SOME ENTIRE FUNCTIONS WITH FIXPOINTS OF EVERY ORDER

I. N. BAKER

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1. Introduction

In this paper $f(z)$ will always stand for an entire transcendental function of the complex variable z. For $p = 1, 2, \cdots$ the natural iterate $f_p(z)$ of $f(z)$ is defined by

$$f_1(z) = f(z), \quad f_p(z) = f_{p-1}(f(z)) = f(f_{p-1}(z)).$$

These natural iterates are themselves entire transcendental functions; they have been studied by various writers, notably Fatou [3]. References to many papers on iterates will be found in [1].

A fixpoint of $f(z)$ is a zero of $f(z) - z$; more generally a fixpoint of order p of $f(z)$ is a zero of $f_p(z) - z$. A fixpoint of order p is said to have order exactly p when it is not a fixpoint of order less than p.

The fixpoints are of great importance in the theory of iteration so that a discussion of their existence and distribution is interesting. In [2] it is pointed out that very little is known about the existence of fixpoints of the various orders and a few results are derived in the case where $f(z)$ has order less than $\frac{1}{2}$. Although it is known that any $f(z)$ has fixpoints of arbitrarily high exact order no examples seem to have been given of functions having fixpoints of order exactly p for every natural number p. In this paper it is shown that the class C_p of functions $\{f(z); f_p(z)$ has finite defect values the sum of whose defects is greater than $\frac{1}{2}\}$ has fixpoints of order exactly p. The class formed by the intersection of classes C_p, $p = 1, 2, \cdots$ has fixpoints of all exact orders. In particular any function $f(z)$ with a Picard exceptional value is of this type and there are others as shown by Lemma 4. Finally one may conjecture that any $f(z)$ has fixpoints of every exact order from a certain order on.

2. Preliminary Lemmas

The following notation will be used (c.f. Nevanlinna [4]):

$$M_p(r) = M(f_p, r) = \max_{|z|=r} |f_p(z)|$$

$$n_p(r, a) = n(f_p, r, a) = \text{number of solutions of } f_p(z) = a \text{ in } |z| \leqq r$$

$$N_p(r, a) = \int_0^r \frac{n_p(t, a) - n_p(0, a)}{t} dt + n_p(0, a) \log r$$

$$T_p(r) = T(f_p, r) = \frac{1}{2\pi} \int_0^{2\pi} \overset{+}{\log} |f_p(re^{i\varphi})| d\varphi = m_p(r, \infty)$$

$$\delta_p(a) = 1 - \overline{\lim_{r\to\infty}} \frac{N_p(r, a)}{T_p(r)} = \underline{\lim_{r\to\infty}} \frac{m_p(r, a)}{T_p(r)}.$$

LEMMA 1 (Pólya [5]). *Let $e(z)$, $g(z)$ and $h(z)$ be entire functions satisfying*

(1) $e(z) = g(h(z))$
(2) $h(0) = 0.$

There is a constant c independent of e, g, h with

(3) $M(e, r) > M\left[g, cM\left(h, \frac{r}{2}\right)\right].$

Further it is clear that the condition (2) can be dropped provided (3) is to hold only for all sufficiently great r and this is the form we shall use in the proof of Lemma 3.

LEMMA 2 (e.g. Baker [1, p. 124]). *If $f(z)$ is an entire function and $k > 1$, $a > 1$ are constants, then for all sufficiently large r one has*

$$M(f, ar^k) > M^k(f, r). \tag{4}$$

LEMMA 3. *If $f(z)$ has an exceptional value b (taken only a finite number k of times) then b is a value of defect one for $f_p(z)$, $p = 1, 2, \cdots$.*

PROOF: Let the roots of $f(z) = b$ be $d_1, d_2, \cdots d_k$. The roots of $f_p(z) = b$ are the roots of $f_{p-1}(z) = d_i$, $i = 1, 2, \cdots, k$. We assume they are counted according to the usual multiplicities so that

$$n_p(r, b) = \sum_{i=1}^{k} n_{p-1}(r, d_i)$$

and

$$N_p(r, b) = \sum_{i=1}^{k} N_{p-1}(r, d_i) \leqq kT_{p-1}(r) + O(1) \tag{5}$$

by the first fundamental theorem [4].

Now [4, p. 220]

$$T_p(r) \geqq \tfrac{1}{3} \log M_p\left(\frac{r}{2}\right)$$

$$\geqq \tfrac{1}{3} \log M_{p-1}\left(cM_1\left(f, \frac{r}{4}\right)\right) \text{ by Lemma 1,}$$

and however small $\varepsilon > 0$ is, this becomes greater than

$$\tfrac{1}{3} \log M_{p-1}(r^{(3k+1)/\varepsilon})$$

which by Lemma 2 is greater than

$$\tfrac{1}{3} \log M_{p-1}^{3k/\varepsilon}(r) \geqq \frac{k}{\varepsilon} T_{p-1}(r), \tag{6}$$

all of these inequalities being understood to hold only for sufficiently large r. Then from (5), (6)

$$\overline{\lim_{r\to\infty}} \frac{N_p(r, b)}{T_p(r)} \leqq \varepsilon \text{ and hence } \delta_p(b) = 1.$$

LEMMA 4. *There are functions $f(z)$ other than those of Lemma 3 such that a value b is of defect one for $f_p(z)$, $p = 1, 2, \cdots$*

PROOF: Consider a function

$$f(z) = b + e^{e^z} h(z), \quad b > 0 \tag{7}$$

where $h(z)$ is a function of order 1 with the properties:

$$M(h, r) = h(r) > 0 \tag{8}$$

$$\exp\left(\frac{r}{2}\right) < h(r) < e^r \text{ for large } r \tag{9}$$

$$h(z) \text{ has an infinity of zeros.} \tag{10}$$

We could take $h(z) = \sinh z$ but the proof is just as simple with general $h(z)$. We see that

$$M(f, r) = f(r)$$
$$M(f_p, r) = f_p(r)$$

and

$$\exp\exp r < f(r) < \exp(2\exp r) \tag{11}$$

all hold for large r. Now

$$f_p(z) = b + \{\exp\exp f_{p-1}(z)\} h(f_{p-1}(z))$$

and

$$N_p(r, b) = N(h(f_{p-1}), r) \leqq T(h(f_{p-1}), r) < \log h(f_{p-1}(r)),$$
$$N_p(r, b) < f_{p-1}(r). \tag{12}$$

On the other hand [4, p. 220]:

$$T_p(r) = T(f_p, r) > \tfrac{1}{3} \log f_p\left(\frac{r}{2}\right) > \tfrac{1}{3}\log f_{p-1}\left\{h_2\left(\frac{r}{2}\right)\right\} \tag{13}$$

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from (9), (11),

$$h_2\left(\frac{r}{2}\right) > \exp\left(\tfrac{1}{2}h\left(\frac{r}{2}\right)\right) > \exp(r^2)$$

by Liouville's theorem and

$$f\left(h_2\left(\frac{r}{2}\right)\right) > f(\exp(r^2)) > \exp\exp\exp(r^2) > \exp\exp(4e^r)$$
$$= \exp\{(\exp 2e^r)^2\} > \exp\{f^2(r)\}.$$

By induction

$$f_{p-1}\left(h_2\left(\frac{r}{2}\right)\right) > \exp\{f^2_{p-1}(r)\},$$

and from (13)

$$T_p(r) > \tfrac{1}{3}f^2_{p-1}(r).$$

Together with (12) this gives

$$\frac{N_p(r, b)}{T_p(r)} < \frac{3}{f_{p-1}(r)} \to 0 \text{ as } r \to \infty.$$

Thus $\delta_p(b) = 1$. All the inequalities above are supposed to hold only for sufficiently large r.

3. The Results on Fixpoints

THEOREM. *Suppose $f(z)$ has defect values b_i, $i = 1, 2, \cdots, k$ so that*

$$\sum_{i=1}^{k} \delta_p(b_i) = \tfrac{1}{2} + d; \quad d > 0, \quad b_i \neq \infty.$$

Then $f(z)$ has fixpoints of order exactly p.

If $m_p(r, b_i)$ is the "Schmiegungsfunktion" defined by

$$m_p(r, b_i) = \frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log}\left|\frac{1}{f_p(re^{i\varphi}) - b_i}\right| d\varphi$$

we have

$$\sum_{i=1}^{k} m_p(r, b_i) > \frac{1+d}{2} T_p(r) = \frac{1+d}{2} T(f_p, r)$$

for all sufficiently large r. Denote by m'_p, m''_p, T'_p, T''_p the Schmiegung and characteristic functions for $f'_p(z)$ and $f''_p(z)$ respectively and by $\bar{m}_p$, $\bar{T}_p$ the functions for $f_p(z) - z$.

In his discussion of the second fundamental theorem Ullrich [6, p. 598 equation (20)] has proved a result which we write as

$$(14) \qquad m'_p(r, 0) \geqq \sum_{i=1}^{k} m_p(r, b_i) - O\left[\log\left(rT_p(r)\right)\right] \qquad (E)$$

where the symbol (E) means that the given estimate of the remainder term holds with the possible exception of a set of r-intervals whose total length is finite. Thus in our case

$$(15) \qquad m'_p(r, 0) \geqq \frac{1+d}{2} T_p(r) - O\left[\log\left(rT_p(r)\right)\right] \qquad (E).$$

We note that

$$\begin{aligned} T'_p(r) &= m'_p(r, \infty) + O(1) \\ (16) \qquad &\leqq m_p(r, \infty) + m\left(\frac{f'_p}{f_p}, r, \infty\right) + O(1) \\ &= T_p(r) + O[\log\left(rT_p(r)\right)] \qquad (E). \end{aligned}$$

by the theorem of the logarithmic derivative [e.g. 7 p. 594]. From the second fundamental theorem and (16):

$$\begin{aligned} (17) \qquad m'_p(r, 0) + m'_p(r, 1) &\leqq T'_p(r) + O\left[\log\left(rT'_p(r)\right)\right] \qquad (E). \\ &\leqq T_p(r) + O\left[\log\left(rT_p(r)\right)\right]. \qquad (E). \end{aligned}$$

Applying the result of Ullrich used in (14) but this time to the function $f(z) - z$ and its derivative we obtain

$$\bar{m}_p(r, 0) \leqq m'_p(r, 1) + O\left[\log rT_p(r)\right] \qquad (E),$$

and using (15), (17):

$$\begin{aligned} \bar{m}_p(r, 0) &\leqq m'_p(r, 1) + O\left[\log\left(rT_p(r)\right)\right] \qquad (E) \\ &\leqq \left(\frac{1-d}{2}\right) T_p(r) + O\left[\log\left(rT_p(r)\right)\right]. \qquad (E). \end{aligned}$$

Using the first fundamental theorem and $T_p^{\cdot}(r) \simeq T_p(r)$ it follows that

$$(18) \qquad \bar{N}_p(r, 0) \geqq \left(\frac{1+d}{2}\right) T_p(r) - O\left[\log\left(rT_p(r)\right)\right] \qquad (E).$$

Now by a further application of the result of (14) and (15):

$$\begin{aligned} m''_p(r, 0) &\geqq m'_p(r, 0) - O\left[\log\left(rT'_p(r)\right)\right] \qquad (E) \\ &\geqq \frac{1+d}{2} T_p(r) - O\left[\log rT_p(r)\right] \qquad (E) \end{aligned}$$

and

$$(19) \qquad N''_p(r, 0) \leqq T''_p(r) - \left(\frac{1+d}{2}\right) T_p(r) + O\left[\log\left(rT_p(r)\right)\right] \qquad (E),$$

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while from (16):

$$T_p''(r) \leqq T_p(r) + O\left[\log\left(rT(r)\right)\right] \qquad \text{(E)}$$

which reduces (19) to

$$N_p''(r, 0) \leqq \left(\frac{1-d}{2}\right) T_p(r) + O\left[\log\left(rT_p(r)\right)\right] \qquad \text{(E)}.$$

Thus from (18), (19):

$$\text{(20)} \qquad \tfrac{1}{2}\{\bar{N}_p(r, 0) - N_p''(r, 0)\} \geqq \frac{d}{2} T_p(r) - O\left[\log\left(rT_p(r)\right)\right] \qquad \text{(E)}$$

and for some values of r the quantity on the left hand side of (20) will take large values of the same order as $dT_p(r)/2$.

A k-fold ($k \geqq 1$) zero of $f_p(z) - z$ is counted k times in $\bar{N}_p(r, 0)$ but only $\operatorname{Max}\{0, k-2\}$ times in N_p'' so that the left hand side of (20) is not greater than

$$N_+(r, 0) = \int_0^r \frac{n_+(t, 0) - n_+(0, 0)}{t}\, dt + n_+(0, 0) \log r$$

where $n_+(t, 0)$ counts the number of *different* solutions of $f_p(z) = z$ in $|z| \leqq t$. For all sufficiently large r

$$\text{(21)} \qquad \sum_{j=1}^{p-1} \bar{N}_j(r, 0) \leqq \sum_{j=1}^{p-1} T_j(r, 0) + O(1)$$

and by Lemma 1 (as applied in Lemma 3) this right hand side of (21) is $o(T_p(r, 0)$ for large r. The left hand side of (21) is an upper bound for the contribution of fixpoints of orders less than p to $N_+(r, 0)$ since each of them is counted at least once there. Thus from (20) the counting function of different fixpoints of order p is $> dT_p(r)/3$ for some arbitrarily large values of r, while from (21) this is not caused by the fixpoints which are of exact order less than p. It follows that the fixpoints of exact order p have a counting function $\tilde{N}(r)$ which satisfies

$$\overline{\lim_{r\to\infty}} \frac{\tilde{N}(r)}{T(r)} > 0$$

and that a great many such fixpoints exist.

APPLICATION: The functions of Lemma 3 and 4 afford examples of functions which have *fixpoints of exact order p for all natural numbers p.*

Note: Professor W. K. Hayman has pointed out that the constant $\frac{1}{2} + d$ in the above theorem can be replaced by d alone if $d > 0$, i.e. it is sufficient to suppose that $f_p(z)$ has some defective value b. One has only to apply Nevanlinna's theory to the function $(f_p(z) - z)/(f_p(z) - b)$.

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Imperial College of Science and Technology, London