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RESEARCH ARTICLE

Characterization of totally geodesic foliations with integrable and parallelizable normal bundle

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Abstract

In this work, we study foliations of arbitrary codimension $\mathfrak{F}$ with integrable normal bundles on complete Riemannian manifolds. We obtain a necessary and sufficient condition for $\mathfrak{F}$ to be totally geodesic. For this, we introduce a special number $\mathfrak{G}^{\alpha}_{\mathfrak{F}}$ that measures when the foliation ceases to be totally geodesic. Furthermore, applying some maximum principle we deduce geometric properties for $\mathfrak{F}$. We conclude with a geometrical version of Novikov's theorem (*Trans. Moscow Math. Soc.* (1965), 268–304), for Riemannian compact manifolds of arbitrary dimension.

1. Introduction

The study of foliations by codimension one hypersurfaces on Riemannian or Lorentzian manifolds has been carried out by many authors. For the purpose of this paper, we point out [1, 3, 7, 10, 14, 9]. These works are focused on the geometry of the leaves in order to answer if they are totally geodesic, umbilical, or stable hypersurfaces, among other results. The totally geodesic foliations were studied in [1, 3, 10]. The analysis of umbilicity can be found in [7, 14].

The interest in foliations of arbitrary codimension on a Riemannian manifold is well known at works [12, 19, 21, 17]. In 1986, Brito and Walczak [4], studied totally geodesic foliations with integrable normal bundles on Riemannian manifolds. They showed that the manifold ambient is locally a Riemannian product of orthogonal leaves, provided that each leaf is totally geodesic. In 1994, Rovenski in [18] also studied the relation between the curvature and topology of totally geodesic foliations. Almeida et al. [2], in 2017, have given a characterization of totally umbilical foliations on a constant curvature space. They proved that on odd-dimensional unit spheres, there is no umbilical foliation with integrable normal bundle and divergence free mean curvature vector.

Based on the above discussion, the following questions arise naturally:

(1) *For a given arbitrary codimension foliation with integrable normal bundle on a Riemannian manifold, which conditions should be imposed on the leaves to be totally geodesic?*
(2) *Are there obstructions for the existence of a totally geodesic foliation on a Riemannian manifold?*

In this work, we analyze and answer the questions above.

In Section 2, we state some preliminaries.

In Section 3, we use a key equation that relates the principal curvatures of the leaves and the curvature of the ambient space. We define the number $\mathfrak{G}^{\alpha}_{\mathfrak{F}}$, for all $\alpha \in \{n+1, \cdots, n+p\}$ by

$$\mathfrak{G}^{\alpha}_{\mathfrak{F}} = \inf_{M} \left\{ \frac{1}{n} \sum_{i=1}^{n} H^{i}_{\mathfrak{F}^{\perp}}\left(e_{\alpha}, \nabla^{\perp}_{e_i} e_{\alpha} - [e_{\alpha}, e_i]^{\perp}\right) + \frac{1}{n}\mathrm{tr}\left(K^{\alpha}_{ij}\right) - \frac{1}{n}\mathrm{div}_{\mathfrak{F}}\left(\nabla_{e_{\alpha}} e_{\alpha}\right) \right\},$$

where $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}\}$ is an adapted frame in a neighborhood of a point. By using the method of matrix Riccati ODE, we answer Question (1). That is, with this number we estimate the mean curvature function and we characterize totally geodesic foliations with the integrable normal bundle on Riemannian manifolds. More precisely, we prove the following result.

Let $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$ be orthogonal foliations of complementary dimensions on a complete Riemannian manifold M. If $\mathfrak{F}^{\perp}$ is parallelizable, then for all α

$$\mathfrak{G}^{\alpha}_{\mathfrak{F}} \leq 0,$$

$$\left(\mathcal{H}^{\alpha}_{\mathfrak{F}}\right)^2 \leq -\mathfrak{G}^{\alpha}_{\mathfrak{F}},$$

and

$$\mathfrak{F} \textit{ is totally geodesic} \Leftrightarrow \mathfrak{G}^{\alpha}_{\mathfrak{F}} = 0.$$

The following corollary gives us a result of obstruction to the existence of umbilical foliations on Riemannian manifolds of positive curvature. Precisely,

Let $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$ be orthogonal foliations of complementary dimensions on a complete Riemannian manifold M with constant sectional curvature c. Assume that $\mathfrak{F}^{\perp}$ is parallelizable and totally umbilical. If the vector fields $\nabla_{e_{\alpha}} e_{\alpha}$ are divergence free, then

(1) $c \leq 0$;
(2) $(\mathcal{H}^{\alpha}_{\mathfrak{F}})^2 \leq c$.

In Section 4, we create a differential equation that is crucial for answering Question (2) and other related questions. Assuming that $\mathfrak{G}^{\alpha}_{\mathfrak{F}}$ is finite, and using a maximum principle due to Yau, we prove the following result.

Let $\mathfrak{F}$ be a foliation on a manifold M of codimension p with integrable parallelizable normal bundle. If M has Ricci curvature bounded from below, then there exists a sequence of points $\{p_k\} \in M$ such that

(1) $\lim_{k\to\infty} \mathcal{H}^{\alpha}_{\mathfrak{F}}(p_k) = \sup_{M} \mathcal{H}^{\alpha}_{\mathfrak{F}}$;
(2) $\lim_{k\to\infty} \|\nabla\mathcal{H}^{\alpha}_{\mathfrak{F}}(p_k)\| = 0$;
(3) $\lim_{k\to\infty} \Delta\mathcal{H}^{\alpha}_{\mathfrak{F}}(p_k) \leq 0$.

Furthermore, we give a sufficient condition for an umbilical foliation with integrable normal bundle on an orientable Riemannian manifold to be totally geodesic.

Let $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$ be two orthogonal foliations of complementary dimensions on a complete Riemannian manifold M. Suppose that $\mathfrak{F}^{\perp}$ is parallelizable and totally umbilical. If $\mathfrak{F}$ has a complete leaf L such that tr$\left(K^{\alpha}_{ij}\right) \geq 0$ on L and the mean curvature of L is constant and satisfies

$$\mathcal{H}^{\alpha}_{L} = \left(-\mathfrak{G}^{\alpha}_{\mathfrak{F}}\right)^{\frac{1}{2}},$$

then $\mathfrak{F}$ is totally geodesic.

Let M be a compact 3-manifold such that the fundamental group $\pi_1(M)$ is finite. It follows from classical Novikov's theorem [16] that there is no foliation on M by closed curves with integrable normal bundle. In particular, there is no foliation of $\mathbb{S}^3$ by circles with integrable normal bundle (for more details, see [2]).

However, this theorem does not apply to $\mathbb{S}^{2k+1}$, $k \geq 2$. In [2], the authors prove that on odd-dimensional unit spheres there is no umbilical foliation with integrable normal bundle and divergence free mean

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curvature vector. In Section 5, we conclude this paper with a geometric version of the Novikov's theorem and a generalization of [2] for compact manifolds and arbitrary dimensions.

On a compact manifold M, there is no umbilical foliation with an integrable and parallelizable normal bundle, condition $tr(K_{ij}^{\alpha}) > 0$ *and a divergence free mean curvature vector field.*

2. Preliminaries

We shall now present the main tools we use in this work.

Let M be a $(n+p)$-dimensional manifold equipped with a Riemannian metric $\langle\,,\rangle$. Denote by ∇ the Levi-Civita connection and by R the curvature tensor of M. Let $\mathfrak{F}$ be a foliation of codimension p on M and $\mathfrak{F}^{\perp}$ be an orthogonal foliation to $\mathfrak{F}$. Consider an adapted frame $\{e_1, ..., e_n, e_{n+1}, ..., e_{n+p}\}$ in a neighborhood of a point $x \in M$. We shall make use of the following convention on the range of indices

$$1 \leq A, B, C \leq n+p,$$

$$1 \leq i, j, k \leq n,$$

$$n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

Let L be a submanifold of M and let A_{ν} be the shape operator of L defined by an arbitrary normal vector ν. Note that A_{ν} and II are related by

$$\langle II(v, w), \nu\rangle = \langle A_{\nu}(v), w\rangle.$$

The submanifold L is said to be totally umbilical if the second fundamental form is $II = \vec{h}\langle\cdot,\cdot\rangle$ which is equivalent to say that the shape operators are always multiple of the identity I. In this case, for each normal vector ν, $A_{\nu} = \xi_{\nu} I$, where $\xi_{\nu} = \langle\vec{h}, \nu\rangle$.

We define the second fundamental form of $\mathfrak{F}$ in the direction e_{α} by

$$H_{\mathfrak{F}}^{\alpha}(e_i, e_j) = \langle -\nabla_{e_i} e_{\alpha}, e_j\rangle.$$

Analogously, we define the second fundamental form of $\mathfrak{F}^{\perp}$ in the direction of e_i by

$$H_{\mathfrak{F}^{\perp}}^{i}(e_{\alpha}, e_{\beta}) = \langle -\nabla_{e_{\alpha}} e_i, e_{\beta}\rangle.$$

The Weingarten operators of $H_{\mathfrak{F}}^{\alpha}$ and $H_{\mathfrak{F}^{\perp}}^{i}$ are given, respectively, by

$$A_{e_{\alpha}}(e_i) = -\left(\nabla_{e_i} e_{\alpha}\right)^{\top} \quad \text{and} \quad A_{e_{\alpha}}(e_i) = -\left(\nabla_{e_{\alpha}} e_i\right)^{\perp}.$$

We define the mean curvature vector of $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$, respectively, by

$$\vec{h} = \sum_i \left(\nabla_{e_i} e_i\right)^{\perp} \quad \text{and} \quad \vec{h}^{\perp} = \sum_{\alpha} \left(\nabla_{e_{\alpha}} e_{\alpha}\right)^{\top},$$

and finally, the mean curvature function in direction e_{α} is defined by

$$\mathcal{H}_{\mathfrak{F}}^{\alpha} = \frac{1}{n}\langle\vec{h}, e_{\alpha}\rangle.$$

We define the norms of the second fundamental form $H_{\mathfrak{F}}^{\alpha}$ and $H_{\mathfrak{F}^{\perp}}^{i}$ of $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$, respectively, by

$$\|H_{\mathfrak{F}}^{\alpha}\| = \left(\sum_{i,j} \langle -\nabla_{e_i} e_{\alpha}, e_j\rangle^2\right)^{1/2}$$

and

$$\|H_{\mathfrak{F}^{\perp}}^{i}\| = \left(\sum_{\alpha,\beta} \langle -\nabla_{e_{\alpha}} e_i, e_{\beta}\rangle^2\right)^{1/2}.$$

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We say that $\mathfrak{F}^{\perp}$ is totally umbilical if the mean vector field $\vec{h}^{\perp}$ is

$$\vec{h}^{\perp} = \sum_i \lambda(x, e_i)e_i,$$

where $\lambda : T\mathfrak{F}^{\perp} \to \mathbb{R}$ is the eigenvalue of the Weingarten operator

$$A^{\nu} : T\mathfrak{F}^{\perp} \to T\mathfrak{F}^{\perp}$$

defined in the smooth field $T\mathfrak{F}^{\perp}$ of p-planes tangent to the leaves of $T\mathfrak{F}^{\perp}$. Note that $\lambda(x, v) = \langle \vec{h}^{\perp}, v\rangle$.

For each fixed α, we denote by (K^{α}_{ij}) the $n \times n$ matrix with entries given by $R(e_\alpha, e_i, e_j, e_\alpha)$. The trace of the matrix (K^{α}_{ij}) is then given by

$$\mathrm{tr}(K^{\alpha}_{ij}) = \sum_i R(e_\alpha, e_i, e_i, e_\alpha).$$

The following theorem was proved in [11] where the authors find an differential equation that relates the foliations with the ambient manifold.

Proposition 2.1. *Let $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$ be complementary foliations of dimensions n and p, respectively, on a $(n+p)$-dimensional Riemannian manifold. Then, for all α*

$$e_\alpha \left\langle \vec{h}, e_\alpha \right\rangle - \left\| H^{\alpha}_{\mathfrak{F}} \right\|^2 - tr\left(K^{\alpha}_{ij}\right) = \sum_{i=1}^{n} H^{i}_{\mathfrak{F}^{\perp}}\left(e_\alpha, \nabla^{\perp}_{e_i} e_\alpha - [e_\alpha, e_i]^{\perp}\right) - \mathrm{div}_{\mathfrak{F}}\left(\nabla_{e_\alpha} e_\alpha\right)$$

where

(i) $\vec{h}$ *represents the mean curvature vector of* $\mathfrak{F}$;
(ii) $\|H^{\alpha}_{\mathfrak{F}}\|$ *is the norm of the second fundamental form of* $\mathfrak{F}$;
(iii) $\mathrm{div}_{\mathfrak{F}}(\nabla_{e_\alpha} e_\alpha) = \sum_{i=1}^{n} \langle e_i, \nabla_{e_i}\nabla_{e_\alpha} e_\alpha\rangle$.

Corollary 2.1. *Let $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$ be two orthogonal foliations of complementary dimensions on a Riemannian manifold M. Suppose that $\mathfrak{F}^{\perp}$ is totally umbilical. Then,*

$$n\left\langle \nabla_{e_\alpha}\vec{h}, e_\alpha \right\rangle - \left\| H^{\alpha}_{\mathfrak{F}} \right\|^2 - tr\left(K^{\alpha}_{ij}\right) - \left\| \vec{h}^{\perp} \right\|^2 + \mathrm{div}_{\mathfrak{F}}\vec{h}^{\perp} = 0.$$

Using the Cauchy-Schwarz inequality, it follows that:

Proposition 2.2. *Let $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$ be complementary foliations of dimensions n and p, respectively, on a $(n+p)$-dimensional Riemannian manifold. Then,*

$$e_\alpha\left(\mathcal{H}^{\alpha}_{\mathfrak{F}}\right) \geq \frac{1}{n}\sum_{i=1}^{n} H^{i}_{\mathfrak{F}^{\perp}}\left(e_\alpha, \nabla^{\perp}_{e_i} e_\alpha - [e_\alpha, e_i]^{\perp}\right) + \frac{1}{n} tr\left(K^{\alpha}_{ij}\right)$$
$$-\frac{1}{n}\mathrm{div}_{\mathfrak{F}}\left(\nabla_{e_\alpha} e_\alpha\right) + \left(\mathcal{H}^{\alpha}_{\mathfrak{F}}\right)^2,$$

where $n+1 \leq \alpha \leq n+p$.

3. Characterization of totally geodesic foliations

Let us analyze orthogonal foliations of complementary dimensions, $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$ defined on a complete Riemannian manifold M. We say that $\mathfrak{F}$ is transversely parallelizable (or $\mathfrak{F}^{\perp}$ is parallelizable) if its normal bundle is trivial, that is, there is a global frame $\{e_{n+1}, \cdots, e_{n+p}\}$ on M (for more details see [13]). Now define the number $\mathfrak{G}^{\alpha}_{\mathfrak{F}}$ for all $\alpha \in \{n+1, \cdots, n+p\}$ by

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$$\mathfrak{G}^{\alpha}_{\mathfrak{F}} = \inf_{M}\left\{\frac{1}{n}\sum_{i=1}^{n} H^{i}_{\mathfrak{F}^{\perp}}\left(e_\alpha, \nabla^{\perp}_{e_i} e_\alpha - [e_\alpha, e_i]^{\perp}\right) + \frac{1}{n}\mathrm{tr}\left(K^{\alpha}_{ij}\right) - \frac{1}{n}\mathrm{div}_{\mathfrak{F}}\left(\nabla_{e_\alpha} e_\alpha\right)\right\},$$

where $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}\}$ is an adapted frame in a neighborhood of a point. With this number, we will estimate the mean curvature function and give a characterization of the totally geodesic foliations of complementary dimensions on a complete Riemannian manifold M. In fact, we generalize for arbitrary codimension, Theorem 4.2 in [7] and Theorem 2.4 in [10], following the steps of the proof of the latter and using $\mathfrak{G}^{\alpha}_{\mathfrak{F}}$.

Theorem 3.1. *Let $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$ be orthogonal foliations of complementary dimensions on a complete Riemannian manifolds M. If $\mathfrak{F}^{\perp}$ is parallelizable then, for all α*

$$\mathfrak{G}^{\alpha}_{\mathfrak{F}} \le 0, \tag{3.1}$$

$$\left(\mathcal{H}^{\alpha}_{\mathfrak{F}}\right)^2 \le -\mathfrak{G}^{\alpha}_{\mathfrak{F}}, \tag{3.2}$$

and

$$\mathfrak{F} \text{ is totally geodesic} \Leftrightarrow \mathfrak{G}^{\alpha}_{\mathfrak{F}} = 0.$$

Proof. We will use a proof by contradiction. So we assume that

$$\frac{1}{n}\sum_{i=1}^{n} H^{i}_{\mathfrak{F}^{\perp}}\left(e_\alpha, \nabla^{\perp}_{e_i} e_\alpha - [e_\alpha, e_i]^{\perp}\right) + \frac{1}{n}\sum_{i=1}^{n} R(e_\alpha, e_i, e_i, e_\alpha) - \frac{1}{n}\mathrm{div}_{\mathfrak{F}}\left(\nabla_{e_\alpha} e_\alpha\right) > 0,$$

on M. By Proposition 2.2, we have that $d\mathcal{H}^{\alpha}_{\mathfrak{F}}(e_\alpha) > \left(\mathcal{H}^{\alpha}_{\mathfrak{F}}\right)^2$ holds on M. Let $\gamma(s)$ be an integral curve of the unit vector field e_α. Since M is complete, γ may be extended to all $\mathbb{R}$. Thus, along γ, the above inequality has the form

$$\left(\mathcal{H}^{\alpha}_{\gamma(s)}\right)' > \left(\mathcal{H}^{\alpha}_{\gamma(s)}\right)^2 \quad \forall s \in \mathbb{R}.$$

We can choose the field e_α of such a way that $\mathcal{H}_{\gamma(0)} \ge 0$. Note that such choose does not change the expression

$$\frac{1}{n}\sum_{i=1}^{n} H^{i}_{\mathfrak{F}^{\perp}}\left(e_\alpha, \nabla^{\perp}_{e_i} e_\alpha - [e_\alpha, e_i]^{\perp}\right) + \frac{1}{n}\sum_{i=1}^{n} R(e_\alpha, e_i, e_i, e_\alpha) - \frac{1}{n}\mathrm{div}_{\mathfrak{F}}\left(\nabla_{e_\alpha} e_\alpha\right).$$

Therefore, the following inequalities are valid for every $s > 0$,

$$\left(\mathcal{H}^{\alpha}_{\gamma(s)}\right)' > \left(\mathcal{H}^{\alpha}_{\gamma(s)}\right)^2 > 0 \quad \text{and} \quad \frac{\left(\mathcal{H}^{\alpha}_{\gamma(s)}\right)'}{\left(\mathcal{H}^{\alpha}_{\gamma(s)}\right)^2} > 1. \tag{3.3}$$

Now consider the real function G given by

$$G(s) = -\frac{1}{\mathcal{H}^{\alpha}_{\gamma(s)}}, \quad s > 0.$$

Take a fixed $b > 0$ and apply the Mean Value Theorem for the function G on the interval $[b, s]$, we obtain

$$-\frac{1}{\mathcal{H}^{\alpha}_{\gamma(s)}} + \frac{1}{\mathcal{H}^{\alpha}_{\gamma(b)}} = \frac{\left(\mathcal{H}^{\alpha}_{\gamma(\xi)}\right)'}{\left(\mathcal{H}^{\alpha}_{\gamma(\xi)}\right)^2}(s-b),$$

where $\xi \in (b, s)$. Consequently, for all $s > b$, we have

$$-\frac{1}{\mathcal{H}^{\alpha}_{\gamma(s)}} + \frac{1}{\mathcal{H}^{\alpha}_{\gamma(b)}} > s - b.$$

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As s tends to infinity, the right side of this inequality is unlimited, while the left side is limited, which is a contradiction. This Proves (3.1).

To prove (3.2), suppose by contradiction that there exists a point $p \in M$ such that,

$$\left(\mathcal{H}_p^\alpha\right)^2 > -\mathfrak{G}_{\mathfrak{F}}^\alpha.$$

If $\mathfrak{G}_{\mathfrak{F}}^\alpha = -\infty$, then there is nothing to be proved. By (3.1), for some $a \geq 0$ we have

$$\mathfrak{G}_{\mathfrak{F}}^\alpha = -a^2,$$

where $a = 0$ or $a > 0$. If $a > 0$, we have by hypothesis that, $\left(\mathcal{H}_p^\alpha\right)^2 - a^2 > 0$. Let γ be an integral curve of e_α such that $\gamma(0) = p$. As previously we can choose a direction, e_α such that $\mathcal{H}_p^\alpha = \mathcal{H}_{\gamma(0)}^\alpha \geq 0$ and then,

$$\mathcal{H}_p^\alpha = \mathcal{H}_{\gamma(0)}^\alpha > a.$$

By continuity, there exists a maximal interval $[0, b)$ where

$$\left(\mathcal{H}_p^\alpha\right)^2 - a^2 > 0 \quad \forall s \in [0, b).$$

We claim that $b = +\infty$.

In fact, if $b < +\infty$, by continuity we should have $(\mathcal{H}_{\gamma(b)}^\alpha)^2 = a^2$. But, from proposition (2.2), we have:

$$\left(\mathcal{H}_{\gamma(s)}^\alpha\right)' \geq \left(\mathcal{H}_{\gamma(s)}^\alpha\right)^2 - a^2 > 0, \quad \forall s \in [0, b).$$

Thus, we conclude that $\mathcal{H}_{\gamma(s)}^\alpha$ is a strictly increasing function in $[0, b]$, contradiction. Therefore, the following inequality are valid for every $s > 0$,

$$\mathcal{H}_{\gamma(s)}^\alpha > 0, \quad \left(\mathcal{H}_{\gamma(s)}^\alpha\right)' \geq \left(\mathcal{H}_{\gamma(s)}^\alpha\right)^2 - a^2 > 0 \text{ and } \frac{\left(\mathcal{H}_{\gamma(s)}^\alpha\right)'}{\left(\mathcal{H}_{\gamma(s)}^\alpha\right)^2 - a^2} \geq 1.$$

Considering the function G defined by

$$G(s) = \frac{1}{2a} \ln\left(\frac{\mathcal{H}_{\gamma(s)}^\alpha - a}{\mathcal{H}_{\gamma(s)}^\alpha + a}\right), \quad s > 0.$$

For $b > 0$ fixed and by the Mean Value Theorem, we have that there exists $c \in [b, s]$ such that

$$\frac{1}{2a} \ln\left(\frac{\mathcal{H}_{\gamma(s)}^\alpha - a}{\mathcal{H}_{\gamma(s)}^\alpha + a}\right) - \frac{1}{2a} \ln\left(\frac{\mathcal{H}_{\gamma(b)}^\alpha - a}{\mathcal{H}_{\gamma(b)}^\alpha + a}\right) = \frac{\left(\mathcal{H}_{\gamma(c)}^\alpha\right)'}{\left(\mathcal{H}_{\gamma(c)}^\alpha\right)^2 - a^2}(s - b).$$

Consequently, for all $s > b$, we have

$$\frac{1}{2a} \ln\left(\frac{\mathcal{H}_{\gamma(s)}^\alpha - a}{\mathcal{H}_{\gamma(s)}^\alpha + a}\right) - \frac{1}{2a} \ln\left(\frac{\mathcal{H}_{\gamma(b)}^\alpha - a}{\mathcal{H}_{\gamma(b)}^\alpha + a}\right) \geq s - b.$$

Letting $s \longrightarrow +\infty$ we have a contradiction, because the left side is limited while the right side is unlimited. The case $a = 0$ is similar.

Finally, suppose

$$\mathfrak{G}_{\mathfrak{F}}^\alpha = 0.$$

It follows from Theorem 3.1 that

$$\left(\mathcal{H}_{\mathfrak{F}}^\alpha\right)^2 \leq -\mathfrak{G}_{\mathfrak{F}}^\alpha = 0.$$

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Therefore, $\mathcal{H}^{\alpha}_{\mathfrak{F}} \equiv 0$. By Proposition 2.2,

$$0 \geq \frac{1}{n}\sum_{i=1}^{n} H^{i}_{\mathfrak{F}^{\perp}}\left(e_{\alpha}, \nabla^{\perp}_{e_i} e_{\alpha} - [e_{\alpha}, e_i]^{\perp}\right) + \frac{1}{n}\sum_{i=1}^{n} R(e_{\alpha}, e_i, e_i, e_{\alpha}) - \frac{1}{n}\mathrm{div}_{\mathfrak{F}}\left(\nabla_{e_{\alpha}} e_{\alpha}\right).$$

Thus,

$$0 = \frac{1}{n}\sum_{i=1}^{n} H^{i}_{\mathfrak{F}^{\perp}}\left(e_{\alpha}, \nabla^{\perp}_{e_i} e_{\alpha} - [e_{\alpha}, e_i]^{\perp}\right) + \frac{1}{n}\sum_{i=1}^{n} R(e_{\alpha}, e_i, e_i, e_{\alpha}) - \frac{1}{n}\mathrm{div}_{\mathfrak{F}}\left(\nabla_{e_{\alpha}} e_{\alpha}\right),$$

on M. By using the Proposition 2.1, we conclude that $\|H^{\alpha}_{\mathfrak{F}}\| = 0$, i.e., $\mathfrak{F}$ is a totally geodesic foliation. The converse follows from Proposition 2.1. □

Corollary 3.1. *Let $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$ be orthogonal foliations of complementary dimensions on a complete Riemannian manifold M with constant sectional curvature c. Assume that $\mathfrak{F}^{\perp}$ is parallelizable and totally umbilical. If the vector fields $\nabla_{e_{\alpha}} e_{\alpha}$ are divergence free, then*

(1) $c \leq 0$;
(2) $(\mathcal{H}^{\alpha}_{\mathfrak{F}})^2 \leq c$.

Proof. Since M has constant sectional curvature, it follows that $\mathrm{tr}(K^{\alpha}_{ij}) = nc$ for all $n+1 \leq \alpha \leq n+p$. Then,

$$k \leq \frac{1}{n}\sum_{i=1}^{n}\left\{(\lambda(x, e_i))^2 + \frac{1}{n}\mathrm{tr}\left(K^{\alpha}_{ij}\right) - \frac{1}{n}\mathrm{div}_{\mathfrak{F}}\left(\nabla_{e_{\alpha}} e_{\alpha}\right)\right\},$$

where in the last identity we use the fact that $\mathfrak{F}^{\perp}$ is umbilical and the vector field $\nabla_{e_{\alpha}} e_{\alpha}$ is free divergent. Consequently $k \leq \mathfrak{G}^{\alpha}_{\mathfrak{F}}$. □

4. Some Applications of the maximum principle on umbilical and minimal foliations

Suppose that one of the foliation is totally umbilical, as in [11] and [2] on a complete Riemannian manifold M.

From now on, we assume that $\mathfrak{G}^{\alpha}_{\mathfrak{F}}$ is finite. Then, we use some well-known results of the geometric analysis to get further results for umbilical foliations with integrable normal bundle. First, let us apply the maximum principle due to Yau [22].

Corollary 4.1. *Let $\mathfrak{F}$ be a foliation on a manifold M of codimension p with integrable parallelizable normal bundle. If M has Ricci curvature bounded from below, then there exists a sequence of points $\{p_k\} \in M$ such that*

(1) $\lim_{k\to\infty} \mathcal{H}^{\alpha}_{\mathfrak{F}}(p_k) = \sup_M \mathcal{H}^{\alpha}_{\mathfrak{F}}$;
(2) $\lim_{k\to\infty} \|\nabla \mathcal{H}^{\alpha}_{\mathfrak{F}}(p_k)\| = 0$;
(3) $\lim_{k\to\infty} \Delta \mathcal{H}^{\alpha}_{\mathfrak{F}}(p_k) \leq 0$.

Proof. From Theorem 3.1, it follows that $\left(\mathcal{H}^{\alpha}_{\mathfrak{F}}\right)^2$ is bounded, since $\mathfrak{G}_{\mathfrak{F}}$ is finite. By Yau [22], we finish the proof. □

Denote by $\mathcal{L}^1(N)$ the space of Lebesgue integrable functions on a manifold N. In [7], the authors give the definition of Lebesgue integrable vector fields on foliations. Precisely, a vector field X is *Lebesque integrable on a foliation* $\mathfrak{F}$, if and only if $\|X^{\top}\| \in \mathcal{L}^1(L)$ for each leaf L of $\mathfrak{F}$, where $X^{\top}$ means the tangent projection of X, on L. In this case, we will denote $\|X\| \in \mathcal{L}^1(\mathfrak{F})$.

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The following result gives a sufficient condition for an umbilical foliation with integrable normal bundle on an orientable Riemannian manifold to be totally geodesic.

Theorem 4.1. *Let $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$ be two orthogonal foliations of complementary dimensions on a complete Riemannian manifold M. Suppose that $\mathfrak{F}^{\perp}$ is parallelizable and totally umbilical. If $\mathfrak{F}$ has a complete leaf L such that $tr(K^{\alpha}_{ij}) \geq 0$ on L and the mean curvature of L is constant and satisfies*

$$\mathcal{H}^{\alpha}_{L} = \left(-\mathfrak{G}^{\alpha}_{\mathfrak{F}}\right)^{\frac{1}{2}},$$

then $\mathfrak{F}$ is totally geodesic.

Proof. We will do the proof into two steps: the compact case and the complete non-compact case.

Compact case: Let $p \in M$ and $\{e_1, ..., e_n, e_{n+1}, ..., e_{n+p}\}$ be a orthonormal adapted frame, in a neighborhood of p. If $\vec{h}(p) \neq 0$, choose e_α as the mean curvature vector normalized, that is, $e_\alpha = \dfrac{\vec{h}}{\|\vec{h}\|}$.

By Theorem 3.1 the mean curvature function $\mathcal{H}^{\alpha}_{\mathfrak{F}}$ attains a maximum on L. Therefore, $d\mathcal{H}^{\alpha}_{L}(e_\alpha) = 0$ on L. Therefore by Corollary 2.1 we have

$$\mathrm{div}_L \vec{h}^{\perp} = \left\|H^{\alpha}_{L}\right\|^2 + \mathrm{tr}\left(K^{\alpha}_{ij}\right) + \left\|\vec{h}^{\perp}\right\|^2.$$

on L. By the Stokes Theorem and from the fact that $\mathrm{tr}(K^{\alpha}_{ij}) \geq 0$ on L, we conclude $H^{\alpha}_{L} = 0$ on L. Therefore, L is totally geodesic and

$$\mathfrak{G}^{\alpha}_{\mathfrak{F}} = 0.$$

By Theorem 3.1 we conclude the result.

Now if p is a singularity of the field $\vec{h}^{\perp}$, using the same equation (2.1), we get the equation for the leaf that passes through p say L_p and the same argument is valid.

Complete and non-compact case: Let $p \in M$ and $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}\}$ be a orthonormal adapted frame, in a neighborhood of p. If $\vec{h}(p) \neq 0$ choose e_α as the mean curvature vector normalized, that is, $e_\alpha = \dfrac{\vec{h}}{\|\vec{h}\|}$.

By Theorem 3.1, the mean curvature function of L attains its maximum on L. Thus, $d\mathcal{H}^{\alpha}_{L}(e_\alpha) = 0$ on L. By Corollary 2.1 we have the equation

$$\mathrm{div}_L \vec{h}^{\perp} = \left\|H^{\alpha}_{L}\right\|^2 + \mathrm{tr}(K^{\alpha}_{ij}) + \left\|\vec{h}^{\perp}\right\|^2,$$

on L. Since L is a complete non-compact and oriented leaf of $\mathfrak{F}$ and $\mathrm{div}_L \vec{h}^{\perp}$ does not change signal on L, we can apply Proposition 1 in [6] to obtain

$$0 = \mathrm{div}_L \vec{h}^{\perp} = \left\|H^{\alpha}_{L}\right\|^2 + \mathrm{tr}(K^{\alpha}_{ij}) + \left\|\vec{h}^{\perp}\right\|^2.$$

By the same argument above, we conclude that $H^{\alpha}_{L} = 0$ on L. Therefore, L is totally geodesic and

$$\mathfrak{G}^{\alpha}_{\mathfrak{F}} = 0.$$

Again by Theorem 3.1, the result follows. □

Corollary 4.2. *Let $\mathfrak{F}$ and $\mathfrak{F}^{\perp}$ be two orthogonal foliations of complementary dimensions on a complete Riemannian manifold M. Suppose that $\mathfrak{F}^{\perp}$ is parallelizable and totally umbilical. If $\mathcal{H}^{\alpha}_{\mathfrak{F}}$ is subharmonic and there is $p \in M$ such that $\mathcal{H}^{\alpha}_{\mathfrak{F}}(p) = \left(-\mathfrak{G}^{\alpha}_{\mathfrak{F}}\right)^{\frac{1}{2},}$ then $\mathfrak{F}$ has constant mean curvature. Furthermore, the leaves have the same mean curvature.*

Proof. The proof follows directly from Hopf-Calabi's maximum principle in [5] and the fact that $H^{\alpha}_{\mathfrak{F}}(p) = \left(-\mathfrak{G}^{\alpha}_{\mathfrak{F}}\right)^{\frac{1}{2}}$. □

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5. Umbilical foliations with integrable normal bundle

Lemma 5.1. *Let $\mathcal{F}$ be a foliation of codimension p on a compact Riemannian manifold M and $f : M \longrightarrow \mathbb{R}$ be a continuous function, non-constant on M, and constant along the leaves of $\mathcal{F}$. Then, the set $A = \{x \in M : f(x) = \max_M\{f(x)\}$ has at least one leaf L of $\mathcal{F}$.*

Proof. Since $f : M \longrightarrow \mathbb{R}$ is continuous defined on a compact Riemannian manifold M, then, there is a maximum and a minimum point. Let $x \in M$ be a point of maximum of f. Denote by L_x the leaf through x. As f is constant along L_x, we have that $f(y) = \max_M\{f(x)\}$ for any $y \in L_x$. Therefore, $L_x \subset A$. □

Theorem 5.1. *Let $\mathfrak{F}$ and $\mathfrak{F}^\perp$ be orthogonal foliations of complementary dimensions on a compact Riemannian manifold M with $tr(K_{ij}^\alpha) \geq 0$. If $\mathcal{F}$ has constant mean curvature and $\operatorname{div}_{\mathcal{F}} h^\perp \leq tr\big(K_{ij}^\alpha\big)$ for each α, then $\mathcal{F}$ is totally geodesic foliation.*

Proof. Let $p \in M$ and $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}\}$ an orthonormal adapted frame in a neighborhood of p. If $\vec{h}(p) \neq 0$ choose e_α as the normalized mean curvature vector, that is, $e_\alpha = \frac{\vec{h}}{\|\vec{h}\|}$. The mean curvature function $\mathcal{H}_{\mathfrak{F}}^\alpha : M \to \mathbb{R}$ is constant on M or there exists a leaf $L \in \mathfrak{F}$ having the property that

$$\mathcal{H}_L^\alpha = \max_M \mathcal{H}_{\mathfrak{F}}^\alpha(x). \tag{5.1}$$

First, assume that $\mathcal{H}_{\mathfrak{F}}^\alpha$ is not constant on M. This implies that $e_\alpha(\mathcal{H}_L^\alpha) = 0$ along L. Then, it follows from Corollary 2.1 that

$$ne_\alpha(\mathcal{H}_L^\alpha) = \left\|H_L^\alpha\right\|^2 + \operatorname{tr}(K_{ij}^\alpha) + \left\|\vec{h}^\perp\right\|^2 - \operatorname{div}_L \vec{h}^\perp. \tag{5.2}$$

Since $tr(K_{ij}^\alpha) - \operatorname{div}_{\mathcal{F}} \vec{h}^\perp \geq 0$, we have:

$$\left\|H_L^\alpha\right\|^2 \leq 0.$$

Therefore, L is totally geodesic, in particular, $\mathcal{H}_L^\alpha = 0$. From (5.1), we conclude that $\mathcal{H}_{\mathfrak{F}}^\alpha \leq 0$.

Similarly, considering the function $(-\mathcal{H}_{\mathfrak{F}}^\alpha)$ we conclude $\mathcal{H}_{\mathfrak{F}}^\alpha \geq 0$. Therefore, $\mathcal{H}_{\mathfrak{F}}^\alpha = 0$. Contradiction. Hence, $\mathcal{H}_{\mathfrak{F}}^\alpha$ is constant along M. Since $e_\alpha(\mathcal{H}_{\mathfrak{F}}^\alpha) = 0$, it follows from (5.2) that

$$\operatorname{div}_{\mathfrak{F}} \vec{h}^\perp = \left\|H_{\mathfrak{F}}^\alpha\right\|^2 + \operatorname{tr}(K_{ij}^\alpha) + \left\|\vec{h}^\perp\right\|^2. \tag{5.3}$$

If L is compact, applying Stokes' theorem we obtain

$$H_{\mathfrak{F}}^\alpha \equiv 0, \operatorname{tr}(K_{ij}^\alpha) \equiv 0 \text{ and } \vec{h}^\perp \equiv 0.$$

If L is complete and non-compact, note that $\operatorname{div}_L \vec{h}^\perp$ does not change sign and since $\left\|\vec{h}^\perp\right\| \in \mathcal{L}^1(\mathfrak{F})$ then $\operatorname{div}_L \vec{h}^\perp = 0$. Therefore by equation (5.3), we have

$$H_{\mathfrak{F}}^\alpha \equiv 0.$$

Finally if p is a singularity of the field $\vec{h}^\perp$, using (2.1) we get the equation for the leaf that passes through p and the same argument is valid. □

As an immediate consequence of our results, we obtain a geometric version of Novikov's theorem [16].

Corollary 5.1. *On a compact manifold M, there is no umbilical foliation with an integrable and parallelizable normal bundle, condition $tr\big(K_{ij}^\alpha\big) > 0$ and a divergence free mean curvature vector field.*

Since this holds for compact manifolds with no restriction on dimension, the above is a generalization of the result for odd-dimensional unit spheres, proved in [2].

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