Mathematics in 'the news': number theory and number sense

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Time spent in national pandemic 'lockdowns' and 'tiers' for most of 2020 has created an opportunity to revisit some ideas previously committed to paper, but unfinished. I now return to one of them, to continue and share the line of thought, if not to complete it.

My local newspaper, the *Cambridge News*, was preceded (until 2007) by the *Cambridge Evening News* (CEN). In common with many daily newspapers, both include (-ed) a whole-page spread of puzzles of various kinds. Along with the crossword, this can include a vocabulary test, a general knowledge section–and a brain twister. Beneath the light wrappings of a domestic scenario of some kind, this last feature is overtly mathematical, and always numerical. The problem setter frequently shows considerable ingenuity and originality, and has certainly brought some properties of, and relationships between, particular numbers to my attention for the first time. Occasionally the problem posed is less demanding, and the process of solution relatively elementary, for example:

> "Yes, we've got four boys," said John. "They're spaced evenly, two years apart." Martha smiled. "I've only met Tim, I gather he's the eldest." "That's right." John nodded. "He's three times as old as his youngest brother." How old was Tim? (*Cambridge Evening News*, 31 July 2003, p. 38).

What interests me is the fact that a great many of the problems set–and I've not been regular in trying them myself–seem to call upon some knowledge of number theory in their solution. For example, several can be reduced to a linear Diophantine equation $ax + by = c$ requiring positive integer solutions x, y, usually subject to some condition, such as a restriction on their size. Number theory is, of course, a source of fascination and recreation for a great many individuals, and by no means all of them have any formal education in the subject. In any case, the readership of the *Cambridge (Evening) News* probably has its fair share of mathematicians. Nevertheless, I still wonder what proportion of its readers are drawn to these brain twisters, and what methods they use in their solutions.

Many of the problems that I've looked at lend themselves to a computational solution, using a spreadsheet or some programming language, whereby a large number of possibilities is considered in order to arrive at one (or more) that fit the conditions. Programming the computer to find solutions in this way can be both enjoyable and satisfying, and my guess–unsupported by empirical evidence–would be that many readers have enjoyed` the brain twisters for such opportunities. However, I have become interested to see which problems I can solve analytically, without recourse to anything more sophisticated than simple arithmetic, and without the need to test a large number of possible cases. I couldn't resist offering a few of these problems to my own number theory students; one of my problem

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sheets included the following:

> A woman presents a cheque for cash in a bank. The cashier misreads the amount on the cheque, and pays out the cash as if the pounds and pence were reversed. Outside the bank, the woman drops 2 pence down a drain. She realises that she now has twice as much as she would have received in the bank if the cashier had not made the error. What sum was written on the cheque?

For me, and I hope, for my students too, this invites a Diophantine equation, and its solution. I'm not sure what I would do otherwise. I'm also thinking that readers with secondary school mathematics might be able to form the algebraic equation: but the coefficients of the two variables (numbers of pounds and pence) are quite large. So how would one solve it without the Euclidean algorithm?

A twister–doughnuts and eclairs

Here's an example of a *Cambridge Evening News* brain twister, with my solution. It pleased me to find numbers that ‘do’ what the problem requires of them, and I won't forget them for a long time. It's also been interesting to speculate how it could be solved without knowledge of at least some of the number theory that I used, short of brute computation. I'll come back to that later.

Here goes.

“Doughnuts and éclairs, that's fine for the party”, said Susan. “You must have got about 50 in all”, Nancy shook her head. “Not that many, but I got all they had”, she replied. “I got as many of each as its price in pence, and the lot came to as many pounds as the number of éclairs and as many pence as the number of doughnuts.” How many of each? (*Cambridge Evening News*, 14 August 2003)

A solution

If the numbers of doughnuts and éclairs are d, e, the word problem translates to the Diophantine equation

$$d^2 + e^2 = 100e + d$$

d and e must be positive, and their sum is required to be less than 50.

I'm drawn to complete the square, multiplying by 4 so as to keep integer terms throughout.

$$(2e - 100)^2 + (2d - 1)^2 = 100^2 + 1^2$$

This looks manageable: I need to express 10001 as a sum of two squares.

The most obvious solution is $2e - 100 = 100$, $2d - 1 = 1$ or $e = 100$, $d = 1$. This amounts to 100 éclairs at £1 each and 1 doughnut at 1p. Now this would cost £100.01, and so meets the requirement about the pounds and pence in the total, but not that there be fewer than 50 in all. So 10001 must be the sum of two squares in another way. In that case, I know that 10001 cannot be prime, for the following reason: 10001 is of the form

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$4k + 1$, so if it were prime, it could be represented as a sum of two squares, but in only one way–a result due to Fermat. Finding its factors would be useful if I am to find another sum of two squares equal to 10001.

Now a composite natural number must have at least one prime factor less than its square root – less than 100 in this case–and there are 25 prime numbers less than 100. But if p divides $100^2 + 1$, then -1 is a *quadratic residue* of p; that is to say, -1 is congruent to some perfect square, modulo p. A standard result in the theory of quadratic residues then says that p must be of the form $4k + 1$, i.e. p is in the list 5, 13, 17, …, 73, 89, 97: with only 11 primes less than 100. Checking through them with a calculator, it didn't take long to find that $10001 = 73 \times 137$. Since these primes are of the form $4k + 1$, Fermat then assures me that each is (uniquely) expressible as a sum of two squares*. Indeed, $73 = 8^2 + 3^2$ and $137 = 11^2 + 4^2$. Two identities attributed to Euler each show that the products of sums of squares are themselves sums of squares:

$$\left(x^2 + y^2\right)\left(m^2 + n^2\right) = (xm + yn)^2 + (xn - ym)^2,$$
$$\left(x^2 + y^2\right)\left(m^2 + n^2\right) = (xm - yn)^2 + (xn + ym)^2.$$

It follows from the first of these identities that $73 \times 137 = 100^2 + 1^2$, which I already knew. The second identity, however, gives $10001 = 76^2 + 65^2$–an interesting arithmetical result, and a new one to me.

Then $2e - 100 = 76$, $2d - 1 = 65$ gives $e = 88$, $d = 33$, and I could now easily check that 88 éclairs at 88p and 33 doughnuts at 33p come to £88.33–very nice indeed! except that there are supposed to be fewer than 50 altogether.

I was somewhat blinkered about what to do next until it occurred to me that, since we are dealing with squares, $2e - 100 = -76$ will fit, with $e = 12$. Twelve éclairs at 12p and 33 doughnuts at 33p come to £12.33, fitting all the conditions of the problem as posed. The same line of thought opens up other possibilities, as follows.

With $e = 12$ or 88, I could have $2d - 1 = -65$, but then $d = -32$ is

* Fermat says that an odd prime number can be expressed as a sum of two squares if, and only if, it is of the form $4k + 1$. I find this quite a remarkable result. One night as I lay awake, I passed the time by verifying it with examples new to me. So, for example, the prime number 89 is of the required form, and $89 = 64 + 25$. But I 'knew' that, so let's try a prime that I haven't thought about before. 421, say. Now, thinking about the possible units digits of perfect squares, the sum of two of them can only end in 1 if the units digits of the two squares are 0 and 1, or 5 and 6. So one of them is a multiple of 5. It didn't take long to arrive at $421 = 225 + 196$. Moving on from 421, I asked myself 'so what about 437'. Reasoning as before, the units digits of the two squares would have to be 1 and 6. And one of the two squares must be less than 220. I subtract 1, 81, 121, from 437, but none leaves me with a square. Fermat's theorem then forces me to conclude that 437 is not prime. This comes as a surprise, because I do not recall seeing it written as a product. I run through possible prime factors: 3, 5, 7, 11 … certainly not, but I know that I won't have far to go: and very soon I arrive at $437 = 380 + 57 = 19 \times 23$. At that point I'm annoyed that I didn't notice that $437 = 441 - 4$, because the difference between two non-consecutive squares has to be composite.

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negative. This is not much use with the éclairs and doughnuts, but it is of theoretical interest since $12^2 + (-32)^2 = 100(12) + 1(-32)$.

Returning to $(2e - 100)^2 + (2d - 1)^2 = 100^2 + 1^2$, I have four possibilities, i.e. $2e - 100 = \pm 100$ and $2d - 1 = \pm 1$. The solution with 100 doughnuts and 1 éclair has already been mentioned, and also that with 12 doughnuts and 33 éclairs. The other two solutions: $e = 0$, $d = 0$ and $e = 0$, $d = 1$ also fit the conditions of the problem, albeit rather implausibly.

Number sense?

Looking back at what I had done, the only part of the solution that specifically makes use of number-theoretic know-how is the chain of reasoning that leads to the expression of 10001 as $76^2 + 65^2$. Could the reader with good 'number sense' do it any other way? We are looking for solutions to $a^2 + b^2 = 10001$. Well, squares end in 0, 1, 4, 5, 6, 9. The only pairs with sum ending in 1 are {0, 1} and {5, 6}: given the need for 0 or 5, we see that either a or b must be a multiple of 5. That reduces the number of possibilities to be tested to just 20. Even better, if the first square is a multiple of 10 then the second must end in 01 i.e. it must be the square of 1, 49, 51 or 99. If it is an odd multiple of 5 then the second must end in 76 i.e. it must be the square of 24, 26, 74 or 76. [In case it is helpful, this argument is 'spelled out' in a footnote.* So there are just 8 possibilities to consider, and so 'number sense' is at least as efficient as number theory!

Pythagoras?

And so, over a decade later, in an edition of *Cambridge News* whose exact date I did not record we come to a folded paper problem.

> "Look Dad," Stan said. "I folded this rectangular sheet of paper so that one corner is exactly on top of the opposite corner". The short side is 24 cm. How long is the other side? Both sides and the fold line are whole numbers of centimetres.

Looks interesting. I make a note of the problem, and six years later–the 2020 'lockdown' somehow created an afternoon when I felt motivated to try to solve it.

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* Looking for perfect squares a^2 ending in 01 or 76: values of $x < 100$ will suffice since if a^2 is such a square, then so will $100k + a$, for any integer k. But then we need only check up to 50, because expanding $(50 - a)^2$ and $(50 + a)^2$ shows that they differ by a multiple of 100. Even better, expanding $(25 - a)^2$ and $(25 + a)^2$ shows that they also differ by $100a$–for example, with $a = 4$, $21^2 = 441$ and $29^2 = 841$. (In both cases, the factors of the difference of two squares lead to the same conclusion). To find all possible tens-units in integer squares, it therefore suffices to evaluate $1^2, 2^2, \ldots, 25^2$, and if we want the units-digit to be 1 or 6, then the squares of 1, 4, 6, 9, 11, 14, 16, 19, 21, 24 suffice, with final digits 01, 16, 36, 81, 21, 96, 56, 31, 41, and 76. Thus in $1 < a < 25$ only 1 and 24 have squares ending in 01 or 76. The arguments about $25 \pm a$ and $50 \pm a$ now show that the squares of 1, 49, 51 and 99 end in 01, and the squares of 24, 26, 74 and 76 end in 76.

Let's start with a diagram; I made a sketch of that piece of folded paper, in portrait orientation.

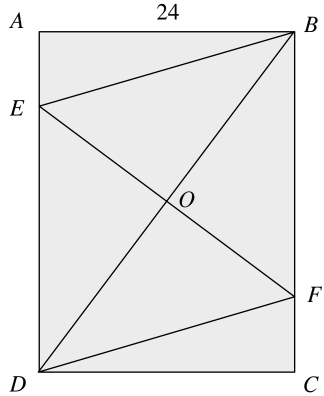

FIGURE 1

The short sides AB and CD are then the top and bottom edges of my rectangle $ACBD$, with length 24 cm, as prescribed.

Let EF be the fold line, so that D will fold over to B. B is the reflection-image of D in the fold line, so BD must be perpendicular to EF (and now we have some right-angled triangles). Let O be the point of intersection of the diagonal and the fold line: the four right angles are at O.

At this point I cannot see how to 'calculate' any lengths. But I know that ABE is right-angled with $AB = 24$, and (7, 24, 25) is in my Pythagorean triple repertoire. Well then, let's just *try* $EA = 7$ and $BE = 25$.

Now, looking at BOE, with the right angle at O, we have $BE = 25$, so this time let's try the Pythagorean triple (15, 20, 25): just 5 times the familiar (3, 4, 5). Then $EO = 15$ and the fold line has length 30, an integer, as required.

But it gets even better, in that (15, 20, 25) triangle BOE, $BO = 20$, and so the diagonal BD of the rectangle has length 40, another integer. And that diagonal, BD, is the hypotenuse of the right-angled triangle BCD, with $CD = 24$. So now we have 8 times (3, 4, 5), i.e. (24, 32, 40), and so the BC, the 'other' side of the rectangle, has length 32 cm – an integer, as required.

So, using a kind of educated guesswork arising from familiarity with Pythagorean triples, I seem to have arrived at a solution. The corner-onto-corner fold certainly will 'work' with those integer dimensions.

However, I'm now left wondering whether there are any other integer solutions. I no longer have access to any solution published in the *Cambridge News*. It somehow seems unlikely that my solution is unique. But in order to find out, I need to take a more analytical approach to the problem. This time I switch the orientation of the rectangle to landscape.

As before, EF is the fold line, which meets the diagonal BD at O. Drop a perpendicular from O onto AB, meeting AB at R; and another perpendicular from O onto BC, meeting BC at M; and let $MF = x$.

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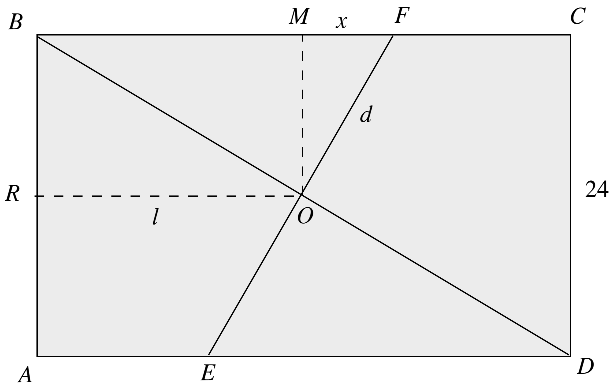

FIGURE 2

Now $OM = 12$: let $BC = 2l$ (so $OR = l$) and let $OF = d$. We require that $2l$ and $2d$ be integers.

Until now I seemed to have overlooked the fact that triangles OMF and ORB are similar. It follows that $12/l = x/12$. Hence $lx = 144$.

Now $x < l$, and therefore $x^2 < 144$, and $x < 12$. For the moment, we don't know whether x must be an integer, but we do know that $x = 144/l$, i.e. $x = 288/2l$, and that $2l$ is required be an integer. So x must (at least) be a rational number.

Also, from triangle OMF, $144 + x^2 = d^2$. We know that $x < 12$, so $d^2 < 288$. Thus $d < 17$ and so the length of the fold line, $2d$, is an integer less than 34. It was 30 in my earlier, educated guesswork solution.

Of course, the fold line EF must be longer than the width of the sheet of paper, so $24 < 2d$. So now we know that $2d$ lies strictly between 24 and 34. But $2d$ is an integer, so $2d \in \{25, 26, \ldots, 33\}$.

We also know (above) that $144 + x^2 = d^2$. Therefore $(2x)^2 = (2d)^2 - 576$, and we can find $2x$ for each of those integer values of $2d$ from 25 to 33. In each case, $2x$ is the square root of an integer, and this can only be rational when it is an integer, because square roots (such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$) of natural numbers that are not perfect squares are irrational. We find that x is an integer when the fold $2d$ has length 25, 26 or 30, when $2x = 7, 10, 18$ respectively. Then, since $lx = 144$, we have $2l = 576/2x$, and so the rational paper lengths $2l$ are $82\frac{2}{7}$, $56\frac{3}{5}$ and 32 respectively.

My conclusion, then, is that $2l = 32$ is indeed the only integer ‘other side’ solution, the one arrived at earlier by educated guesswork. However, the other two rational side lengths, $57\frac{3}{5}$ and $82\frac{2}{7}$, also turn out to be of interest. I'll take them one at a time.

When the fold length ($2d$) is 26, an integer, the corresponding paper side-length $2l = 57\frac{3}{5}$ is not. But looking at triangle OMF, we have a (5, 12, 13) Pythagorean triple, with $x = 5$. Then triangle BCD, being similar to OMF, is also 5−12−13, enlarged by a factor 24/5. So the diagonal BD has

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length $13 \times \frac{24}{5}$, or $62\frac{2}{5}$. Overall, then, we have a fold with integer-length 26, as Stan had required, but a non-integer length $57\frac{3}{5}$. However, it is pleasing to find the rational-length diagonal, and the 5–12–13 triple $\left(24,\ 57\frac{3}{5},\ 62\frac{2}{5}\right)$. The paper width here is 24, and if only we enlarged it by a factor 1.25 (to become 30), both the length (72) and the fold (65) would be integers. And also, for good measure, the paper diagonal too, with length 78.

Finally, when the fold length ($2d$) is 25, an integer, the corresponding paper side-length $2l = 82\frac{2}{7}$ is not. But then, since $lx = 144$, we find $x = 3\frac{1}{2}$. So FMO is a half-size 7–24–25 triangle, and we already knew that the fold-line EF has length 25. The similarity reasoning in the previous paragraph then establishes that the diagonal BD has length $25 \times \frac{24}{7}$, or $85\frac{5}{7}$. The paper width here is 24, and in this case, if we enlarge it by a factor $3\frac{1}{2}$ (to become 84), both the length (288) and the fold (91) are integers. So also is the paper diagonal, with length 300.

We find, then, that all three solutions to this folded paper problem with rational length fold lines and rational length long edges also have a rational length diagonal. These three solutions represent, in turn, the Pythagorean Triples (3, 4, 5), (5, 12, 13) and (7, 24, 25).

Now I'm left with a new problem: the short side of the paper in Stan's problem had length 24. So what other short side lengths would result in integer-length fold lines and long sides? But I'll leave it there for now …

Conclusion

I need hardly say that I was pleasantly surprised to discover that these everyday word problems connected so well with topics in the Theory of Numbers that have interested me for many years. Again, it has been interesting to speculate how readers not already familiar with quadratic residues and Pythagorean triples might go about solving these problems.

For your own enjoyment, I conclude with the brain twister from the 20 January 2011 edition of the *Cambridge News*.

> "I've forgotten your house number!", said Bill, "but you did tell me it was on the odd number side of the road." "Yes. There are less than a hundred numbers in all," Clem replied, "and on our side the numbers below ours total the same as those above ours, with no numbers missing." What was his number?

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