coherent sheaf of $\mathcal{O}_X$-modules of* $\dim(\mathrm{Supp})\mathcal{Q} = d-2$ *and with the following respective Hilbert polynomials (where* $\dim \mathrm{supp} \mathcal{Q} \leqslant d-2$*):*

$$\chi(X, \mathcal{O}_X(m)) = \tilde{e}_0 \binom{m+d-1}{d-1} - \tilde{e}_1 \binom{m+d-2}{d-2} + \cdots + (-1)^{d-1} \tilde{e}_{d-1}$$

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*and*

$$\chi(X, \mathcal{Q}(m)) = q_0\binom{m+d-2}{d-2} - q_1\binom{m+d-3}{d-3} + \cdots + (-1)^{d-2}q_{d-2},$$

*then we have the following.*

(1) *For* $0 \leqslant i \leqslant d-2$, *we have* $|q_i| \leqslant p^{d-1}P_i^d(\tilde{e}_0, \ldots, \tilde{e}_{i+1})$, *if there is a short exact sequence of* $\mathcal{O}_X$*-modules*

$$0 \longrightarrow \oplus^{p^{d-1}}\mathcal{O}_X(-d) \longrightarrow F_*\mathcal{O}_X \longrightarrow \mathcal{Q} \longrightarrow 0.$$

(2) *For* $0 \leqslant i \leqslant d-2$, *we have* $|q_i| \leqslant m_0^{i+1}P_i'^d(\tilde{e}_0, \ldots, \tilde{e}_i)$, *if* $\mathcal{Q}$ *fits in the short exact sequence*

$$0 \longrightarrow \mathcal{O}_X(-m_0) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{Q} \longrightarrow 0$$

*or in the short exact sequence*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(m_0) \longrightarrow \mathcal{Q} \longrightarrow 0$$

*of* $\mathcal{O}_X$*-modules.*

*Proof.* Assertion (1): Note that for $m \in \mathbb{Z}$, we have

$$\chi(X, \mathcal{Q}(m)) = \chi(X, O_X(mp)) - p^{d-1}\chi(X, \mathcal{O}_X(m)). \tag{A.1}$$

We can express, for $1 \leqslant n \leqslant d-1$,

$$(Y+n)\cdots(Y+2)(Y+1) = \sum_{j=0}^{n} C_j^n Y^j,$$

where $C_n^n = 1$, and, for $j < n$, the coefficient $C_j^n$ is in the set

$$\left\{\sum x_1^{i_1}\cdots x_n^{i_n} \;\middle|\; \sum i_l = n-j,\ 0 \leqslant j < n \leqslant d-1, \{x_1, \ldots, x_n\} \right.$$
$$\left. = \{1, \ldots, n\}\right\}.$$

Now, expanding Equation (A.1), we get

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$$
\begin{aligned}
&\frac{\tilde{e}_0}{(d-1)!}\,[C^{d-1}_{d-2}m^{d-2}(p^{d-2}-p^{d-1})+C^{d-1}_{d-3}m^{d-3}(p^{d-3}-p^{d-1})\\
&\quad+\cdots+C^{d-1}_{0}(1-p^{d-1})]\\
&\quad+\cdots+\frac{(-1)^i\tilde{e}_i}{(d-1-i)!}\,[C^{d-1-i}_{d-1-i}m^{d-1-i}(p^{d-1-i}-p^{d-1})\\
&\quad+C^{d-1-i}_{d-2-i}m^{d-2-i}(p^{d-2-i}-p^{d-1})\\
&\quad+\cdots+C^{d-1-i}_{0}(1-p^{d-1})]+\cdots+(-1)^{d-1}\tilde{e}_{d-1}[(1-p^{d-1})]\\
&=\frac{q_0}{(d-2)!}[C^{d-2}_{d-2}m^{d-2}+C^{d-2}_{d-3}m^{d-3}+\cdots+C^{d-2}_{0}]\\
&\quad-\frac{q_1}{(d-3)!}[C^{d-3}_{d-3}m^{d-3}+C^{d-3}_{d-4}m^{d-4}+\cdots+C^{d-3}_{0}]+\cdots\\
&\quad+\frac{(-1)^{i-1}q_{i-1}}{(d-1-i)!}[C^{d-1-i}_{d-1-i}m^{d-1-i}+C^{d-1-i}_{d-2-i}m^{d-2-i}+\cdots+C^{d-1-i}_{0}]\\
&\quad+\cdots+(-1)^{d-2}q_{d-2}.
\end{aligned}
$$

We prove the result for $q_i$, by induction on $i$. For $i=0$, comparing the coefficients of $m^{d-2}$ on both sides, we get

$$(p^{d-2}-p^{d-1})\left[\frac{\tilde{e}_0}{(d-1)!}C^{d-1}_{d-2}-\frac{\tilde{e}_1}{(d-2)!}\right]=\frac{q_0}{(d-2)!},$$

which implies

$$|q_0|\leqslant p^{d-1}(|\tilde{e}_0|C^{d-1}_{d-2}+|\tilde{e}_1|)\leqslant p^{d-1}(\tilde{e}_0^2C^{d-1}_{d-2}+\tilde{e}_1^2).$$

Comparing coefficients of $m^{d-i}$, we get

$$
\begin{aligned}
&(p^{d-i}-p^{d-1})\\
&\quad\times\left[\frac{\tilde{e}_0}{(d-1)!}C^{d-1}_{d-i}-\frac{\tilde{e}_1}{(d-2)!}C^{d-2}_{d-i}+\cdots+(-1)^{i-1}\frac{\tilde{e}_{i-1}}{(d-i)!}C^{d-1-i}_{d-i}\right]\\
&=\frac{q_0}{(d-2)!}C^{d-2}_{d-i}-\frac{q_1}{(d-3)!}C^{d-3}_{d-i}+\cdots+(-1)^i\frac{q_{i-2}}{(d-i)!}C^{d-i}_{d-i}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|q_{i-2}|\leqslant p^{d-1}&[|\tilde{e}_0|C^{d-1}_{d-i}+|\tilde{e}_1|C^{d-2}_{d-i}+\cdots+|\tilde{e}_{i-1}|C^{d-1-i}_{d-i}]\\
&+[|q_0|C^{d-2}_{d-i}+|q_1|C^{d-3}_{d-i}+\cdots+|q_{i-3}|C^{d+1-i}_{d-i}].
\end{aligned}
$$

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However,

$$p^{d-1}[|\tilde{e}_0|C^{d-1}_{d-i} + |\tilde{e}_1|C^{d-2}_{d-i} + \cdots + |\tilde{e}_{i-1}|C^{d-1-i}_{d-i}] \\ \leqslant p^{d-1}[\tilde{e}_0^2 C^{d-1}_{d-i} + \tilde{e}_1^2 C^{d-2}_{d-i} + \cdots + \tilde{e}_{i-1}^2 C^{d-1-i}_{d-i}].$$

Now, the proof follows by induction.

Assertion (2): For $m_0 = 0$, the statement is true vacuously. Therefore, we can assume that $m_0 \geqslant 1$. Now,

$$\begin{aligned}\chi(X, \mathcal{Q}(m)) &= q_0\binom{m+d-2}{d-2} - q_1\binom{m+d-3}{d-3} + \cdots + (-1)^{d-2}q_{d-2}, \\ &= \frac{q_0}{(d-2)!}[D^{d-2}_{d-2}m^{d-2} + D^{d-2}_{d-3}m^{d-3} + \cdots + D^{d-2}_0] \\ &\quad - \frac{q_1}{(d-3)!}[D^{d-3}_{d-3}m^{d-3} + D^{d-3}_{d-4}m^{d-4} + \cdots + D^{d-3}_0] \\ &\quad + \cdots + \frac{(-1)^{i-1}q_{i-1}}{(d-1-i)!}\,[D^{d-1-i}_{d-1-i}m^{d-1-i} + D^{d-1-i}_{d-2-i}m^{d-2-i} \\ &\quad + \cdots + D^{d-1-i}_0] + \cdots + (-1)^{d-2}q_{d-2},\end{aligned}$$

where $D^k_j$ belongs to the set

$$\left\{\sum x^{i_1}\cdots x_k^{i_k} \,\middle|\, \sum i_l = k-j \quad 0 \leqslant j \leqslant k \leqslant d-2, \right. \\ \left. \{x_1, \ldots, x_k\} = \{1, \ldots, k\}\right\}.$$

On the other hand,

$$\begin{aligned}&\chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}_X(m-m_0)) \\ &= \frac{\tilde{e}_0}{(d-1)!}\,[C^{d-1}_{d-1}(m_0)(m^{d-2} + \cdots m^{d-3}m_0 + \cdots + m_0^{d-2}) \\ &\quad + C^{d-1}_{d-2}(m_0)(m^{d-3} + m^{d-4}m_0 + \cdots + m_0^{d-3}) + \cdots + C^{d-1}_1(m_0)] \\ &\quad - \frac{\tilde{e}_1}{(d-2)!}\,[C^{d-2}_{d-2}(m_0)(m^{d-3} + \cdots m^{d-4}m_0 + \cdots + m_0^{d-3}) \\ &\quad + C^{d-2}_{d-3}(m_0)(m^{d-4} + m^{d-5}m_0 + \cdots + m_0^{d-4}) + \cdots + C^{d-2}_1(m_0)] \\ &\quad + \cdots .\end{aligned}$$

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Again, we prove the result for $q_i$, by induction on $i$. Comparing the coefficients for $m^{d-2}$, we get

$$\frac{q_0}{(d-2)!}D_{d-2}^{d-2} = \frac{\tilde{e}_0}{(d-1)!}C_{d-1}^{d-1}m_0 \Rightarrow |q_0| \leqslant \tilde{e}_0 \frac{C_{d-1}^{d-1}m_0}{|D_{d-2}^{d-2}|} \leqslant \tilde{e}_0^2 m_0.$$

Comparing the coefficients of $m^{d-i}$, where $2 \leqslant i \leqslant d$, we get

$$\begin{aligned} &\frac{q_0}{(d-2)!}D_{d-i}^{d-2} - \frac{q_1}{(d-3)!}D_{d-i}^{d-3} + \cdots + (-1)^{i-2}\frac{q_{i-2}}{(d-i)!}D_{d-i}^{d-i} \\ &= \frac{\tilde{e}_0}{(d-1)!}(C_{d-1}^{d-1}m_0^{i-1} + C_{d-2}^{d-1}m_0^{i-2} + \cdots + C_{d-i+1}^{d-1}m_0) \\ &\quad - \frac{\tilde{e}_1}{(d-2)!}(C_{d-2}^{d-2}m_0^{i-2} + C_{d-3}^{d-2}m_0^{i-3} + \cdots + C_{d-i+1}^{d-2}m_0) \\ &\quad + \cdots (-1)^{i-2}\frac{\tilde{e}_{i-2}}{(d+1-i)!}(C_{d+1-i}^{d-i+1}). \end{aligned}$$

This implies that

$$\begin{aligned} |q_{i-2}||D_{d-i}^{d-i}| \leqslant\ & |\tilde{e}_0|(C_{d-1}^{d-1}m_0^{i-1} + \cdots + C_{d-i+1}^{d-1}m_0) \\ & + |\tilde{e}_1|(C_{d-2}^{d-2}m_0^{i-2} + \cdots + C_{d-i+1}^{d-2}m_0) \\ & + \cdots + |\tilde{e}_{i-2}|(C_{d+1-i}^{d+1-i}) \\ & + (|q_0||D_{d-i}^{d-2}| + |q_1||D_{d-i}^{d-3}| + \cdots + |q_{i-3}||D_{d-i}^{d+1-i}|). \end{aligned}$$

Now, the proof follows by induction.

For $\mathcal{Q}$ such that $0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(m_0) \longrightarrow \mathcal{Q} \longrightarrow 0$, we have $\chi(X, \mathcal{Q}(m - m_0)) = \chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}(m - m_0))$, so we get the same bounds for the $q_i$ in terms of the $\tilde{e}_j$ as above except that now $D_j^n$ is in the set

$$\left\{ \sum x_1^{i_1} \cdots x_n^{i_n} \;\middle|\; \sum_{l=1}^{n} i_l = n - j, 0 \leqslant j \leqslant n, \right. \\ \left. \{x_1, \ldots, x_n\} = \{1 - m_0, \ldots, n - m_0\} \right\}.$$

Hence, the lemma follows. ∎

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