Y. Kitaoka
Nagoya Math. J.
Vol. 70 (1978), 173-181

TENSOR PRODUCTS OF POSITIVE DEFINITE QUADRATIC FORMS III

YOSHIYUKI KITAOKA

In the previous papers [2], [3] we treated the following two questions. Let L, M, N be positive definite quadratic lattices over $\boldsymbol{Z}$:

(i) If L, M are indecomposable, then is $L \otimes M$ indecomposable?

(ii) Does $L \otimes M \cong L \otimes N$ imply $M \cong N$?

In this paper we discuss the uniqueness of decompositions with respect to tensor products. Our aim is to prove the following two theorems.

THEOREM 1. *Let L_i, M_i be indecomposable positive definite binary quadratic lattices with $L_i = \tilde{L}_i$, $M_i = \tilde{M}_i$, $m(L_i) = m(M_i) = 1$. For any isometry $\sigma: \bigotimes_{i=1}^n L_i \cong \bigotimes_{i=1}^n M_i$, we have $\sigma = \bigotimes_{i=1}^n \sigma_i$ where σ_i is an isometry from L_i on M_i, changing the suffix if necessary.*

THEOREM 2. *Let L_i, M_i be positive definite quadratic lattices with $[L_i : \tilde{L}_i] < \infty$, $[M_i : \tilde{M}_i] < \infty$. Assume that*

(i) *L_i (resp. M_i) is of E-type except at most one,*

(ii) *$sL_i = sM_i = \boldsymbol{Z}$, and $m(L_i), m(M_i)$ are prime numbers, and*

(iii) *$\tilde{L}_i, \tilde{M}_i$ are indecomposable.*

Then for any isometry $\sigma: \bigotimes_{i=1}^n L_i \cong \bigotimes_{i=1}^m M_i$ we have $n = m$ and $\sigma = \otimes \sigma_i$, where σ_i is an isometry from L_i on M_i, changing the suffix if necessary.

We must explain the notations and terminologies in two theorems. By a positive definite quadratic lattice we mean a lattice in a positive definite quadratic space over the rational number field $\boldsymbol{Q}$. For any quadratic space we use the same letter Q, B which are the corresponding quadratic form and bilinear form ($2B(x, y) = Q(x + y) - Q(x) - Q(y)$). Let L be a positive definite quadratic lattice; then sL denotes $\{\sum B(x_i, y_i);\ x_i, y_i \in L\}$ and we put $m(L) = \min Q(x)$ where x runs over non-zero elements of L. $\mathfrak{M}(L)$ stands for $\{x \in L;\ Q(x) = m(L)\}$, and $\tilde{L}$ is the sub-

Received June 21, 1977.

module of L spanned by elements of $\mathfrak{M}(L)$. L is called E-type if every element of $\mathfrak{M}(L \otimes M)$ is of the form $x \otimes y$ $(x \in L,\ y \in M)$ for any positive definite quadratic lattice M. If either $sL \subseteq \boldsymbol{Z}$, $m(L) \leq 6$ or rank $L \leq 42$, then L is of E-type [1].

§1. In this section we define a weighted graph and prove some properties.

DEFINITION. Let A be a finite set, and $[\ ,\ ]$ be a mapping from $A \times A$ into $\{t; 0 \leq t \leq 1\}$ such that

(i) $[a, a'] = 1$ if and only if $a = a'$, and

(ii) $[a, a'] = [a', a]$ for a, a' in A.

Then we call $(A, [\ ,\ ])$ or simply A a weighted graph. A weighted graph A is called connected if for any x, y in A there are elements z_i of A such that $x = z_1$, $y = z_r$ and $[z_i, z_{i+1}] \neq 0$ $(i = 1, \cdots, r-1)$. For weighted graphs A, B we define the direct product $A \times B$ by $[(a, b), (a', b')] = [a, a'][b, b']$ $(a, a' \in A,\ b, b' \in B)$; then $A \times B$ is clearly a weighted graph. It is also clear that the direct product of connected weighted graphs is connected. A bijection f from A on B is called an isometry if f satisfies $[f(a), f(a')] = [a, a']$ for $a, a' \in A$.

LEMMA 1. *Let A, B, C be connected weighted graphs, and let σ be an isometry from $A \times B$ on $A \times C$. If there are $b_0 \in B$, $c_0 \in C$ such that $\sigma(x, b_0) = (f(x), c_0)$ for every x in A, then f is an isometry from A on A and there is an isometry g from B on C with $\sigma(x, y) = (f(x), g(y))$ $(x \in A,\ y \in B)$.*

Proof. Since σ is a bijection and A is a finite set, f is a bijection of A. Moreover for a, a' in A we have $[a, a'] = [(a, b_0), (a', b_0)] = [(f(a), c_0), (f(a'), c_0)] = [f(a), f(a')]$. This means that f is an isometry of A. Multiplying $f^{-1} \times \mathrm{id}_C$ to σ, we have only to prove the lemma in case of $f = 1$. Put $S = \{\tilde{B} \subset B; \sigma(a, b) = (a, c)$ for every $a \in A$ and $b \in \tilde{B}$, where c is only dependent of $b\}$. S is not empty since $S \ni \{b_0\}$. Take an element B' in S such that $\sharp B' \geq \sharp \tilde{B}$ for $\tilde{B}$ in S. If $B' = B$, then we have $\sigma(a, b) = (a, g(b))$ for $a \in A$, $b \in B$. It is easy to see that g is an isometry from B on C, and this completes the proof. Now we assume $B' \neq B$. We have to show that this implies a contradiction. Define a subset C' by $\sigma(A, B') = (A, C')$. Put $m = \max[b, b']$ where $b \in B'$, $b' \notin B'$, and we may assume $m \geq \max[c, c']$ where $c \in C'$, $c' \notin C'$, taking σ^{-1} instead of σ if necessary. Since B is connected, m is positive.

Put $m = [b, b']$ ($b \in B'$, $b' \notin B'$) and take any element x of A. Put $\sigma(x, b') = (x', c_1)$; then c_1 is not in C' since $c_1 \in C'$ implies $(x, b') \in \sigma^{-1}(A, C') = (A, B')$. Putting $\sigma(x, b) = (x, c)$, we have $m = [b, b'] = [(x, b), (x, b')] = [(x, c), (x', c_1)] = [x, x'][c, c_1]$. If $x \neq x'$, then $0 < [x, x'] < 1$ implies a contradiction $m < [c, c_1] \leq m$. Hence $x' = x$ follows. Thus we get $\sigma(x, b') = (x, c(x))$ ($c(x) \in C$) for every x in A. For x, y in A with $[x, y] \neq 0$, $[x, y] = [(x, b'), (y, b')] = [(x, c(x)), (y, c(y))] = [x, y][c(x), c(y)]$ implies $[c(x), c(y)] = 1$, and so $c(x) = c(y)$. Since A is connected, this yields that $c(x)$ in C is independent of x in A, and then it implies a contradiction $B' \cup \{b'\} \in S$ and $\sharp(B' \cup \{b'\}) > \sharp B'$.

LEMMA 2. *Let L be a positive definite quadratic lattice. For x, y in L we put $[x, y] = |B(x, y)|/m(L)$. Then $(\mathfrak{M}(L)/\pm 1, [\ ,\ ])$ is a weighted graph and it is connected if and only if $\tilde{L}$ is indecomposable.*

Proof. Take x, y in $\mathfrak{M}(L)$; then $x = \pm y$ if and only if $|B(x, y)| = m(L)$. Moreover $B(x, y)^2 \leq Q(x)Q(y) = m(L)^2$ implies that $\mathfrak{M}(L)/\pm 1$ is a weighted graph. The latter part is obvious.

We say that $(\mathfrak{M}(L)/\pm 1, [\ ,\ ])$ is a weighted graph associated to L.

§2. Let L_i, M_j be positive definite quadratic lattices and let σ be an isometry from $\otimes_{i=1}^n L_i$ on $\otimes_{j=1}^m M_j$. Suppose that

(i) $\mathfrak{M}(\otimes L_i) = \otimes \mathfrak{M}(L_i)$, $\mathfrak{M}(\otimes M_j) = \otimes \mathfrak{M}(M_j)$,

(ii) $[L_i : \tilde{L}_i]$, $[M_j : \tilde{M}_j] < \infty$ for every i, j,

(iii) $\mathfrak{M}(L_i)/\pm 1$, $\mathfrak{M}(M_j)/\pm 1$ are connected weighted graphs for every i, j.

Let A, B, A_i, B_i be weighted graphs associated to $\otimes L_i$, $\otimes M_i$, L_i, M_i respectively. Then σ induces an isometry from $A = \prod_{i=1}^n A_i$ on $B = \prod_{i=1}^m B_i$ which is denoted by the same letter σ.

THEOREM. *If it follows that $n = m$, $\sigma = \prod_{i=1}^n \sigma_i$ where σ_i is an isometry from A_i on B_i, changing the suffix if necessary, then we have $\sigma = \otimes_{i=1}^n \mu_i$ where μ_i is an isometry from L_i on M_i, changing the suffix if necessary.*

Proof. We may assume $\sigma = \prod \sigma_i$ where σ_i is an isometry from A_i on B_i. By the same letter σ_i we denote a mapping from $\mathfrak{M}(L_i)$ on $\mathfrak{M}(M_i)$ which induces an isometry σ_i from $A_i = \mathfrak{M}(L_i)/\pm 1$ on $B_i = \mathfrak{M}(M_i)/\pm 1$. Fix any element e_i in $\mathfrak{M}(L_i)$ ($i \geq 2$). Then $\sigma(e \otimes e_2 \otimes \cdots \otimes e_n) = \pm \sigma_1(e)$

https://doi.org/10.1017/S0027763000021863 Published online by Cambridge University Press

$\otimes \sigma_2(e_2) \otimes \cdots \otimes \sigma_n(e_n)$ holds for every e in $\mathfrak{M}(L_1)$. Putting $\pm\sigma_1(e) = \mu_1(e)$, then $\sigma(e \otimes e_2 \otimes \cdots \otimes e_n) = \mu_1(e) \otimes \sigma_2(e_2) \otimes \cdots \otimes \sigma_n(e_n)$ for any e in $\mathfrak{M}(L_1)$. This means that μ_1 is an isometry from $\tilde{L}_1$ onto $\tilde{M}_1$. Since $M_1 \otimes \sigma_2(e_2) \otimes \cdots \otimes \sigma_n(e_n)$ is a direct summand of $\otimes M_i$ and $[L_1 : \tilde{L}_1] < \infty$, μ_1 is an isometry from L_1 into M_1. Similarly we get an isometry μ_i from L_i into M_i so that $\sigma(e_1 \otimes \cdots \otimes e_n) = \pm\mu_1(e_1) \otimes \cdots \otimes \mu_n(e_n)$ for e_i in $\mathfrak{M}(L_i)$, where $\pm$ may depend on the choice of e_i. Since $\mathfrak{M}(L_i)/\pm 1$ is connected and moreover $\delta = \delta'$ if $\sigma(e_1 \otimes \cdots \otimes e_n) = \delta\mu_1(e_1) \otimes \cdots \otimes \mu_n(e_n)$, $\sigma(e'_1 \otimes \cdots \otimes e'_n) = \delta'\mu_1(e'_1) \otimes \cdots \otimes \mu_n(e'_n)$ are not orthogonal, $\pm$ does not depend on the choice of e_i. Thus we get $\sigma = \otimes \mu_i$, taking $-\mu_1$ if necessary. Since σ is an onto-mapping, μ_i is an isometry from L_i on M_i. This completes the proof.

§ **3.** First we discuss the case of Theorem 1. Let L be an indecomposable binary positive definite quadratic lattice with $L = \tilde{L}$, $m(L) = 1$. Then L has a basis $\{e_1, e_2\}$ so that $Q(e_1) = Q(e_2) = 1$, $0 < B(e_1, e_2) \leq \frac{1}{2}$, and moreover we have $\mathfrak{M}(L) = \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}$ ($\pm(e_1 - e_2)$ happens only when $B(e_1, e_2) = \frac{1}{2}$). Let A_L be a weighted graph associated to L; then A_L is connected. $\# A_L$ is two for $B(e_1, e_2) < \frac{1}{2}$. If $B(e_1, e_2) = \frac{1}{2}$, then $\# A_L = 3$ and $[a_i, a_j] = \frac{1}{2}$ for $i \neq j$ where we put $A_L = \{a_1, a_2, a_3\}$.

Let L_i, M_i, σ be as in Theorem 1; then L_i, M_i are of E-type, and define A, A_i, B, B_i and σ as in § 2; then we have

LEMMA 3. *$\sigma = \prod \sigma_i$ where σ_i is an isometry from A_i on B_i, changing the suffix if necessary.*

Proof. We prove this by the induction with respect to $\# A$. Put $m = \max[a, a'] = \max[b, b']$ where $a, a' \in A$, $a \neq a'$ and $b, b' \in B$, $b \neq b'$. Since A_i, B_i are indecomposable, we get $0 < m \leq \frac{1}{2}$. Take $a \neq a'$ in A with $[a, a'] = m$. Putting $a = \prod a_i$, $a' = \prod a'_i$, $m = \prod [a_i, a'_i]$ follows. Noting $[a_i, a'_i] < 1$ for $a_i \neq a'_i$, the maximality of m implies that there is an index j such that $[a_i, a'_i] = 1$, i.e., $a_i = a'_i$ for $i \neq j$, and $a_j \neq a'_j$. We may assume $j = 1$, and similarly $\sigma(a) = \prod b_i$, $\sigma(a') = \prod b'_i$, $b_i = b'_i$ for $i > 1$ and $b_1 \neq b'_1$. Then $m = [a_1, a'_1] = [b_1, b'_1]$ follows. If $m < \frac{1}{2}$, then $A_1 = \{a_1, a'_1\}$, $B_1 = \{b_1, b'_1\}$ and $\sigma(A_1 \times \prod_{i=2}^n a_i) = B_1 \times \prod_{i=2}^n b_i$. Hence Lemma 1 and the assumption of the induction completes the proof. Suppose $m = \frac{1}{2}$; then there is an element a''_1 in A_1 so that $A_1 = \{a_1, a'_1, a''_1\}$ and $[a_1, a''_1] = [a'_1, a''_1] = \frac{1}{2}$. Put $\sigma(a''_1 \times \prod_{i=2}^n a_i) = \prod b''_i$; then $[a_1, a''_1] =$

https://doi.org/10.1017/S0027763000021863 Published online by Cambridge University Press

$[a_1', a_1''] = \frac{1}{2}$ implies $[b_1, b_1''] \prod_{i=2}^n [b_i'', b_i] = [b_1', b_1''] \prod_{i=2}^n [b_i'', b_i] = \frac{1}{2}$. Suppose $b_1 = b_1''$; then $\prod_{i=2}^n [b_i'', b_i] = \frac{1}{2}$, and so $[b_1', b_1''] = 1$, that is, $b_1' = b_1'' = b_1$. This is a contradiction. Hence we have $b_1 \neq b_1''$, and then $[b_1, b_1''] = \frac{1}{2}$. Therefore $b_i'' = b_i$ for $i \geq 2$ and $\sigma(A_1 \times \prod_{i=2}^n a_i) = B_1 \times \prod_{i=2}^n b_i$. This completes the proof as above.

Now Theorem 1 follows from Theorem in § 2.

Next we discuss the case of Theorem 2.

LEMMA 4. *Let $a_i, b_i \in \boldsymbol{Z}$ and $0 < b_i < a_i$, and let a_i be prime. Put $\prod_{i=1}^n (b_i/a_i) = b/a$, $(a, b) = 1$. Then $a > a_i$ for some i if $n \geq 2$.*

Proof. We may suppose $a_1 \leq \cdots \leq a_n$, and assume $a \leq a_i$ for any i. Since a divides $\prod a_i$, we have $a = a_1$. $b_1 \prod_{i=2}^n (b_i/a_i) = b$ and $a_i \nmid b_1$ imply $\prod_{i=2}^n a_i \mid \prod_{i=2}^n b_i$. This contradicts $0 < b_i < a_i$.

LEMMA 5. *Let A_i, B_i be connected weighted graphs with $\sharp A_i > 1$, $\sharp B_i > 1$, and let p_i, q_i be primes. Suppose*

$$\{[x, y]\,;\, x, y \in A_i\} \subset \{a/p_i\,;\, a = 0, 1, \cdots, p_i\}$$

and

$$\{[x, y]\,;\, x, y \in B_i\} \subset \{b/q_i\,;\, b = 0, 1, \cdots, q_i\}\,.$$

If σ is an isometry from $\prod_{i=1}^n A_i$ on $\prod_{i=1}^m B_i$, then $n = m$ and $\sigma = \prod \sigma_i$ where σ_i is an isometry from A_i on B_i, changing the suffix if necessary.

Proof. We prove by the induction with respect to $\sharp \prod_{i=1}^n A_i$. Since A_i is connected and $\sharp A_i > 1$, for any element a in A_i there is an element a' in A_i such that $0 < [a, a'] < 1$. If $[a, a'] \neq 0, 1$ for a, a' in A_i, then the denominator of $[a, a']$ is a prime p_i. Without loss of generality we may assume $p_1 = \cdots = p_k < p_{k+1} \leq \cdots \leq p_n$, $q_1 = \cdots = q_h < q_{h+1} \leq \cdots \leq q_m$. Put $A = \prod_{i=1}^n A_i$, $B = \prod_{i=1}^m B_i$, and fix any element $a = \prod a_i$ of A. Suppose that the minimal value of the denominator of $[a, a']$ with $[a, a'] \neq 0, 1$ $(a' \in A)$ is taken by $a' = \prod a_i' \in A$. Then the above remark and Lemma 4 imply $a_i' = a_i$ for $i \neq j$, and $a_j' \neq a_j$ for some j and so the minimal value is obviously p_1, and $j \leq k$. On the other hand, by virtue of Lemma 4 and the connectedness of A_i, it is easy to see that $A_1 \times \cdots \times A_k \times a_{k+1} \times \cdots \times a_n$ is a subset of A consisting of elements z such that there are elements $z_1 = a, \cdots, z_r = z$ of A satisfying that the denominator of $[z_i, z_{i+1}]$ is p_1 for $i = 1, \cdots, r-1$. From the similar argument for $\sigma(a) = \prod b_i$ in B follows that the corresponding minimal denominator is q_1, and the corresponding subset of B for $q_1, \sigma(a)$ instead of p_1, a is $B_1 \times \cdots \times B_h \times b_{h+1} \times \cdots \times b_m$. Since σ is an isometry, we have $p_1 = q_1$, and so $\sigma(A_1 \times \cdots \times A_k \times a_{k+1} \times \cdots \times a_n) = B_1 \times \cdots \times B_h \times b_{h+1} \times \cdots \times b_m$ by their definitions. This implies that $A_1 \times \cdots \times A_k$ and $B_1 \times \cdots \times B_h$ are isometric. Therefore Lemma 1 and the assumption of the induction completes the proof if $n > k$. Thus we may suppose $n = k$. Then $\sigma(A) = \prod_{i=1}^{h} B_i \times b_{h+1} \times \cdots \times b_m$ implies $h = m$. Moreover we have $n = m$ since the maximal value of the denominators of $[a, a']$ $(a, a' \in A)$ (resp. $[b, b']$ $(b, b' \in B)$) is p_1^n (resp. p_1^m), and they are equal. For simplicity we put $p_1 = p$ in the following.

(i) Assume that A_1 contains distinct three elements x_1, x_2, x_3 such $[x_1, x_2][x_2, x_3][x_3, x_1] \neq 0$. Fix any element a_i in A_i $(i \geq 2)$, and put $\sigma(x_k \prod_{i=2}^{n} a_i) = \prod_{j=1}^{n} b_{k,j}$ $(b_{k,j} \in B_j)$; then $[x_i, x_k] = \prod_{j=1}^{n} [b_{i,j}, b_{k,j}] \neq 0$. Since $0 < [b_{i,j}, b_{k,j}] \leq 1$ and the denominator of $[b_{i,j}, b_{k,j}]$ is p if $b_{i,j} \neq b_{k,j}$, compairing the denominators of both sides, we have $b_{i,j} = b_{k,j}$ for any j except one index if $i \neq k$. Without loss of generality we may assume $b_{1,1} \neq b_{2,1}$, $b_{1,i} = b_{2,i}$ $(i \geq 2)$. Similarly we may assume $b_{2,k} = b_{3,k}$ for $k \neq t$. If $t \geq 2$, then $b_{1,j} = b_{2,j} = b_{3,j}$ for $j \neq 1, t$. This implies $[x_1, x_3] = [b_{1,1}, b_{3,1}][b_{1,t}, b_{3,t}] = [b_{1,1}, b_{2,1}][b_{2,t}, b_{3,t}]$. The denominator of the left (resp. right) side is p (resp. p^2) since $b_{1,1} \neq b_{2,1}$, $b_{2,t} \neq b_{3,t}$. This is a contradiction. Hence we get $t = 1$, and so $b_{2,1} \neq b_{3,1}$, $b_{2,j} = b_{3,j}$ $(j \geq 2)$. Thus we may put $\sigma(x_k \times \prod_{i=2}^{n} a_i) = y_k \times \prod_{i=2}^{n} b_i$ $(y_k \in B_1, b_i \in B_i)$. Take an element a_n' in A_n such that $[a_n, a_n'] \neq 0, 1$. Similarly we get $\sigma(x_k \times \prod_{i=2}^{n-1} a_i \times a_n') = z_k \times \prod_{i \neq j} b_i'$ for some z_k in B_j and b_i' in B_i. Suppose $j \neq 1$, then $[a_n, a_n'] = [x_k \times \prod_{i=2}^{n} a_i, x_k \times \prod_{i=2}^{n-1} a_i \times a_n'] = [y_k, b_1'][b_j, z_k] \times \prod_{i \neq 1, j} [b_i, b_i']$. We note that the denominator of the left side is p. If $b_i \neq b_i'$ for $i \neq 1, j$, then $[y_k, b_1'] = 1$, and so $y_1 = y_2 = y_3$. This implies a contradiction $x_1 = x_2 = x_3$. Hence $b_i = b_i'$ for $i \neq 1, j$. $b_1' \neq y_1$ implies $b_j = z_1$ $(\neq z_2, z_3)$, and so we get $y_2 = y_3 = b_1'$, taking $k = 2$ or 3. This is a contradiction. Hence we have $b_1' = y_1$, and similarly $b_1' = y_2$. This contradicts $x_1 \neq x_2$. Hence j equals 1, and we may put $\sigma(x_k \times \prod_{i=2}^{n-1} a_i \times a_n') = z_k \times \prod_{i=2}^{n} b_i'$ $(z_k \in B_1, b_i' \in B_i)$. $\sigma(x_k \times \prod_{i=2}^{n} a_i) = y_k \times \prod_{i=2}^{n} b_i$ implies $[x_k, x_h][a_n, a_n'] = [y_k, z_h] \prod_{i=2}^{n} [b_i, b_i']$. Putting $k = h$, and compairing the denominators we have $b_i = b_i'$ for any $i \geq 2$ except at most one i. Putting $k \neq h$, the denominator of the left hand equals p^2. Hence the exceptional suffix exists. Then putting $k = h$ again, we have $y_k = z_k$

https://doi.org/10.1017/S0027763000021863 Published online by Cambridge University Press

for $k=1,2,3$. Thus we have $\sigma(x_k\times\prod_{i=2}^{n-1}a_i\times a'_n)=y_k\times\prod_{i=2}^n b'_i$. Doing the similar operations for a_i, a'_n, we have $\sigma(x_k\times\prod_{i=2}^n A_i)\subset y_k\times\prod_{i=2}^n B_i$ since A_i is connected. Similarly $\sigma^{-1}(y_k\prod_{i=2}^n b_i)=x_k\prod_{i=2}^n a_i$ and $[y_1,y_2]\times[y_2,y_3][y_3,y_1]\neq 0$ imply $\sigma^{-1}(y_k\prod_{i=2}^n B_i)\subset x_k\times\prod_{i=2}^n A_i$, and so $\sigma(x_k\times\prod_{i=2}^n A_i)=y_k\times\prod_{i=2}^n B_i$. This implies $\prod_{i=2}^n A_i\cong\prod_{i=2}^n B_i$, and then Lemma 1 and the assumption of the induction completes the proof.

(ii) Suppose that A_1 contains distinct four elements x_i such that $[x_1,x_2][x_1,x_3][x_1,x_4]\neq 0$, $[x_2,x_3]=[x_2,x_4]=[x_3,x_4]=0$. Fix any element a_i in A_i $(i\geq 2)$. Put $\sigma(x_k\times\prod_{i=2}^n a_i)=\prod_{i=1}^n b_{k,i}$; then $[x_k,x_1]=\prod_{i=1}^n[b_{k,i},b_{1,i}]\neq 0$. Since the denominator of the left hand is p for $k\neq 1$, there is a number t_k such that $b_{k,i}=b_{1,i}$ for $i\neq t_k$, and $b_{k,t_k}\neq b_{1,t_k}$.

a) Suppose that t_2,t_3,t_4 are distinct.

$[x_3,x_4]=0$ implies $[b_{3,i},b_{4,i}]=0$ for some i. Since $b_{k,j}=b_{1,j}$ for $j\neq t_2,t_3,t_4$, i equals t_2, t_3 or t_4. If $i=t_2$, then $b_{3,t_2}=b_{1,t_2}=b_{4,t_2}$ implies a contradiction $[b_{3,t_2},b_{4,t_2}]=1$. Similarly $i=t_3$ or $i=t_4$ implies a contradiction.

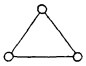
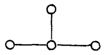
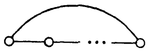
b) Suppose that $t_2=t_3\neq t_4$.

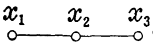
$[x_3,x_4]=0$ implies $[b_{3,i},b_{4,i}]=0$ for some i. $b_{k,j}=b_{1,j}$ for $j\neq t_k$ yields $i=t_2$ or t_4. $i=t_2$ implies $b_{4,t_2}=b_{1,t_2}$, and so $[b_{3,i},b_{1,i}]=0$. This contradicts $[x_3,x_1]\neq 0$. Similarly $i=t_4$ is a contradiction.

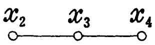
Similarly $t_2\neq t_3=t_4$ or $t_2=t_4\neq t_3$ implies a contradiction. Hence we have $t_2=t_3=t_4=1$ (say). Thus we may assume $\sigma(x_k\times\prod_{i=2}^n a_i)=y_k\times\prod_{i=2}^n b_i$ $(y_k\in B_1,\ b_i\in B_i)$. Take an element a'_n in A_n with $[a_n,a'_n]\neq 0,1$, and put $\sigma(x_k\times\prod_{i=2}^{n-1}a_i\times a'_n)=z_k\prod_{i\neq j}b'_i$ $(z_k\in B_j,\ b'_i\in B_i)$. Assume $j\neq 1$; then $[x_k\times\prod_{i=2}^n a_i,\ x_t\times\prod_{i=2}^{n-1}a_i\times a'_n]=[x_k,x_t]\times[a_n,a'_n]=[y_k,b'_1][b_j,z_t]\prod_{i\neq 1,j}[b_i,b'_i]$. $[x_1,x_t][a_n,a'_n]\neq 0$ implies $[b_j,z_t]\neq 0$ $(t=1,2,3,4)$, $[b_i,b'_i]\neq 0$ for $i\neq 1,j$. Similarly $[x_k,x_1]\neq 0$ implies $[y_k,b'_1]\neq 0$ $(k=1,2,3,4)$. This means $[x_k,x_t][a_n,a'_n]\neq 0$ for any k,t and contradicts $[x_2,x_3]=0$. Thus we have $j=1$, and $[x_k,x_t]$. $[a_n,a'_n]=[y_k,z_t]\times\prod_{i=2}^n[b_i,b'_i]$. Since the denominator of the left hand for $k=1$, $t=2$ is p^2, there is at least one suffix i such that $b_i\neq b'_i$. Moreover the denominator of the left side for $k=t$ is p. Hence there is no such suffix except i, and this yields $[y_k,z_k]=1$, i.e., $y_k=z_k$. As the proof of the case (i) we have $\sigma(x_k\times\prod_{i=2}^n A_i)=y_k\times\prod_{i=2}^n B_i$ and complete the proof for the case (ii) by the induction and Lemma 1.

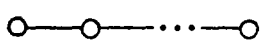
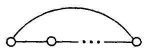
For a weighted graph W we make a usual graph, joining two ele-

https://doi.org/10.1017/S0027763000021863 Published online by Cambridge University Press

ments x, y with $[x, y] \neq 0$. Then, by virtue of (i), (ii), we may assume that A_i, B_i do not contain subgraphs  , . Hence A_i, B_i are  or  as graphs.

(iii) Suppose that A_1 contains three distinct elements x_1, x_2, x_3 such that $[x_1, x_2] \neq 0$, $[x_2, x_3] \neq 0$, $[x_1, x_3] = 0$, i.e., 

. Take any element a_i in A_i, and put $\sigma(x_k \prod_{i=2}^n a_i) = \prod_{i=1}^n b_{k,i}$ $(b_{k,i} \in B_i)$. Comparing the denominators of $[x_k, x_t] = \prod_{i=1}^n [b_{k,i}, b_{t,i}]$, there are numbers q, s so that $b_{1,i} = b_{2,i}$ for $i \neq q$, $b_{2,i} = b_{3,i}$ for $i \neq s$. $q \neq s$ implies $b_{1,i} = b_{2,i} = b_{3,i}$ for $i \neq q, s$, $b_{2,q} = b_{3,q}$ and $b_{1,s} = b_{2,s}$, and then we have $[x_1, x_3] = \prod_{i=1}^n [b_{1,i}, b_{3,i}] = [b_{1,q}, b_{3,q}][b_{1,s}, b_{3,s}] = [b_{1,q}, b_{2,q}][b_{2,s}, b_{3,s}] = 0$. This contradicts $[x_1, x_2][x_2, x_3] \neq 0$. Thus we may assume $q = s = 1$ (say), and $\sigma(x_k \prod_{i=2}^n a_i) = y_k \prod_{i=2}^n b_i$ $(y_k \in B_1,\ b_i \in B_i)$. Doing the similar thing for 

, we have $\sigma(x_k \times \prod_{i=2}^n a_i) = z_k \prod_{i \neq j} b'_i$ $(z_k \in B_j,\ b'_i \in B_i)$ for $k = 2, 3, 4$. Comparing the case $k = 2, 3$, we get $z_2 = b_j = z_3$ if $j \neq 1$. This is a contradiction, and so $j = 1$. This means $b'_i = b_i$ for $i \geq 2$ and $\sigma(x_4 \prod_{i=2}^n a_i) = z_4 \prod_{i=2}^n b_i$. Since A_1 is  or , we have $\sigma(x \times \prod_{i=2}^n a_i) = f(x) \times \prod_{i=2}^n b_i$ for any x in A_1, that is, $\sigma(A_1 \times \prod_{i=2}^n a_i) \subset B_1 \times \prod_{i=2}^n b_i$. Similarly we have $\sigma^{-1}(B_1 \times \prod_{i=2}^n b_i) \subset A_1 \times \prod_{i=2}^n a_i$ and so $\sigma(A_1 \times \prod_{i=2}^n a_i) = B_1 \times \prod_{i=2}^n b_i$. Lemma 1 and the induction complete the proof.

(iv) By virtue of (i), (ii), (iii) we have only to prove the case that $\# A_i = \# B_i = 2$. Put $m = \max [a, a']$ $(a, a' \in A,\ a \neq a')$ and assume $m = [a, a']$ for $a = \prod_{i=1}^n a_i$, $a' = \prod_{i=1}^n a'_i$. Since $[a_i, a'_i] < 1$ if $a_i \neq a'_i$, by the definition, there is a suffix t so that $a_i = a'_i$ for $i \neq t$ and $a_t \neq a'_t$. Putting $\sigma(a) = \prod b_i$, $\sigma(a') = \prod b'_i$, there is a suffix s so that $b_i = b'_i$ for $i \neq s$, and $b_s \neq b'_s$. Without loss of generality we may assume $t = s = 1$; then $A_1 = \{a_1, a'_1\}$, $B_1 = \{b_1, b'_1\}$ and $[a_1, a'_1] = [b_1, b'_1] = m$. Hence $A_1 \cong B_1$ and $\sigma(A_1 \times \prod_{i=2}^n a_i) = B_1 \times \prod_{i=2}^n b_i$. Lemma 1 and the assumption of the induction complete the proof of Lemma 4.

To complete the proof of Theorem 2 we need only to prove that the cardinalities of weighted graphs associated to L_i, M_i are not 1. It follows immediately from the assumption (ii).

Let L be an indecomposable positive definite quadratic lattice, and put $A = \mathfrak{M}(L)/\pm 1$ and we consider A as a weighted graph by $[x, y] = |B(x, y)|/m(L)$ for $x, y \in \mathfrak{M}(L)/\pm 1$ as above. We call such a weighted graph a quadratic weighted graph associated to L. Then the following questions arise.

(i) Let A_i, B_i be connected quadratic weighted graphs and f be an isometry from $\prod_{i=1}^n A_i$ on $\prod_{i=1}^m B_i$. What is a sufficient condition to the following assertion?

$n = m$ and $f = \prod f_i$ (changing the suffix if necessary), where f_i is an isometry from A_i on B_i.

(ii) Let L be an indecomposable positive definite quadratic lattice with $L = \tilde{L}$, and let A be an associated quadratic weighted graph. If $A \cong B \times C$ where B, C are quadratic weighted graphs, then is there a decomposition $L \cong M \otimes N$ so that B (resp. C) is a quadratic weighted graph associated to M (resp. N)?

Remark 1. For $M \cong \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$, $N \cong \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}$, associated quadratic graphs are isometric but M, N are not isometric.

Remark 2. Let L be a positive definite quadratic lattice with $L = \tilde{L}$, $m(L) = 1$, and assume that $\mathfrak{M}(L)/\pm 1 = A \times B$ where A, B are weighted graphs with $\sharp A$, $\sharp B > 1$. Put $\mathfrak{M}(L)/\pm 1 = \{e_i\}$ and $e_i = (a_i, b_i)$ $(a_i \in A,\ b_i \in B)$. Suppose that there is a mapping s_1 (resp. s_2) from $A \times A$ (resp. $B \times B$) into $\{\pm 1\}$ so that $s_1(a, a) = s_2(b, b) = 1$ for every a in A and every b in B, and $B(e_i, e_j) = s_1(a_i, a_j)s_2(b_i, b_j)[a_i, a_j][b_i, b_j]$ for any i, j. Then we can show that there are positive definite quadratic lattices M, N such that $L \cong M \otimes N$, $M = \tilde{M}$, $N = \tilde{N}$, $m(M) = m(N) = 1$ and A, B are quadratic graphs associated to M, N respectively. The assumption on s_1, s_2 is not satisfied for a decomposable lattice $M \perp N$ in Remark 1.

References

[1] Y. Kitaoka, Scalar extension of quadratic lattices II, Nagoya Math. J. vol. **67** (1977), 159–164.

[2] ——, Tensor products of positive definite quadratic forms, Göttingen Nachr. Nr. **4** (1977).

[3] ——, Tensor products of positive definite quadratic forms II, to appear.

[4] O. T. O'Meara, Introduction to quadratic forms, Springer-Verlag, 1963.

Department of Mathematics
Nagoya University