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Uniqueness Implies Existence and Uniqueness Conditions for a Class of $(k + j)$-Point Boundary Value Problems for n-th Order Differential Equations

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Abstract. For the n-th order nonlinear differential equation, $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$, we consider uniqueness implies uniqueness and existence results for solutions satisfying certain $(k + j)$-point boundary conditions for $1 \le j \le n-1$ and $1 \le k \le n-j$. We define $(k; j)$-point unique solvability in analogy to k-point disconjugacy and we show that $(n-j_0; j_0)$-point unique solvability implies $(k; j)$-point unique solvability for $1 \le j \le j_0$, and $1 \le k \le n-j$. This result is analogous to n-point disconjugacy implies k-point disconjugacy for $2 \le k \le n-1$.

1 Introduction

In this paper we are concerned with uniqueness and existence of solutions for a class of boundary value problems for the n-th order ordinary differential equation, $n \ge 3$,

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \quad a < x < b, \tag{1.1}$$

subject to $n-j$ conjugate boundary conditions followed by j nonlocal boundary conditions. In particular, given $1 \le j \le n-1$, $1 \le k \le n-j$, positive integers $m_1, \dots, m_k$ such that $m_1 + \dots + m_k = n-j$, points $a < x_1 < x_2 < \dots < x_k < x_{k+1} < \dots < x_{k+2j} < b$, real values y_{il}, $1 \le i \le m_l$, $1 \le l \le k$, and real values y_n, y_{n-1}, $y_{n-(j-1)}$, we are concerned with uniqueness implies uniqueness and existence questions for solutions of (1.1) satisfying the conjugate and nonlocal boundary conditions of the type,

$$\begin{aligned} (1.2)\qquad & y^{(i-1)}(x_l) = y_{il},\ 1 \le i \le m_l,\ 1 \le l \le k, \qquad \text{conjugate conditions},\\ & (a_1 y(x_{k+1}) - a_2 y(x_{k+2}), \dots, a_{2j-1} y(x_{k+2j-1}) - a_{2j} y(x_{k+2j}))\\ & \qquad = (y_n, y_{n-1}, \dots, y_{n-(j-1)}), \qquad \text{nonlocal conditions}, \end{aligned}$$

where $a_1, a_2, \dots, a_{2j}$ are positive real numbers. We shall refer to the boundary conditions, (1.2), as $(k; j)$-point boundary conditions. The boundary conditions $(k; 0)$ are referred to as *conjugate* type boundary conditions [15].

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We shall also refer to the $(k; j)$-*point unique solvability* of (1.1) on (a, b) where (1.1) is $(k; j)$-point uniquely solvable on (a, b) if given $1 \leq j \leq n-1$ and $1 \leq k \leq n-j$, positive integers $m_1, \ldots, m_k$ such that $m_1 + \cdots + m_k = n - j$, points $a < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_{k+2j} < b$, real values y_{il}, $1 \leq i \leq m_l$, $1 \leq l \leq k$, real values y_{n-j+l}, $1 \leq l \leq j$, and positive real numbers $a_1, a_2, \ldots, a_{2j}$, then the boundary value problem, (1.1), (1.2), is uniquely solvable.

Questions of the types with which we deal in this paper have been considered for solutions of (1.1) satisfying α-point conjugate boundary conditions. In particular, for boundary value problems for (1.1) satisfying, for $2 \leq \alpha \leq n$, conjugate boundary conditions of the form,

$$y^{(i-1)}(t_l) = r_{il}, 1 \leq i \leq p_l, 1 \leq l \leq \alpha, \tag{1.3}$$

where $p_1, \ldots, p_\alpha$ are positive integers such that $p_1 + \cdots + p_\alpha = n$, $a < t_1 < \cdots < t_\alpha < b$, and $r_{ij} \in \mathbb{R}$, $1 \leq i \leq p_j$, $1 \leq j \leq \alpha$. These questions have involved:

(i) whether uniqueness of solutions of (1.1), (1.3) for $\alpha = n$ implies uniqueness of solutions of (1.1), (1.3) for $2 \leq \alpha \leq n-1$,

(ii) whether uniqueness of solutions of (1.1), (1.3) for $\alpha = n$ implies existence of solutions of (1.1), (1.3) for $2 \leq \alpha \leq n$.

Of course, a primary reason for considering question (i) would be to resolve question (ii).

Hypothesis 1.1 *With respect to equation* (1.1)*, we assume throughout that*

(A) $f(t, s_1, \ldots, s_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ *is continuous;*

(B) *solutions of initial problems for* (1.1) *are unique and extend to* (a, b).

Given Hypothesis 1.1, Jackson [15] established that (i) is true. In independent works, Hartman [6, 7] and Klaasen [18] provided a positive answer to question (ii).

Several other papers have been devoted to uniqueness questions of these types as well as uniqueness implies existence questions for boundary value problems. These works have dealt not only with ordinary differential equations [2, 8, 9, 16, 19, 20], but also with boundary value problems for finite difference equations [10, 11], and recently with dynamic equations on time scales [5, 14]. Some questions of these types have also received recent attention for nonlocal boundary value problems for (1.1), for the cases of $n = 2, 3, 4$; see [1, 4, 12, 13]. Recently, the case of nonlocal conditions for equations of arbitrary order n with $j = 1$ has been addressed [3, 17].

2 Uniqueness of Solutions

Let $j_0 \in \{1, \ldots, n-1\}$. Under Hypothesis 1.1, we establish in this section that uniqueness of solutions for the $(n - j_0, j_0)$-point boundary value problem implies uniqueness of solutions for the (k, j)-point boundary value problem for $0 \leq j \leq j_0$, $1 \leq k \leq n - j$.

First we first shall obtain continuous dependence of solutions of (1.1) on boundary conditions.

Theorem 2.1 *Let $j \in \{1, \dots, n-1\}$. Assume that for some $1 \le k \le n-j$, and positive integers $m_1, \dots, m_k$ such that $m_1 + \cdots + m_k = n - j$, solutions of the corresponding boundary value problem* (1.1), (1.2) *are unique when they exist. Given a solution $y(x)$ of* (1.1)*, an interval $[c, d]$, points $c < x_1 < \cdots < x_k < \cdots < x_{k+2j} < d$, and an $\epsilon > 0$, there exists $\delta(\epsilon, [c, d]) > 0$ such that, if $|x_i - \xi_i| < \delta$, $1 \le i \le k + 2j$, and $c < \xi_1 < \cdots < \xi_k < \cdots < \xi_{k+2j} < d$, and if $|y^{(i-1)}(x_l) - z_{il}| < \delta$, $1 \le i \le m_l$, $1 \le l \le k$, and*

$$|a_{2i-1} y(x_{k+2i-1}) - a_{2i} y(x_{k+2i}) - z_{n-(i-1)}| < \delta, \quad 1 \le i \le j,$$

then there exists a solution $z(x)$ of (1.1) *satisfying $z^{(i-1)}(\xi_l) = z_{il}$ for $1 \le i \le m_l, 1 \le l \le k$,*

$$(a_1 z(\xi_{k+l}) - a_2 z(\xi_{k+2}), \dots, a_{2j-1} z(\xi_{k+2j-1}) - a_{2j} z(\xi_{k+2j})) = (z_n, \dots, z_{n-(j-1)}),$$

and $|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon$ on $[c, d]$, $1 \le i \le n$.

Proof Fix a point $p_0 \in (c, d)$ and define the set

$$G = \{(s_1, \dots, s_{k+2j}, c_1, \dots, c_n) \mid c < s_1 < \cdots < s_{k+2j} < d, \ c_1, \dots, c_n \in \mathbb{R}\}.$$

Then G is an open subset of $\mathbb{R}^{k+2j+n}$. Let $u(x)$ be a solution of the initial value problem for (1.1) satisfying the initial conditions $u^{(i-1)}(p_0) = c_i$, $1 \le i \le n$. Define a mapping $\phi\colon G \to \mathbb{R}^{k+1+n}$ by

$$\begin{aligned} &\phi(s_1, \dots, s_{k+2j}, c_1, \dots, c_n) = \\ &\qquad \big(s_1, \dots, s_{k+2j}, u(s_1), \dots, u^{(m_1-1)}(s_1), \dots, u(s_k), \dots, u^{(m_k-1)}(s_k), \\ &\qquad\qquad a_1 u(s_{k+1}) - a_2 u(s_{k+2}), \dots, a_{2j-1} u(s_{k+2j-1}) - a_{2j} u(s_{k+2j})\big). \end{aligned}$$

Condition (B) in Hypothesis 1.1 implies the continuity of solutions of initial value problems for (1.1) with respect to initial conditions, from which we have the continuity of ϕ. In addition, the uniqueness assumption on solutions of (1.1) and (1.2) for the given k and $m_1, \dots, m_k$ in the present context implies that ϕ is one-to-one. It follows from the Brouwer theorem on invariance of domain [22] that $\phi(G)$ is an open subset of $\mathbb{R}^{k+2j+n}$ and that ϕ is a homeomorphism from G to $\phi(G)$. The statement of the theorem follows directly from the continuity of ϕ^{-1} and the fact that $\phi(G)$ is open. ∎

We now establish that for $k = n - j$, uniqueness of solutions of the $(n - j; j)$-point BVP (1.1), (1.2), implies uniqueness of solutions of the $(n - j + i; j - i)$-point BVP (1.1), (1.2), for $i = 1, 2, \dots, j$.

Theorem 2.2 *Assume that for $k = n - j$, solutions of the $(n - j; j)$-point BVP* (1.1), (1.2) *are unique when they exist. Then for each $i = 1, 2, \dots, j$, solutions of the $(n - j + i; j - i)$-point BVP* (1.1), (1.2) *are unique when they exist.*

Proof Assume uniqueness of solutions of the $(n-j;j)$-point BVP (1.1), (1.2). The proof is by induction on i, and we begin by showing that solutions of the $(n-j+1;j-1)$-point BVP are unique. Assume the conclusion is not true and there exist points $a < x_1 < \cdots < x_{n-j+1} < \cdots < x_{n+j-1} < b$ for which there exist distinct solutions $y(x)$ and $z(x)$ of the $(n-j+1;j-1)$-point BVP such that

$$
\begin{aligned}
&y(x_l) = z(x_l),\ 1 \le l \le n-j+1,\\
&a_1 y(x_{n-j+1+1}) - a_2 y(x_{n-j+1+2}) = a_1 z(x_{n-j+1+1}) - a_2 z(x_{n-j+1+2}),\\
&\quad\vdots\\
&a_{2j-3} y(x_{n-j+1+2j-3}) - a_{2j-2} y(x_{n-j+1+2j-2})\\
&\quad = a_{2j-3} z(x_{n-j+1+2j-3}) - a_{2j-2} z(x_{n-j+1+2j-2}).
\end{aligned}
$$

Define $w = y - z$; then we obtain

$$
\begin{aligned}
&w(x_l) = 0,\ 1 \le l \le n-j+1,\\
&a_1 w(x_{n-j+1+1}) - a_2 w(x_{n-j+1+2}) = 0,\\
&\quad\vdots\\
&a_{2j-3} w(x_{n-j+1+2j-3}) - a_{2j-2} y(x_{n-j+1+2j-2}) = 0.
\end{aligned}
$$

If there exists some $t_1 \in (x_{n-j+1}, x_{n-j+2})$ such that $w(t_1) = 0$, then we have

$$aw(x_{n-j+1}) - bw(t_1) = 0, a, b \in \mathbb{R}.$$

This implies that $y(x)$ and $z(x)$ are distinct solutions of the $(n-j;j)$-point BVP at the points $x_1, \dots, x_{n-j}, x_{n-j+1}, t_1, x_{n-j+2}, x_{n-j+2}, \dots, x_{n+j-2}, x_{n+j-1}$, which is a contradiction. Hence, $w(t) \neq 0$ on (x_{n-j+1}, x_{n-j+2}). Let $w(t) > 0$ on (x_{n-j+1}, x_{n-j+2}). The case $w(t) < 0$ on (x_{n-j+1}, x_{n-j+2}) can be dealt with similarly. Then there exists $\tau \in [x_{n-j+1}, (x_{n-j+1} + x_{n-j+2})/2]$ such that

$$\max\Big\{ w(t) : t \in \Big[x_{n-j+1}, \frac{x_{n-j+1} + x_{n-j+2}}{2}\Big]\Big\} = w(\tau) > 0.$$

Define

$$v(t) = \begin{cases} aw(t) - bw(\tau) & \text{if } a > b,\\ bw(t) - aw(\tau) & \text{if } a \le b. \end{cases}$$

Then $v(\tau) \ge 0$ and $v(x_{n-j+1}) < 0$. By the intermediate value theorem, there exists $t' \in (x_{n-j+1}, \tau)$ such that $v(t') = 0$ which implies that $aw(t') - bw(\tau) = 0$. Hence, there are distinct solutions of the $(n-j;j)$-point BVP at the points

$$x_1, \dots, x_{n-j}, t', \tau, x_{n-j+2}, x_{n-j+2}, \dots, x_{n+j-2}, x_{n+j-1},$$

which is again a contradiction. Hence solutions of the $(n-j+1;j-1)$-point BVP (1.1), (1.2) are unique. Now the theorem is proved inductively. ■

Corollary 2.3 *Assume that for $k = n-j$, solutions of the $(n-j;j)$-point BVP are unique when they exist. Then solutions of the $(n;0)$-point conjugate BVP are unique when they exist.*

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In view of the uniqueness implies existence results due to Hartman [6, 7] and Klassen [18] as discussed in regard to question (ii), we have an immediate corollary concerning existence of solutions for $(l; 0)$-point conjugate boundary value problems for (1.1).

Corollary 2.4 *Assume that for $k = n - j$, solutions of the $(n - j; j)$-point BVP are unique when they exist. Then for $2 \le l \le n$, solutions of the $(l; 0)$-point conjugate BVP* (1.3) *are unique when they exist.*

We now establish that uniqueness of solutions of the $(n - j; j)$-point BVP implies uniqueness of solutions of the $(k; j)$-point BVP, when $1 \le k \le n - j - 1$.

Theorem 2.5 *Assume that for $k = n - j$, solutions of the $(n - j; j)$-point BVP are unique when they exist. Then for each $1 \le k \le n - j - 1$, solutions of the $(k; j)$-point BVP are unique when they exist.*

Proof Assume that solutions of the $(n - j; j)$-point BVP are unique. Assume that, for some $1 \le k \le n - j - 1$, some $(k; j)$-point BVP has distinct solutions. Let

$$h = \max\{k = 1, \dots, n - j - 1 \mid (k; j)\text{-point BVP has distinct solutions}\}.$$

Then there are positive integers $m_1, \dots, m_h$ such that $m_1 + \cdots + m_h = n - j$, points $a < x_1 < \cdots < x_h < \cdots < x_{h+2j} < b$, and positive reals, $a_1, \dots, a_{2j}$, for which there exist distinct solutions $y(x)$ and $z(x)$ of the corresponding $(h; j)$-point boundary value problem (1.1), (1.2). In particular,

$$\begin{aligned}
&y^{(i-1)}(x_l) = z^{(i-1)}(x_l),\ 1 \le i \le m_l, 1 \le l \le h,\\
&a_1 y(x_{h+1}) - a_2 y(x_{h+2}) = a_1 z(x_{h+1}) - a_2 z(x_{h+2}), \dots\\
&\qquad \dots, a_{2j-1} y(x_{h+2j-1}) - a_{2j} y(x_{h+2j}) = a_{2j-1} z(x_{h+2j-1}) - a_{2j} z(x_{h+2j}).
\end{aligned}$$

Since $h \le n - j - 1$, some $m_l \ge 2$. Let

$$m_{l_0} = \max\{m_l \mid 1 \le l \le h\} \ge 2.$$

At this point, we need to argue that each x_l is a zero of $y - z$ of exact multiplicity m_l, $1 \le l \le h$. This argument is done by induction on j, and in fact the proof of this theorem is truly completed by induction on j. For $j = 1$, if any of the next higher order derivatives vanish at x_l, then y and z are distinct solutions of an $(h; 0)$-point conjugate boundary value problem. Complete the proof of this theorem below for $j = 1$. Now for $j > 1$, if any of the next higher order derivatives vanish at x_l, then y and z are distinct solutions of an $(h; j - 1)$-point BVP. So, we complete this proof by assuming that each x_l, is a zero of $y - z$ of exact multiplicity m_l, $1 \le l \le h$.

Thus, we assume, with no loss of generality, that

$$y^{(m_{l_0})}(x_{l_0}) > z^{(m_{l_0})}(x_{l_0}).$$

Now fix $a < \tau < x_1$. By the maximality of h, solutions of the $(h+1; j)$ problems (1.1), (1.2) at the points $\tau, x_1, \dots x_h, \cdots x_{h+2j}$ are unique. Hence, it follows from Theorem

2.1 that for each $\epsilon > 0$, there is a $\delta > 0$ and there is a solution $z_\delta(x)$ of the $(h+1; j)$ problem (1.1), (1.2), satisfying the conditions,

$$
\begin{aligned}
&z_\delta(\tau) = z(\tau),\\
&z_\delta^{(i-1)}(x_l) = z^{(i-1)}(x_l) = y^{(i-1)}(x_l), \quad 1 \le i \le m_l, \quad 1 \le l \le h, \quad l \ne l_0,\\
&z_\delta^{(i-1)}(x_{l_0}) = z^{(i-1)}(x_{l_0}) = y^{(i-1)}(x_{l_0}), \quad 1 \le i \le m_{l_0} - 2, \quad (\text{if } m_{l_0} > 2),\\
&z_\delta^{(m_{l_0}-2)}(x_{l_0}) = z^{(m_{l_0}-2)}(x_{l_0}) + \delta = y^{(m_{l_0}-2)}(x_{l_0}) + \delta,\\
&a_1 z_\delta(x_{h+1}) - a_2 z_\delta(x_{h+2}) = a_1 z(x_{h+1}) - a_2 z(x_{h+2}) = a_1 y(x_{h+1}) - a_2 y(x_{h+2}),\\
&\qquad\vdots\\
&a_{2j-1} z_\delta(x_{h+2j-1}) - a_{2j} z_\delta(x_{h+2j}) = a_{2j-1} z(x_{h+2j-1}) - a_{2j} z(x_{h+2j})\\
&\quad = a_{2j-1} y(x_{h+2j-1}) - a_{2j} y(x_{h+2j}),
\end{aligned}
$$

and $|z_\delta(x) - z(x)| < \epsilon$ on $[\tau, x_{h+2j}]$. For $\epsilon > 0$, sufficiently small, there exist points $x_{l_0-1} < \rho_1 < x_{l_0} < \rho_2 < x_{l_0+1}$ such that

$$
\begin{aligned}
&z_\delta^{(i-1)}(x_l) = y^{(i-1)}(x_l), \quad 1 \le i \le m_l, \quad 1 \le l \le l_0 - 1,\\
&z_\delta(\rho_1) = y(\rho_1),\\
&z_\delta^{(i-1)}(x_{l_0}) = y^{(i-1)}(x_{l_0}), \quad 1 \le i \le m_{l_0} - 2, \quad (\text{if } m_{l_0} > 2),\\
&z_\delta(\rho_2) = y(\rho_2),\\
&z_\delta^{(i-1)}(x_l) = y^{(i-1)}(x_l), \quad 1 \le i \le m_l, \quad l_0 + 1 \le l \le h,\\
&a_1 z_\delta(x_{h+1}) - a_2 z_\delta(x_{h+2}) = a_1 y(x_{h+1}) - a_2 y(x_{h+2}),\\
&\qquad\vdots\\
&a_{2j-1} z_\delta(x_{h+2j-1}) - a_{2j} z_\delta(x_{h+2j}) = a_{2j-1} y(x_{h+2j-1}) - a_{2j} y(x_{h+2j}).
\end{aligned}
$$

If $m_{l_0} > 2$, $z_\delta(x)$ and $y(x)$ are distinct solutions of the $(h+2; j)$-point boundary value problem at the points $x_1, \dots, x_{l_0-1}, \rho_1, x_{l_0}, \rho_2, x_{l_0+1}, \dots, x_h, \dots, x_{h+2j}$, which is a contradiction, because of the maximality of h. If $m_{l_0} = 2$, then $z_\delta(x)$ and $y(x)$ are distinct solutions of the $(h+1; j)$-point boundary value problem at the points

$$x_1, \dots, x_{l_0-1}, \rho_1, \rho_2, x_{l_0+1}, \dots, x_h, \dots, x_{h+2j},$$

which is again a contradiction. ■

In view of Theorem 2.2 and Theorem 2.5, we have the following corollary.

Corollary 2.6 *Let $j_0 \in \{0, \dots, n-1\}$. Assume that solutions of* (1.1), (1.2), *when $k = n - j_0$, $j = j_0$, are unique. Then for each $1 \le j \le j_0$, $1 \le k \le n - j$, solutions of the $(k; j)$-point BVP are unique when they exist.*

3 Existence of Solutions

Having established in the previous section that uniqueness of solutions of (1.1), (1.2) when $k = n - j_0$, $j = j_0$, implies uniqueness of solutions of (1.1), (1.2) for $1 \le j \le j_0$ and $1 \le k \le n - j$, we now deal with uniqueness implies existence for these problems. For such existence results, continuous dependence as in Theorem 2.1 plays a role. In addition, we shall make use of a Schrader [21] precompactness result on bounded sequences of solutions of (1.1). We begin by stating the Schrader [21] precompactness result.

Theorem 3.1 *Assume the uniqueness of solutions for* (1.1), (1.3) *when* $\ell = n$. *If* $\{y_\nu(x)\}$ *is a sequence of solutions of* (1.1) *that is uniformly bounded on a nondegenerate compact subinterval* $[c, d] \subset (a, b)$, *then there is a subsequence* $\{y_{\nu_l}(x)\}$ *such that* $\{y_{\nu_l}^{(i)}(x)\}$ *converges uniformly on each compact subinterval of* (a, b), *for each* $i = 0, \dots, n-1$.

In view of the Corollary 2.3, we have, as a corollary, a precompactness condition in terms of (1.1), (1.2) when $k = n - j$.

Corollary 3.2 *Let* $j_0 \in \{1, \dots, n-1\}$. *Assume that solutions of the* $(n - j_0; j_0)$*-point BVP* (1.1), (1.2) *are unique. If* $\{y_\nu(x)\}$ *is a sequence of solutions of* (1.1) *that is uniformly bounded on a nondegenerate compact subinterval* $[c, d] \subset (a, b)$, *then there is a subsequence* $\{y_{\nu_l}(x)\}$ *such that* $\{y_{\nu_l}^{(i)}(x)\}$ *converges uniformly on each compact subinterval of* (a, b), *for each* $i = 0, \dots, n-1$.

We now present our uniqueness implies existence result for the $(k; j)$-point boundary value problems.

Theorem 3.3 *Let* $j_0 \in \{0, \dots, n-1\}$. *Assume that solutions of* (1.1), (1.2) *when* $k = n - j_0$, $j = j_0$, *are unique. Then for each* $1 \le j \le j_0$, $1 \le k \le n - j$, *positive integers* $m_1, \dots, m_k$ *such that* $m_1 + \cdots + m_k = n - j$, *points* $a < x_1 < \cdots < x_{k+2j} < b$, *real values* y_{il}, $1 \le i \le m_l$, $1 \le l \le k$, $y_n, y_{n-1}, \dots, y_{n-(j-1)} \in \mathbb{R}$, *and* $a_1, a_2, \dots, a_{2j}$, *positive real numbers, there exists a unique solution of the* $(k; j)$*-point BVP,* (1.1), (1.2).

Proof Let $1 \le j \le j_0$, $1 \le k \le n - j$, positive integers $m_1, \dots, m_k$ such that $m_1 + \cdots + m_k = n - j$, points $a < x_1 < \cdots < x_{k+2j} < b$, real values y_{il}, $1 \le i \le m_l$, $1 \le l \le k$, $y_n, y_{n-1}, \dots, y_{n-(j-1)} \in \mathbb{R}$, and $a_1, a_2, \dots, a_{2j}$, positive real numbers, be given.

Since solutions of the $(n - j_0; j_0)$-point BVP (1.1), (1.2) are unique, it follows from Corollary 2.4 that solutions of the $(l; 0)$-point conjugate BVP for $2 \le l \le n$ are unique; thus, solutions of the $(l; 0)$-point conjugate BVP for $2 \le l \le n$ exist [6,7,18].

Let $1 \le j \le j_0$ and $1 \le k \le n - j$. Let $z(x)$ be the unique solution of (1.1) satisfying $(k + j + 1; 0)$-point conjugate boundary conditions

$$\begin{aligned}
&z^{(i-1)}(x_1) = y_{i1}, \quad 1 \le i \le m_1 - 1,\\
&z^{(i-1)}(x_l) = y_{il}, \quad 1 \le i \le m_l, \quad 2 \le l \le k,\\
&z(x_{k+1}) = \tfrac{y_n}{a_1},\\
&z(x_{k+3}) = \tfrac{y_{n-1}}{a_3},\\
&\qquad \vdots\\
&z(x_{k+2j-1}) = \tfrac{y_{n-(j-1)}}{a_{2j-1}},\\
&z(x_{k+2j}) = 0.
\end{aligned}$$

Note that in the case $m_1 = 1$, z satisfies a $(k + j; 0)$-point problem with boundary conditions beginning at x_2. From the last two conditions

$$z(x_{k+2j}) = 0, \quad z(x_{k+2j-1}) = \frac{y_{n-(j-1)}}{a_{2j-1}},$$

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we obtain

$$a_{2j-1}z(x_{k+j}) - a_{2j}z(t_1) = y_{n-(j-1)}.$$

Define the set

$$\begin{aligned} S = \Big\{ u^{(m_1-1)}(x_1) \;\Big|\; & u \text{ is a solution of (1.1) satisfying} \\ & u^{(i-1)}(x_1) = y_{i1},\ 1 \le i \le m_1 - 1, \\ & u^{(i-1)}(x_l) = y_{il},\ 1 \le i \le m_l,\ 2 \le l \le k, \\ & u(x_{k+1}) = \frac{y_n}{a_1}, u(x_{k+3}) = \frac{y_{n-1}}{a_3}, \dots, u(x_{k+2j-3}) = \frac{y_{n-(j-2)}}{a_{2j-3}}, \\ & a_{2j-1}u(x_{k+2j-1}) - a_{2j}u(x_{k+2j}) = y_{n-(j-1)} \Big\}. \end{aligned}$$

Clearly, $z^{(m_1-1)}(x_1) \in S$, and so S is a nonempty subset of $\mathbb{R}$.

Next, choose $s_0 \in S$. Then there is a solution $u_0(x)$ of (1.1) satisfying

$$\begin{aligned} & u_0^{(i-1)}(x_1) = y_{i1}, \quad 1 \le i \le m_1 - 1, \\ & u_0^{(m_1-1)}(x_1) = s_0, \\ & u_0^{(i-1)}(x_l) = y_{il}, \quad 1 \le i \le m_l, 2 \le l \le k, \\ & u_0(x_{k+1}) = \tfrac{y_n}{a_1}, u_0(x_{k+3}) = \tfrac{y_{n-1}}{a_3}, \\ & \quad \vdots \\ & u_0(x_{k+2j-3}) = \tfrac{y_{n-(j-2)}}{a_{2j-3}},\ a_{2j-1}u_0(x_{k+2j-1}) - a_{2j}u_0(x_{k+2j}) = y_{n-(j-1)}. \end{aligned}$$

By the uniqueness of solutions of the $(k+j-1;1)$-point BVP (Corollary 2.6), and in view of Theorem 2.1, there exists a $\delta > 0$ such that, for each $0 \le |s - s_0| < \delta$, there is a solution $u_s(x)$ of (1.1) satisfying

$$\begin{aligned} & u_s^{(i-1)}(x_1) = y_{i1}, \quad 1 \le i \le m_1 - 1, \\ & u_s^{(m_1-1)}(x_1) = s, \\ & u_s^{(i-1)}(x_l) = y_{il}, \quad 1 \le i \le m_l, \quad 2 \le l \le k, \\ & u_s(x_{k+1}) = \tfrac{y_n}{a_1}, u_s(x_{k+3}) = \tfrac{y_{n-1}}{a_3}, \\ & \quad \vdots \\ & u_s(x_{k+2j-3}) = \tfrac{y_{n-(j-2)}}{a_{2j-3}},\ a_{2j-1}u_s(x_{k+2j-1}) - a_{2j}u_s(x_{k+2j}) = y_{n-(j-1)}, \end{aligned}$$

which implies that $u_s^{(m_1-1)}(x_1) \in S$, that is, $s \in S$. Hence, $(s_0 - \delta, s_0 + \delta) \subset S$ and S is an open subset of $\mathbb{R}$.

Now we show that S is also a closed subset of $\mathbb{R}$. To do this, assume that S is not closed and there exists an $r_0 \in \overline{S} \setminus S$ and a strictly monotone sequence $\{r_\nu\} \subset S$ such that $\lim_{\nu\to\infty} r_\nu = r_0$. We may assume, without loss of generality, that $r_\nu \uparrow r_0$. By the

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definition of S, for each $\nu \in \mathbb{N}$, there exists a unique solution $u_\nu(x)$ of (1.1) satisfying

$$\begin{aligned}
&u_\nu^{(i-1)}(x_1) = y_{i1}, \quad 1 \le i \le m_1 - 1,\\
&u_\nu^{(m_1-1)}(x_1) = r_\nu,\\
&u_\nu^{(i-1)}(x_l) = y_{il}, \quad 1 \le i \le m_l, \quad 2 \le l \le k,\\
&u_\nu(x_{k+1}) = \tfrac{y_n}{a_1}, u_\nu(x_{k+3}) = \tfrac{y_{n-1}}{a_3},\\
&\quad\vdots\\
&u_\nu(x_{k+2j-3}) = \tfrac{y_{n-(j-2)}}{a_{2j-3}},\ a_{2j-1}u_\nu(x_{k+2j-1}) - a_{2j}u_\nu(x_{k+2j}) = y_{n-(j-1)}.
\end{aligned}$$

Set $v = u_\nu - u_{\nu+1}$. Then

$$\begin{aligned}
&v^{(i-1)}(x_1) = 0, \quad 1 \le i \le m_1 - 1,\\
&v^{(m_1-1)}(x_1) = r_\nu - r_{\nu+1} < 0,\\
&v^{(i-1)}(x_l) = 0, \quad 1 \le i \le m_l, \quad 2 \le l \le k,\\
&v(x_{k+1}) = 0,\ v(x_{k+3}) = 0,\\
&\quad\vdots\\
&v(x_{k+2j-3}) = 0,\\
&a_{2j-1}v(x_{k+2j-1}) - a_{2j}v(x_{k+2j}) = y_{n-(j-1)} = 0.
\end{aligned}$$

By the uniqueness of solution of the $(k + j - 1; 1)$-point BVP, we have one of

(i) $u_\nu(x) < u_{\nu+1}(x)$ on $(a, x_2) \setminus \{x_1\}$, if m_1 is odd,

(ii) $u_\nu(x) > u_{\nu+1}(x)$ on (a, x_1) and $u_\nu(x) < u_{\nu+1}(x)$ on (x_1, x_2), if m_1 is even.

We consider only case (i), with case (ii) being completely analogous. So, for case (i), from Corollary 3.2 and the fact that $r_0 \notin S$, we can conclude that $\{u_\nu(x)\}$ is not uniformly bounded above on each compact subinterval of each of (a, x_1) and (x_1, x_2).

Now let $w(x)$ be the solution of (1.1) satisfying $(k+j; 0)$-point conjugate boundary conditions (1.3) at the points $x_1, \dots, x_k, x_{k+1}, x_{k+3}, \dots, x_{k+2j-1}$,

$$\begin{aligned}
&w^{(i-1)}(x_1) = y_{i1}, \quad 1 \le i \le m_1 - 1, (\text{if } m_1 > 1),\\
&w^{(m_1-1)}(x_1) = r_0,\\
&w^{(i-1)}(x_l) = y_{il}, \quad 1 \le i \le m_l, 2 \le l \le k,\\
&w(x_{k+1}) = \tfrac{y_n}{a_1}, w(x_{k+3}) = \tfrac{y_{n-1}}{a_3},\\
&\quad\vdots\\
&w(x_{k+2j-1}) = \tfrac{y_{n-(j-1)}}{a_{2j-1}}.
\end{aligned}$$

From the monotonicity and unboundedness property of the sequence $\{u_\nu(x)\}$, it follows that, for some large ν_0, there exist a solution u_{ν_0} of (1.1) and points $a < \tau_1 < x_1 < \tau_2 < x_2$ such that $u_{\nu_0}(\tau_1) = w(\tau_1)$, $u_{\nu_0}(\tau_2) = w(\tau_2)$. Hence,

$$\begin{aligned}
&u_{\nu_0}(\tau_1) = w(\tau_1),\\
&u_{\nu_0}^{(i-1)}(x_1) = y_{i1} = w^{(i-1)}(x_1), \quad 1 \le i \le m_1 - 1,\\
&u_{\nu_0}(\tau_2) = w(\tau_2),\\
&u_{\nu_0}^{(i-1)}(x_l) = y_{il} = w^{(i-1)}(x_l), \quad 1 \le i \le m_l, 2 \le l \le k\\
&u_{\nu_0}(x_{k+1}) = \tfrac{y_n}{a_1} = w(x_{k+1}), u_{\nu_0}(x_{k+3}) = \tfrac{y_{n-1}}{a_3} = w(x_{k+3}),\\
&\quad\vdots\\
&u_{\nu_0}(x_{k+2j-3}) = \tfrac{y_{n-(j-2)}}{a_{2j-3}} = w(x_{k+2j-3}).
\end{aligned}$$

Thus, $u_{\nu_0}(x)$ and $w(x)$ are distinct solutions of the same $(k+j+1;0)$-point (or if $m_1 = 1$, the same $(k+j;0)$-point) conjugate boundary value problem (1.1), (1.3). This contradicts Corollary 2.4. Thus, S is also a closed subset of $\mathbb{R}$.

As a consequence of S being a nonempty subset of $\mathbb{R}$ that is both open and closed, we have $S \equiv \mathbb{R}$. By choosing $y_{m_1 1} \in S$, there is a corresponding solution $y(x)$ of (1.1) such that

$$
\begin{aligned}
&y^{(i-1)}(x_1) = y_{i1}, \quad 1 \le i \le m_1 - 1,\\
&y^{(m_1-1)}(x_1) = y_{m_1 1},\\
&y^{(i-1)}(x_l) = y_{il}, \quad 1 \le i \le m_l, 2 \le l \le k,\\
&y(x_{k+1}) = \tfrac{y_n}{a_1}, y(x_{k+3}) = \tfrac{y_{n-1}}{a_3},\\
&\quad\vdots\\
&y(x_{k+2j-3}) = \tfrac{y_{n-(j-2)}}{a_{2j-3}},\ a_{2j-1}y(x_{k+2j-1}) - a_{2j}y(x_{k+2j}) = y_{n-(j-1)},
\end{aligned}
$$

which is the desired solution of the $(k+j-1;1)$-point BVP.

Since $1 \le j \le j_0$ and $1 \le k \le n - j_0$ implies $1 \le k+j-1 \le n-1$, we have shown existence for each of the $(k;1)$-point BVPs, $1 \le k \le n-1$.

If $j_0 = 1$, then the proof is complete. If $j_0 > 1$, let $k + j \le n-1$ and let $z_1(x)$ be the unique solution of the $(k+j;1)$-point boundary value problem,

$$
\begin{aligned}
&z_1^{(i-1)}(x_1) = y_{i1}, \quad 1 \le i \le m_1 - 1,\\
&z_1^{(i-1)}(x_l) = y_{il}, \quad 1 \le i \le m_l, 2 \le l \le k,\\
&z_1(x_{k+1}) = \tfrac{y_n}{a_1},\\
&\quad\vdots\\
&z_1(x_{k+2j-3}) = \tfrac{y_{n-(j-2)}}{a_{2j-3}},\\
&z_1(x_{k+2j-2}) = 0,\\
&a_{2j-1}z_1(x_{k+2j-1}) - a_{2j}z_1(x_{k+2j}) = y_{n-(j-1)}.
\end{aligned}
$$

From the two conditions

$$
z(x_{k+2j-2}) = 0, \quad z(x_{k+2j-3}) = \frac{y_{n-(j-2)}}{a_{2j-3}},
$$

we obtain

$$
a_{2j-3}z(x_{k+2j-3}) - a_{2j-2}z(x_{k+2j-2}) = y_{n-(j-1)}.
$$

Define the set

$$
\begin{aligned}
S_1 = \Big\{ u^{(m_1-1)}(x_1) \mid u \text{ is a solution of (1.1) satisfying}\\
&u^{(i-1)}(x_1) = y_{i1},\ 1 \le i \le m_1 - 1,\\
&u^{(i-1)}(x_l) = y_{il},\ 1 \le i \le m_l,\ 2 \le l \le k,\\
&u(x_{k+1}) = \frac{y_n}{a_1}, u(x_{k+2}) = \frac{y_{n-1}}{a_2}, \dots, u(x_{k+2j-5}) = \frac{y_{n-(j-3)}}{a_{2j-5}},\\
&a_{2j-3}u(x_{k+2j-3}) - a_{2j-2}u(x_{k+2j-2}) = y_{n-(j-2)},\\
&a_{2j-1}u(x_{k+2j-1}) - a_{2j}u(x_{k+2j}) = y_{n-(j-1)} \Big\}.
\end{aligned}
$$

Clearly, $z_1^{(m_1-1)}(x_1) \in S_1$, and so S_1 is a nonempty subset of $\mathbb{R}$. A construction, completely analogous to the above argument implies $S_1 = \mathbb{R}$. Hence, $y_{m_1 1} \in S_1$, and there is a corresponding solution, $y(x)$ of (1.1) such that

$$\begin{aligned}
&y^{(i-1)}(x_1) = y_{i1}, \quad 1 \le i \le m_1 - 1,\\
&y^{(m_1-1)}(x_1) = y_{m_1 1},\\
&y^{(i-1)}(x_l) = y_{il}, \quad 1 \le i \le m_l, \quad 2 \le l \le k,\\
&y(x_{k+1}) = \tfrac{y_n}{a_1},\\
&\qquad \vdots\\
&y(x_{k+2j-5}) = \tfrac{y_{n-(j-3)}}{a_{2j-5}},\\
&a_{2j-3}y(x_{k+2j-3}) - a_{2j-2}y(x_{k+2j-2}) = y_{n-(j-2)}\\
&a_{2j-1}y(x_{k+2j-1}) - a_{2j}y(x_{k+2j}) = y_{n-(j-1)},
\end{aligned}$$

which is the desired solution of the $(k + j - 2; 2)$-point BVP.

The proof of Theorem 3.3 is then completed by induction. ■

We restate Theorem 3.3 in the terminology introduced in the Introduction.

Corollary 3.4 *Let $j_0 \in \{0, \dots, n-1\}$. Assume that solutions of the $(n - j_0; j_0)$-point BVP, are unique. Then for each $1 \le j \le j_0$, $1 \le k \le n - j_0$, (1.1) is $(k; j)$-point uniquely solvable.*

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