2. N. Lord, Sequences of averages revisited, *Math. Gaz.* **95** (July 2011) pp. 314-317.
3. N. MacKinnon, Centre of mass by linear transformation, *Math. Gaz.* **72** (March 1988) pp. 34-36.
4. H. Flanders, Averaging sequences again, *Math. Gaz.* **80** (July 1996) pp. 219-222
5. N. Sloane, *The online encyclopedia of integer sequences*, https://oeis.org (accessed Sep. 2022)

10.1017/mag.2024.103

ERIK VIGREN
Swedish Institute of Space Physics,
Uppsala,
Sweden
e-mail: *erik.vigren@irfu.se*

108.29 A geometric mean–arithmetic mean ratio limit

One of the truly delightful results related to the natural numbers is the following limit of the ratio of the geometric and arithmetic means of the first n natural numbers:

$$\lim_{n \to \infty} \frac{\sqrt[n]{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}}{\frac{1}{n}(1 + 2 + 3 + \ldots + n)} = \frac{2}{e}. \tag{1}$$

Obviously, the ratio in (1) approaches its limit really slowly. In fact, the relative difference between the ratio and its limiting value is of order $(n + 1)^{(2n)^{-1}}$, as $n \to \infty$. For example, this is about 2% when $n = 100$.

Some generalisations of the limit can be found in [1], [2] and [3].

In this Note, we offer a short proof and generalisation of limit (1). Our result is narrower here, but the techniques are wholly different from [1], [2] and [3], and rely solely, in theory, on algebraic limit properties. Our proof relies on the following well-known result.

Lemma [See e.g. [4, p. 81]]: Let a_n be a sequence of positive reals with $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L$. Then $\lim_{n \to \infty} \sqrt[n]{a_n} = L$.

We now establish a generalisation of (1) in the following theorem.

Theorem: Let $\{b_n\}$ be a sequence of positive reals with $\lim_{n \to \infty} b_n - n = 0$. Then

$$\lim_{n \to \infty} \frac{\sqrt[n]{b_1 b_2 b_3 \ldots b_n}}{\frac{1}{n}(b_1 + b_2 + b_3 + \ldots + b_n)} = \frac{2}{e}.$$

Proof: We apply the Lemma to

$$a_n = \frac{\prod_{i=1}^{n} b_i}{\left(\frac{1}{n} \sum_{i=1}^{n} b_i\right)^n}.$$

Note that $\frac{1}{n}\sum_{i=1}^{n} b_i = \frac{1}{n}\sum_{i=1}^{n}(b_i - i) + \frac{1}{n}\sum_{i=1}^{n} i$ and define $c_n = \frac{1}{n}\sum_{i=1}^{n}(b_i - i)$. Then

$$\frac{a_{n+1}}{a_n} = b_{n+1}\frac{\left(c_n + \frac{1}{2}(n+1)\right)^n}{\left(c_{n+1} + \frac{1}{2}(n+2)\right)^{n+1}}$$

$$= \frac{b_{n+1}}{c_{n+1} + \frac{1}{2}(n+2)}\left(\frac{n+1+2c_n}{n+2+2c_{n+1}}\right)^n.$$

Noting that $\lim_{n\to\infty} c_n = 0$, we see that the limit of the first part is 2. We may find the limit of the second part directly, or using the main result of [5]:

$$\lim_{n\to\infty}\left(\frac{n+1+2c_n}{n+2+2c_{n+1}}\right)^n = \exp\left(\lim_{n\to\infty} n\frac{-1+2c_n-2c_{n+1}}{n+2+2c_{n+1}}\right) = e^{-1}.$$

This completes the proof.

Note that taking $b_n = n$ in the Theorem gives (1). As a general example, the Theorem applies to any sequence $b_n = n + f(n)$, where $f(n) \to 0$; for example, $b_n = n + \frac{1}{\sqrt{n}}$.

Acknowledgements

The authors would like to thank the Editor and the anonymous reviewer for their valuable suggestions.

References

1. R. P. Kubelka, Means to an end, *Mathematics Magazine* **74** (2001) pp. 141-142.
2. C. Xu, A GM-AM Ratio, *Mathematics Magazine* **83** (2010) pp. 49-50.
3. R. Farhadian, R. Jakimczuk, On the ratio of the arithmetic and geometric means of the first n terms of some general sequences, *Transnational Journal of Mathematical Analysis and Applications* **9** (2022) pp. 67-85.
4. K. A. Ross, J. M. López, *Elementary analysis* (2nd edn). Undergraduate Texts in Mathematics, Springer, (2013).
5. R. Farhadian, V. Ponomarenko, Indeterminate exponentials without tears. *Math. Gaz*. **108** (March 2024) pp. 146-148.

10.1017/mag.2024.104 © The Authors, 2024
Published by Cambridge
University Press on behalf of
The Mathematical Association

REZA FARHADIAN
Department of Statistics,
Razi University,
Kermanshah, Iran
e-mail: *farhadian.reza@yahoo.com*
VADIM PONOMARENKO
Department of Mathematics and Statistics,
San Diego State University, San Diego, USA
e-mail: *vponomarenko@sdsu.edu*