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RADII OF UNIVALENCE, STARLIKENESS, AND CONVEXITY

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Let a function $f(z) = z + \sum_{2}^{\infty} a_n z^n$ be regular in the disk $|z| < 1$. The radius of univalence $0.164 \ldots$ of the family of f with $|a_n| \le n$ $(n \ge 2)$ is, actually, the radius of starlikeness. The radius of univalence $1 - [K/(1+K)]^{\frac{1}{2}}$ of the family of f with $|a_n| \le K$ $(n \ge 2)$, where $K > 0$ is a constant, is, actually, the radius of starlikeness. The radii of convexity of the two families are estimated from below.

1. Introduction

Let N be the family of functions f regular in $D = \{|z| < 1\}$ with the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n . \tag{1.1}$$

Let F be a non-empty subfamily of N. The largest number $u(F)$ of r, $0 < r \le 1$, such that each $f \in F$ is univalent in $D(r) = \{|z| < r\}$, is called the radius of univalence of F. The radius of starlikeness $s(F)$ and that of convexity $c(F)$ of F are defined on adding further the condition that the image $f(D(r))$ is star-shaped with respect to the origin, and the condition that $f(D(r))$ is convex, respectively.

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Evidently, $c(F) \le s(F) \le u(F)$. The determination of $u(F)$, $s(F)$ and $c(F)$ has been one of the subjects in the theory of univalent functions; see, for example, [1]. It is well-known that $s(S) = \tanh(\pi/4)$ and $c(S) = 2 - \sqrt{3}$ for the family S of all univalent members of N .

Let B be the family of f of (1.1) with $|a_n| \le n$ for all $n \ge 2$. Let $K > 0$ be a constant and let $G(K)$ be the family of f of (1.1) with $|a_n| \le K$ for all $n \ge 2$. Gavrilov [3, Theorems 1 and 1'] proved that $u(B) = r_0$, where r_0 is the root in the interval $(0, 1)$ of the equation $2(1-r)^3 - (1+r) = 0$, and that $u(G(K)) = r_1 \equiv 1 - [K/(1+K)]^{\frac{1}{2}}$. Gavrilov's estimate $0.125 < r_0 < 0.130$ is erroneous because $r_0 = 0.164 \ldots$. We first improve his results.

THEOREM 1. *The identities* $u(B) = s(B)$ *and* $u(G(K)) = s(G(K))$ *hold.*

Next we investigate the lower bounds of $c(B)$ and $c(G(K))$.

THEOREM 2. *Let* $r_2 = 0.090 \ldots$ *be the root in* $(0, 1)$ *of the equation* $2(1-r)^4 - (1+4r+r^2) = 0$, *and let* r_3 *be the root in* $(0, 1)$ *of the equation* $(1+K^{-1})(1-r)^3 - (1+r) = 0$. *Then* $c(B) \ge r_2$ *and* $c(G(K)) \ge r_3$.

We note that $(2-\sqrt{3})r_0 = 0.04 \ldots < r_2$ and $(2-\sqrt{3})r_1 < r_3$. The latter inequality needs a proof.

2. Proofs

We shall make use of the following lemma due to Alexander and Remak; see [4, Theorem 1] and [2, Theorem 3].

LEMMA AR. *If* $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ *is a member of* N *and if*

$$\sum_{n=2}^{\infty} n|b_n| \le 1 ,$$

then h *is univalent and starlike in* D , *while if*

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$$\sum_{n=2}^{\infty} n^2|b_n| \leq 1 ,$$

then h *is univalent and convex in* D .

Proof of Theorem 1. Since $s(B) \leq u(B) = r_0$, it suffices to observe that $r_0 \leq s(B)$. For this purpose let $0 < r \leq r_0$, and let f of (1.1) be a member of B . On applying Lemma AR to $h(z) = r^{-1}f(rz)$, together with

$$\sum_{n=2}^{\infty} n|a_n|r^{n-1} \leq \sum_{n=2}^{\infty} n^2r^{n-1} \leq \sum_{n=2}^{\infty} n^2r_0^{n-1} = \left(1+r_0\right)/\left(1-r_0\right)^3 - 1 = 1 ,$$

one can conclude that h is univalent and starlike in D , or, f is starlike in the disk $D(r)$. Therefore $r_0 \leq s(B)$.

For the proof of $s\left(G(K)\right) = u\left(G(K)\right) = r_1$ we note that $s\left(G(K)\right) \leq r_1$. For the proof of the converse we let f of (1.1) be a member of $G(K)$ and let $0 < r \leq r_1$. On applying Lemma AR to $h(z) = r^{-1}f(rz)$, together with

$$\sum_{n=2}^{\infty} n|a_n|r^{n-1} \leq K \sum_{n=2}^{\infty} nr_1^{n-1} = K\left[\left(1-r_1\right)^{-2}-1\right] = 1 ,$$

one observes that h is univalent and starlike in D , or, f is starlike in $D(r)$. Therefore $r_1 \leq s\left(G(K)\right)$.

Proof of Theorem 2. For r , $0 < r \leq r_2$, for $f \in B$ and for $z \in D$, we set $h(z) = r^{-1}f(rz)$. By Lemma AR, together with the estimate

$$\sum_{n=2}^{\infty} n^2|a_n|r^{n-1} \leq \sum_{n=2} n^3r_2^{n-1} = \left(1+4r_2+r_2^2\right)/\left(1-r_2\right)^4 - 1 = 1 ,$$

one observes that h is univalent and convex in D , whence the same is true of f in $D(r)$. Therefore $r_2 \leq c(B)$.

For r , $0 < r \leq r_3$, for $f \in G(K)$ and for $z \in D$, we set $h(z) = r^{-1}f(rz)$. By Lemma AR, together with the estimate

$$\sum_{n=2}^{\infty} n^2|a_n|r^{n-1} \le K \sum_{n=2}^{\infty} n^2 r_3^{n-1} = K\left[(1+r_3)/(1-r_3)^3-1\right] = 1 ,$$

one observes that h is univalent and convex in D, whence the same is true of f in $D(r)$. Therefore $r_3 \le c(G(K))$.

REMARK. For f of (1.1) we set

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k \quad (n \ge 2) .$$

If $f \in B$, then the partial sum $f_n \in B$ for all $n \ge 2$. Therefore Gavrilov's assertion on f_n in [3, Theorem 1] is superfluous. The same is true of f_n for $f \in G(K)$ in [3, Theorem 1'].

It remains to prove that

$$(2-\sqrt{3})(1-\alpha) = (2-\sqrt{3})r_1 < r_3 ,$$

where $\alpha = [K/(1+K)]^{\frac{1}{2}}$. Since the function $\varphi(x) = (1-x)^3/(1+x)$ is decreasing for $0 \le x \le 1$, and since $\varphi(r_3) = \alpha^2$, it suffices to observe that

$$\varphi\left((2-\sqrt{3})(1-\alpha)\right) > \alpha^2 , \text{ or } \Phi(\alpha) > 0 ,$$

where

$$\Phi(x) = (14-8\sqrt{3})x^3 + (-30+17\sqrt{3})x^2 + (21-12\sqrt{3})x + (-5+3\sqrt{3})$$

for $0 \le x \le 1$. As is easily checked, $\Phi'(x) = 0$ has only one solution λ in $0 < x < 1$, and Φ is increasing (decreasing, respectively) in $[0, \lambda]$ ($[\lambda, 1]$, respectively). Since $\Phi(0) > 0 = \Phi(1)$, one can assert that $\Phi(\alpha) > 0$.

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