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HEISENBERG–PAULI–WEYL UNCERTAINTY INEQUALITY FOR THE DUNKL TRANSFORM ON $\mathbb{R}^d$

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Abstract

In this paper, we give analogues of the local uncertainty inequality for the Dunkl transform on $\mathbb{R}^d$, and indicate how the local uncertainty inequality implies a global uncertainty inequality.

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1. Introduction

Uncertainty principles play an important role in harmonic analysis. They state that a function f and its Fourier transform $\widehat{f}$ cannot be simultaneously and sharply localised. That is, it is impossible for a nonzero function and its Fourier transform to be simultaneously small. Many mathematical formulations of this general fact can be found in [4, 5]. In particular, the well-known Heisenberg–Pauli–Weyl uncertainty principle [6, 14] states that for every $f \in L^2(\mathbb{R}^d)$,

$$\Big(\int_{\mathbb{R}^d} x_j^2|f(x)|^2\,dx\Big)\Big(\int_{\mathbb{R}^d} y_j^2|\widehat{f}(y)|^2\,dy\Big) \geq \frac{1}{4}\Big(\int_{\mathbb{R}^d} |f(x)|^2\,dx\Big)^2.$$

This inequality proves that if a function f is highly localised then its Fourier transform $\widehat{f}$ cannot be concentrated near the given fixed point. However, this inequality does not exclude the fact that the Fourier transform $\widehat{f}$ may be concentrated in the neighbourhood of several separated points. In [3, 8, 9], Faris and Price answered this question and showed that this situation cannot occur either. More precisely, they established the so-called local uncertainty principle.

In this paper, we consider $\mathbb{R}^d$ with the Euclidean inner product $\langle\cdot,\cdot\rangle$ and norm $|y| := \sqrt{\langle y, y\rangle}$. For $\alpha \in \mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha y := y - \frac{2\langle\alpha, y\rangle}{|\alpha|^2}\alpha.$$

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A finite set $\mathfrak{R} \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $\mathfrak{R} \cap \mathbb{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \mathfrak{R} = \mathfrak{R}$ for all $\alpha \in \mathfrak{R}$. We assume that it is normalised by $|\alpha|^2 = 2$ for all $\alpha \in \mathfrak{R}$. For a root system $\mathfrak{R}$, the reflections σ_α, $\alpha \in \mathfrak{R}$, generate a finite group $G \subset O(d)$, the reflection group associated with $\mathfrak{R}$. All reflections in G correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d\backslash \bigcup_{\alpha\in\mathfrak{R}} H_\alpha$, we fix the positive subsystem $\mathfrak{R}_+ := \{\alpha \in \mathfrak{R} : \langle\alpha, \beta\rangle > 0\}$. Then for each $\alpha \in \mathfrak{R}$ either $\alpha \in \mathfrak{R}_+$ or $-\alpha \in \mathfrak{R}_+$.

Let $k : \mathfrak{R} \to \mathbb{C}$ be a multiplicity function on $\mathfrak{R}$ (that is, a function which is constant on the orbits under the action of G). As an abbreviation, we introduce the index

$$\gamma = \gamma_k := \sum_{\alpha\in\mathfrak{R}_+} k(\alpha).$$

Throughout this paper, we will assume that the multiplicity is nonnegative, that is, $k(\alpha) \geq 0$ for all $\alpha \in \mathfrak{R}$. Moreover, let w_k denote the weight function

$$w_k(y) := \prod_{\alpha\in\mathfrak{R}_+} |\langle\alpha, y\rangle|^{2k(\alpha)}, \quad y \in \mathbb{R}^d,$$

which is G-invariant and homogeneous of degree 2γ.

We denote by μ_k the measure on $\mathbb{R}^d$ given by $d\mu_k(y) := w_k(y)\, dy$; and by $L^p_k(\mathbb{R}^d)$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L^p_k} := \Big(\int_{\mathbb{R}^d} |f(y)|^p \, d\mu_k(y)\Big)^{1/p} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_{L^\infty_k} := \operatorname*{ess\,sup}_{y\in\mathbb{R}^d} |f(y)| < \infty,$$

and by $L^p_{k,\mathrm{rad}}(\mathbb{R}^d)$ the subspace of $L^p_k(\mathbb{R}^d)$ consisting of radial functions.

For $f \in L^1_k(\mathbb{R}^d)$ the Dunkl transform is defined (see [2]) by

$$\mathcal{F}_k(f)(x) := c_k \int_{\mathbb{R}^d} E_k(-ix, y) f(y)\, d\mu_k(y), \quad x \in \mathbb{R}^d,$$

where c_k is the Mehta-type constant given by

$$c_k := \Big(\int_{\mathbb{R}^d} e^{-|y|^2/2}\, d\mu_k(y)\Big)^{-1}, \tag{1.1}$$

and where $E_k(-ix, y)$ denotes the Dunkl kernel. (For more details see the next section.)

Many uncertainty principles have already been proved for the Dunkl transform, namely by Rösler [11] and Shimeno [12] who established (by two different methods) the Heisenberg–Pauli–Weyl inequality for the Dunkl transform, by showing that, for every $f \in L^2_k(\mathbb{R}^d)$,

$$\Big(\int_{\mathbb{R}^d} |x|^2 |f(x)|^2 \, d\mu_k(x)\Big)\Big(\int_{\mathbb{R}^d} |y|^2 |\mathcal{F}_k(f)(y)|^2 \, d\mu_k(y)\Big) \geq \Big(\gamma + \frac{d}{2}\Big)^2 \|f\|^4_{L^2_k}. \tag{1.2}$$

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This says that if f is highly localised then $\mathcal{F}_k(f)$ cannot be concentrated near a single point, but it does not preclude $\mathcal{F}_k(f)$ from being concentrated in a small neighbourhood or more widely separated points. In fact, the latter phenomenon cannot occur either, and it is the object of the local uncertainty inequality to make this precise. The first such inequalities for the Fourier transform were obtained by Faris [3], and they were subsequently sharpened and generalised by Price [8, 9]. Building on the ideas of Faris and Price, we show a local uncertainty principle for the Dunkl transform $\mathcal{F}_k$. More precisely, we will show the following results.

(a) If $0 < s < \gamma + d/2$, there is a constant $K = K(s, k)$ such that for every $f \in L^2_k(\mathbb{R}^d)$ and every measurable set $E \subset \mathbb{R}^d$ such that $0 < \mu_k(E) < \infty$,

$$\left(\int_E |\mathcal{F}_k(f)(y)|^2 \, d\mu_k(y)\right)^{1/2} \le K(s, k)(\mu_k(E))^{s/(2\gamma+d)} \| (|x|)^s f \|_{L^2_k}.$$

(b) If $s > \gamma + d/2$, there is a constant $K' = K'(s, k)$ such that for every $f \in L^2_k(\mathbb{R}^d)$ and every measurable set $E \subset \mathbb{R}^d$ such that $0 < \mu_k(E) < \infty$,

$$\left(\int_E |\mathcal{F}_k(f)(y)|^2 \, d\mu_k(y)\right)^{1/2} \le K'(s, k)(\mu_k(E))^{1/2} \|f\|_{L^2_k}^{1-((2\gamma+d)/2s)} \| (|x|)^s f \|_{L^2_k}^{(2\gamma+d)/2s}.$$

We shall use the local uncertainty inequality (a) to prove a global uncertainty inequality for the Dunkl transform $\mathcal{F}_k$, that is, for all $f \in L^2_k(\mathbb{R}^d)$ and $s > 0$,

$$\left(\int_{\mathbb{R}^d} |x|^{2s} |f(x)|^2 \, d\mu_k(x)\right)\left(\int_{\mathbb{R}^d} |y|^{2s} |\mathcal{F}_k(f)(y)|^2 \, d\mu_k(y)\right) \ge C(s, k) \|f\|_{L^2_k}^4,$$

where $C(s, k)$ is the constant given in Section 4. This inequality generalises the Heisenberg–Pauli–Weyl inequality given by (1.2).

This paper is organised as follows. In Section 2 we list some basic properties of the Dunkl transform $\mathcal{F}_k$. In Section 3 we show a local uncertainty principle for the Dunkl transform $\mathcal{F}_k$. In the last section we deduce a global uncertainty inequality for $\mathcal{F}_k$.

Throughout this paper, C will always represent a positive constant, not necessarily the same at each occurrence.

2. The Dunkl transform on $\mathbb{R}^d$

The Dunkl operators $\mathcal{D}_j$, $j = 1, \ldots, d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given, for a function f of class C^1 on $\mathbb{R}^d$, by

$$\mathcal{D}_j f(y) := \frac{\partial}{\partial y_j} f(y) + \sum_{\alpha \in \mathfrak{R}_+} k(\alpha)\alpha_j \frac{f(y) - f(\sigma_\alpha y)}{\langle \alpha, y \rangle}.$$

For $y \in \mathbb{R}^d$, the initial problem $\mathcal{D}_j u(\cdot, y)(x) = y_j u(x, y)$, $j = 1, \ldots, d$, with $u(0, y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x, y)$ and called the Dunkl kernel [1, 7]. This kernel has a unique analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

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The Dunkl kernel has the Laplace-type representation [10]

$$E_k(x, y) = \int_{\mathbb{R}^d} e^{\langle y,z\rangle} d\Gamma_x(z), \quad x \in \mathbb{R}^d, \ y \in \mathbb{C}^d,$$

where $\langle y, z\rangle := \sum_{i=1}^d y_i z_i$ and Γ_x is a probability measure on $\mathbb{R}^d$, such that

$$\operatorname{supp}(\Gamma_x) \subset \{z \in \mathbb{R}^d \ : \ |z| \le |x|\}.$$

In our case,

$$|E_k(-ix, y)| \le 1, \quad x, y \in \mathbb{R}^d.$$

If $d = 1$ and $G = \mathbb{Z}_2$ (see [1]),

$$E_k(x, z) = \mathfrak{I}_{\gamma-1/2}(xz) + \frac{xz}{2\gamma + 1}\mathfrak{I}_{\gamma+1/2}(xz), \quad x \in \mathbb{R}, \ z \in \mathbb{C},$$

where

$$\mathfrak{I}_\gamma(xz) := \Gamma(\gamma + 1) \sum_{n=0}^{\infty} \frac{(xz)^{2n}}{2^{2n} n! \, \Gamma(n + \gamma + 1)}$$

is the modified spherical Bessel function of order γ (see [13]).

The Dunkl kernel gives rise to an integral transform, called the Dunkl transform on $\mathbb{R}^d$, introduced by Dunkl in [2], where already many basic properties were established. Dunkl's results were completed and extended later on by de Jeu [7]. The Dunkl transform of a function f in $L^1_k(\mathbb{R}^d)$ is

$$\mathcal{F}_k(f)(x) := c_k \int_{\mathbb{R}^d} E_k(-ix, y) f(y) \, d\mu_k(y), \quad x \in \mathbb{R}^d.$$

We notice that $\mathcal{F}_0$ agrees with the Fourier transform $\mathcal{F}$ that is given by

$$\mathcal{F}(f)(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle x,y\rangle} f(y) \, dy, \quad x \in \mathbb{R}^d.$$

Some of the properties of the Dunkl transform $\mathcal{F}_k$ are collected below (see [2, 7]).

(a) $L^1 - L^\infty$-**boundedness**. For all $f \in L^1_k(\mathbb{R}^d)$, we have $\mathcal{F}_k(f) \in L^\infty_k(\mathbb{R}^d)$ and

$$\|\mathcal{F}_k(f)\|_{L^\infty_k} \le c_k \|f\|_{L^1_k}. \tag{2.1}$$

(b) **Inversion theorem**. Let $f \in L^1_k(\mathbb{R}^d)$, such that $\mathcal{F}_k(f) \in L^1_k(\mathbb{R}^d)$. Then

$$f(x) = \mathcal{F}_k\Big(\mathcal{F}_k(f)\Big)(-x), \quad \text{a.e. } x \in \mathbb{R}^d.$$

(c) **Plancherel theorem**. The Dunkl transform $\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L^2_k(\mathbb{R}^d)$ onto itself. In particular,

$$\|f\|_{L^2_k} = \|\mathcal{F}_k(f)\|_{L^2_k}. \tag{2.2}$$

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3. Local uncertainty inequalities

This section is devoted to establishing a local uncertainty principle for the Dunkl transform $\mathcal{F}_k$. More precisely, we will show the following theorem.

THEOREM 3.1. (i) *If $0 < s < \gamma + d/2$, then for every $f \in L^2_k(\mathbb{R}^d)$ and every measurable set $E \subset \mathbb{R}^d$ such that $0 < \mu_k(E) < \infty$,*

$$\left(\int_E |\mathcal{F}_k(f)(y)|^2 \, d\mu_k(y)\right)^{1/2} \leq K(s,k)(\mu_k(E))^{s/(2\gamma+d)} \| (|x|)^s f \|_{L^2_k}, \tag{3.1}$$

where

$$K(s,k) = \frac{2\gamma + d}{2\gamma + d - 2s}\left(\frac{c_k(\gamma + \frac{d}{2} - s)}{2^{\gamma+(d/2)}\Gamma(\gamma + \frac{d}{2})s^2}\right)^{s/(2\gamma+d)}.$$

(ii) *If $s > \gamma + d/2$, then for every $f \in L^2_k(\mathbb{R}^d)$ and every measurable set $E \subset \mathbb{R}^d$ such that $0 < \mu_k(E) < \infty$,*

$$\left(\int_E |\mathcal{F}_k(f)(y)|^2 \, d\mu_k(y)\right)^{1/2} \leq K'(s,k)(\mu_k(E))^{1/2} \|f\|_{L^2_k}^{1-((2\gamma+d)/2s)} \| (|x|)^s f \|_{L^2_k}^{(2\gamma+d)/2s}, \tag{3.2}$$

where

$$K'(s,k) = \left(c_k \frac{\Gamma(\frac{2\gamma+d}{2s})\Gamma(1 - \frac{2\gamma+d}{2s})}{2^{\gamma+(d/2)}\Gamma(\gamma + (d/2) + 1)}\left(\frac{2s}{2\gamma + d} - 1\right)^{(2\gamma+d-2s)/2s}\right)^{1/2}.$$

To prove this theorem we need the following two lemmas.

LEMMA 3.2. *If $f \in L^1_{k,\mathrm{rad}}(\mathbb{R}^d)$ with $f(x) = F(|x|)$, then*

$$\int_{\mathbb{R}^d} f(x) \, d\mu_k(x) = \frac{1}{c_k 2^\lambda \Gamma(\lambda + 1)} \int_0^\infty F(t) t^{2\lambda+1} dt,$$

where

$$\lambda = \gamma + \frac{d-2}{2}.$$

PROOF. Using the spherical-polar coordinates $x = rx'$, where $x' \in S^{d-1}$,

$$\int_{\mathbb{R}^d} f(x) \, d\mu_k(x) = \int_0^\infty \int_{S^{d-1}} f(tx') w_k(x') \, d\sigma(x') t^{2\gamma+d-1} \, dt.$$

If $f(x) = F(|x|)$, then

$$\int_{\mathbb{R}^d} f(x) \, d\mu_k(x) = \int_{S^{d-1}} w_k(x') \, d\sigma(x') \int_0^\infty F(t) t^{2\gamma+d-1} \, dt.$$

On the other hand, from (1.1),

$$\begin{aligned} c_k^{-1} &= \int_{S^{d-1}} w_k(x') \, d\sigma(x') \int_0^\infty e^{-t^2/2} \, t^{2\gamma+d-1} \, dt \\ &= 2^\lambda \Gamma(\lambda + 1) \int_{S^{d-1}} w_k(x') \, d\sigma(x'). \end{aligned}$$

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Thus,

$$\int_{S^{d-1}} w_k(x')\,d\sigma(x') = \frac{1}{c_k 2^\lambda \Gamma(\lambda+1)}$$

and

$$\int_{\mathbb{R}^d} f(x)\,d\mu_k(x) = \frac{1}{c_k 2^\lambda \Gamma(\lambda+1)} \int_0^\infty F(t) t^{2\lambda+1}\,dt,$$

which completes the proof of the lemma. □

Lemma 3.3. *Let $s > \gamma + d/2$, then for every measurable function f on $\mathbb{R}^d$,*

$$\|f\|_{L^1_k} \le \left(\frac{\Gamma(\frac{\lambda+1}{s})\Gamma(1-\frac{\lambda+1}{s})}{c_k 2^{\lambda+1}\Gamma(\lambda+2)} \left(\frac{s}{\lambda+1} - 1 \right)^{(2\gamma+d-2s)/2s} \right)^{1/2} \|f\|_{L^2_k}^{1-((2\gamma+d)/2s)} \|(|x|)^s f\|_{L^2_k}^{(2\gamma+d)/2s}, \tag{3.3}$$

where

$$\lambda = \gamma + \frac{d-2}{2}.$$

Proof. Inequality (3.3) holds if $\|f\|_{L^2_k} = \infty$ or $\|(|x|)^s f\|_{L^2_k} = \infty$. Assume that $\|f\|_{L^2_k} + \|(|x|)^s f\|_{L^2_k} < \infty$. We write

$$\|f\|_{L^1_k} = \left(\int_{\mathbb{R}^d} (1+|x|^{2s})^{1/2} |f(x)| (1+|x|^{2s})^{-1/2}\,d\mu_k(x) \right).$$

From the hypothesis $s > \gamma + d/2$, we deduce that the function $x \mapsto (1+|x|^{2s})^{-1}$ belongs to $L^1_k(\mathbb{R}^d) \cap L^2_k(\mathbb{R}^d)$, and, by Hölder's inequality,

$$\begin{aligned} \|f\|_{L^1_k}^2 &\le \left(\int_{\mathbb{R}^d} \frac{d\mu_k(x)}{1+|x|^{2s}} \right) \left(\int_{\mathbb{R}^d} (1+|x|^{2s}) |f(x)|^2\,d\mu_k(x) \right) \\ &= \left(\int_{\mathbb{R}^d} \frac{d\mu_k(x)}{1+|x|^{2s}} \right) (\|f\|_{L^2_k}^2 + \|(|x|)^s f\|_{L^2_k}^2). \end{aligned}$$

However, by Lemma 3.2,

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{d\mu_k(x)}{1+|x|^{2s}} &= \frac{1}{c_k 2^\lambda \Gamma(\lambda+1)} \int_0^\infty \frac{t^{2\lambda+1}}{1+r^{2s}}\,dt \\ &= \frac{\Gamma(\frac{\lambda+1}{s})\Gamma(1-\frac{\lambda+1}{s})}{c_k 2^{\lambda+1}\Gamma(\lambda+1)s}. \end{aligned}$$

Replacing $f(x)$ by $f(rx)$, $r > 0$, in the last inequality gives

$$\|f\|_{L^1_k}^2 \le \frac{\Gamma(\frac{\lambda+1}{s})\Gamma(1-\frac{\lambda+1}{s})}{c_k 2^{\lambda+1}\Gamma(\lambda+1)s} (r^{2\gamma+d}\|f\|_{L^2_k}^2 + r^{2\gamma+d-2s}\|(|x|)^s f\|_{L^2_k}^2).$$

However, let g be the function defined on $(0, \infty)$ by

$$g(r) = r^{2\gamma+d}\|f\|_{L^2_k}^2 + r^{2\gamma+d-2s}\|(|x|)^s f\|_{L^2_k}^2.$$

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Then the minimum of g is attained at the point

$$r_0 = \Big(\frac{s}{\lambda+1} - 1\Big)^{1/2s} \Big(\frac{\|(|x|)^s f\|_{L_k^2}}{\|f\|_{L_k^2}}\Big)^{1/s},$$

and

$$g(r_0) = \frac{s}{\lambda+1}\Big(\frac{s}{\lambda+1} - 1\Big)^{(2\gamma+d-2s)/2s} \|f\|_{L_k^2}^{2-((2\gamma+d)/s)} \|(|x|)^s f\|_{L_k^2}^{(2\gamma+d)/s}.$$

Then we have the desired inequality

$$\|f\|_{L_k^1} \le \Big(\frac{\Gamma(\frac{\lambda+1}{s})\Gamma(1-\frac{\lambda+1}{s})}{c_k 2^{\lambda+1}\Gamma(\lambda+2)}\Big(\frac{s}{\lambda+1} - 1\Big)^{(2\gamma+d-2s)/2s}\Big)^{1/2} \|f\|_{L_k^2}^{1-((2\gamma+d)/2s)} \|(|x|)^s f\|_{L_k^2}^{(2\gamma+d)/2s},$$

which completes the proof of the lemma. □

Proof of Theorem 3.1. For $r > 0$, let $B_r = \{x : |x| < r\}$ and $B_r^c = \mathbb{R}^d \backslash B_r$. Denote by χ_{B_r} and $\chi_{B_r^c}$ the characteristic functions.

(i) Let $f \in L_k^2(\mathbb{R}^d)$. By Minkowski's inequality, for all $r > 0$,

$$\begin{aligned}\|\mathcal{F}_k(f)\chi_E\|_{L_k^2} &\le \|\mathcal{F}_k(f\chi_{B_r})\chi_E\|_{L_k^2} + \|\mathcal{F}_k(f\chi_{B_r^c})\chi_E\|_{L_k^2} \\ &\le (\mu_k(E))^{1/2}\|\mathcal{F}_k(f\chi_{B_r})\|_{L_k^\infty} + \|\mathcal{F}_k(f\chi_{B_r^c})\|_{L_k^2};\end{aligned}$$

hence it follows from (2.1) and (2.2) that

$$\|\mathcal{F}_k(f)\chi_E\|_{L_k^2} \le c_k(\mu_k(E))^{1/2}\|f\chi_{B_r}\|_{L_k^1} + \|f\chi_{B_r^c}\|_{L_k^2}. \tag{3.4}$$

On the other hand, by Hölder's inequality,

$$\|f\chi_{B_r}\|_{L_k^1} \le \|(|x|)^{-s}\chi_{B_r}\|_{L_k^2}\|(|x|)^s f\|_{L_k^2}.$$

By Lemma 3.2 and the hypothesis $s < \gamma + d/2$,

$$\|f\chi_{B_r}\|_{L_k^1} \le \Big(c_k\Big(\gamma + \frac{d}{2} - s\Big)2^{\gamma+(d/2)}\Gamma\Big(\gamma + \frac{d}{2}\Big)\Big)^{-1/2} r^{\gamma+(d/2)-s}\|(|x|)^s f\|_{L_k^2}. \tag{3.5}$$

Moreover,

$$\|f\chi_{B_r^c}\|_{L_k^2} \le \|(|x|)^{-s}\chi_{B_r^c}\|_{L_k^\infty}\|(|x|)^s f\|_{L_k^2} \le r^{-s}\|(|x|)^s f\|_{L_k^2}. \tag{3.6}$$

Combining the relations (3.4)–(3.6), we deduce that

$$\|\mathcal{F}_k(f)\chi_E\|_{L_k^2} \le (r^{-s} + a_k(\mu_k(E))^{1/2} r^{\gamma+(d/2)-s})\|(|x|)^s f\|_{L_k^2},$$

where

$$a_k = \Big(\frac{c_k}{(\gamma+\frac{d}{2}-s)2^{\gamma+(d/2)}\Gamma(\gamma+\frac{d}{2})}\Big)^{1/2}.$$

However, let g be the function defined on $(0, \infty)$ by

$$g(r) = r^{-s} + a_k(\mu_k(E))^{1/2} r^{\gamma+(d/2)-s}.$$

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Then the minimum of the function g is attained at the point

$$r_0 = \Big(\frac{2s}{(2\gamma+d-2s)a_k}\Big)^{2/(2\gamma+d)} (\mu_k(E))^{-1/(2\gamma+d)},$$

and

$$g(r_0) = \frac{2\gamma+d}{2\gamma+d-2s}\Big(\frac{(2\gamma+d-2s)a_k}{2s}\Big)^{2s/(2\gamma+d)} (\mu_k(E))^{s/(2\gamma+d)}.$$

Then we have the desired inequality

$$\|\mathcal{F}_k(f)\chi_E\|_{L^2_k} \le K(s,k)(\mu_k(E))^{s/(2\gamma+d)}\|(|x|)^s f\|_{L^2_k},$$

where

$$K(s,k) = \frac{2\gamma+d}{2\gamma+d-2s}\Big(\frac{c_k(\gamma+\frac{d}{2}-s)}{2^{\gamma+(d/2)}\Gamma(\gamma+\frac{d}{2})s^2}\Big)^{s/(2\gamma+d)}.$$

(ii) Suppose that the right-hand side of (3.2) is finite. Then, according to Lemma 3.3, the function f belongs to $L^1_k(\mathbb{R}^d)$ and

$$\begin{aligned}\|\mathcal{F}_k(f)\chi_E\|_{L^2_k} &\le (\mu_k(E))^{1/2}\|\mathcal{F}_k(f)\|_{L^\infty_k}\\ &\le c_k(\mu_k(E))^{1/2}\|f\|_{L^1_k}\\ &\le K'(s,k)(\mu_k(E))^{1/2}\|f\|_{L^2_k}^{1-((2\gamma+d)/2s)}\|(|x|)^s f\|_{L^2_k}^{(2\gamma+d)/2s},\end{aligned}$$

where

$$K'(s,k) = \Big(c_k\frac{\Gamma(\frac{2\gamma+d}{2s})\Gamma(1-\frac{2\gamma+d}{2s})}{2^{\gamma+(d/2)}\Gamma(\gamma+\frac{d}{2}+1)}\Big(\frac{2s}{2\gamma+d}-1\Big)^{(2\gamma+d-2s)/2s}\Big)^{1/2},$$

which completes the proof of the theorem.

Remark 3.4. If we interchange f and $\mathcal{F}_k(f)$ in (3.1) and (3.2), then:

(a) If $0 < s < \gamma + d/2$, then for every $f \in L^2_k(\mathbb{R}^d)$,

$$\sup_{E\subset\mathbb{R}^d,\, 0<\mu_k(E)<\infty} ((\mu_k(E))^{-s/(2\gamma+d)}\|f\chi_E\|_{L^2_k}) \le K(s,k)\|(|y|)^s\mathcal{F}_k(f)\|_{L^2_k}.$$

The left-hand side is known to be an equivalent norm of the Lorentz space $L_k^{p_s,\infty}$, where

$$p_s = \frac{2(2\gamma+d)}{2\gamma+d-2s}.$$

(b) If $s > \gamma + d/2$, then for every $f \in L^2_k(\mathbb{R}^d)$,

$$\begin{aligned}&\sup_{E\subset\mathbb{R}^d,\, 0<\mu_k(E)<\infty} ((\mu_k(E))^{-1/2}\|f\chi_E\|_{L^2_k})\\ &\qquad\le K'(s,k)(\mu_k(E))^{1/2}\|f\|_{L^2_k}^{1-((2\gamma+d)/2s)}\|(|y|)^s\mathcal{F}_k(f)\|_{L^2_k}^{(2\gamma+d)/2s}.\end{aligned}$$

The left-hand side is known to be an equivalent norm of the Lorentz space $L_k^{\infty,\infty}$.

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4. Heisenberg–Pauli–Weyl inequality

In this section, we shall use the local uncertainty inequality (3.1) to prove an analogue of the Heisenberg–Pauli–Weyl uncertainty inequality.

THEOREM 4.1. *For all $f \in L^2_k(\mathbb{R}^d)$ and $s > 0$,*

$$\Big(\int_{\mathbb{R}^d} |x|^{2s}|f(x)|^2\, d\mu_k(x)\Big)\Big(\int_{\mathbb{R}^d} |y|^{2s}|\mathcal{F}_k(f)(y)|^2\, d\mu_k(y)\Big) \geq C(s,k)\|f\|^4_{L^2_k},$$

where $C(s,k)$ is the constant given by

$$C(s,k) = \Big(\frac{2\gamma+d-2s}{4\gamma+2d}\Big)^2\Big(\frac{2^{2\gamma+d}\Gamma(\gamma+\frac{d}{2})\Gamma(\gamma+\frac{d}{2}+1)s^2}{\gamma+\frac{d}{2}-s}\Big)^{2s/(2\gamma+d)}.$$

PROOF. First, suppose that $0 < s < \gamma + d/2$. Let $r > 0$. Then, by (2.2),

$$\|f\|^2_{L^2_k} = \|\mathcal{F}_k(f)\|^2_{L^2_k} = \int_{B_r} |\mathcal{F}_k(f)(y)|^2\, d\mu_k(y) + \int_{B_r^c} |\mathcal{F}_k(f)(y)|^2\, d\mu_k(y). \tag{4.1}$$

Consider

$$\int_{B_r^c} |\mathcal{F}_k(f)(y)|^2\, d\mu_k(y) = \int_{B_r^c} |\mathcal{F}_k(f)(y)|^2|y|^{2s}|y|^{-2s}\, d\mu_k(y);$$

we deduce that

$$\int_{B_r^c} |\mathcal{F}_k(f)(y)|^2\, d\mu_k(y) \leq r^{-2s}\|(|y|)^s\mathcal{F}_k(f)\|^2_{L^2_k}. \tag{4.2}$$

By the local uncertainty inequality (3.1) and Lemma 3.2,

$$\int_{B_r} |\mathcal{F}_k(f)(y)|^2\, d\mu_k(y) \leq Cr^{2s}\|(|x|)^s f\|^2_{L^2_k}, \tag{4.3}$$

where

$$C = \Big(\frac{2\gamma+d}{2\gamma+d-2s}\Big)^2\Big(\frac{\gamma+\frac{d}{2}-s}{2^{2\gamma+d}\Gamma(\gamma+\frac{d}{2})\Gamma(\gamma+\frac{d}{2}+1)s^2}\Big)^{2s/(2\gamma+d)}.$$

Combining the relations (4.1)–(4.3),

$$\|f\|^2_{L^2_k} \leq Cr^{2s}\|(|x|)^s f\|^2_{L^2_k} + r^{-2s}\|(|y|)^s\mathcal{F}_k(f)\|^2_{L^2_k}.$$

However, let g be the function defined on $(0,\infty)$ by

$$g(r) = Cr^{2s}\|(|x|)^s f\|^2_{L^2_k} + r^{-2s}\|(|y|)^s\mathcal{F}_k(f)\|^2_{L^2_k}.$$

Then the minimum of the function g is attained at the point

$$r_0 = \Big(\frac{\|(|y|)^s\mathcal{F}_k(f)\|_{L^2_k}}{\sqrt{C}\|(|x|)^s f\|_{L^2_k}}\Big)^{1/2s},$$

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and

$$g(r_0) = 2\sqrt{C}\|(|x|)^s f\|_{L^2_k}\|(|y|)^s \mathcal{F}_k(f)\|_{L^2_k}.$$

Then we have the desired inequality

$$\|f\|^2_{L^2_k} \leq 2\sqrt{C}\|(|x|)^s f\|_{L^2_k}\|(|y|)^s \mathcal{F}_k(f)\|_{L^2_k}. \tag{4.4}$$

Second, suppose that $s \geq \gamma + d/2$. By Hölder's inequality and (2.2),

$$\int_{\mathbb{R}^d} |x|^{2s}|f(x)|^2\, d\mu_k(x) \leq \|f\|_{L^2_k}\Big(\int_{\mathbb{R}^d} |x|^{4s}|f(x)|^2\, d\mu_k(x)\Big)^{1/2}$$

and

$$\int_{\mathbb{R}^d} |y|^{2s}|\mathcal{F}_k(f)(y)|^2\, d\mu_k(y) \leq \|f\|_{L^2_k}\Big(\int_{\mathbb{R}^d} |y|^{4s}|\mathcal{F}_k(f)(y)|^2\, d\mu_k(y)\Big)^{1/2}.$$

The value of s in (4.4) can be replaced by $2s$, and doing so repeatedly, (4.4) is extended to any positive s. □

REMARK 4.2. The previous inequality generalises the Heisenberg–Pauli–Weyl inequality (1.2) proved by Rösler [11] and Shimeno [12].

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