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COEXISTENCE OF COILED SURFACES AND SPANNING SURFACES FOR KNOTS AND LINKS

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Abstract

It is a well-known procedure for constructing a torus knot or link that first we prepare an unknotted torus and meridian disks in its complementary solid tori, and second we smooth the intersections of the boundaries of the meridian disks uniformly. Then we obtain a torus knot or link on the unknotted torus and its Seifert surface made of meridian disks. In the present paper, we generalize this procedure by using a closed fake surface and show that the two resulting surfaces obtained by smoothing triple points uniformly are essential. We also show that a knot obtained by this procedure satisfies the Neuwirth conjecture and that the distance of two boundary slopes for the knot is equal to the number of triple points of the closed fake surface.

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1. Introduction

1.1. The Neuwirth conjecture. There are not so many geometrical properties satisfied by all nontrivial knots. Any knot bounds a minimal (and hence incompressible) Seifert surface [3, 12], and for any nontrivial knot there exists a properly embedded separating, orientable, incompressible, boundary incompressible and not boundary parallel surface in the exterior of the knot [2]. The following conjecture asserts that any nontrivial knot can be embedded in a closed surface, similarly to the way a torus knot can be embedded in an unknotted torus.

CONJECTURE 1.1 (Neuwirth conjecture, [6], cf. [11, Problem 1.1]). *For any nontrivial knot K in the 3-sphere, there exists a closed surface F containing K as a nonseparating loop such that F is essential in the exterior of K.*

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Recent results on the Neuwirth conjecture can be seen in [9]. Here we summarize all knot classes that are known to satisfy the Neuwirth conjecture:

- alternating knots [1];
- knots satisfying $g_I(K) < 2g(K)$, where $g_I(K)$ is the interpolating genus of a knot K [6];
- generalized alternating knots [7];
- σ-adequate and σ-homogeneous knots for a state σ other than the Seifert state $\vec{\sigma}$ [8];
- knots with nonorientable spanning surfaces obtained by Murasugi sums of essential spanning surfaces [8];
- Montesinos knots [9];
- all knots with 11 crossings or fewer except for $K11_n118$ and $K11_n126$ [9];
- knots with a degree one map to a knot satisfying the Neuwirth conjecture [9].

Since the Neuwirth conjecture originated in torus knots, we go back to the construction of torus knots in the next subsection.

1.2. A procedure for constructing torus knots and links. The following is a well-known procedure for constructing a torus knot or link [5]. Let T be an unknotted torus in the 3-sphere S^3 which decomposes S^3 into two solid tori, V_1 and V_2. Take p mutually disjoint meridian disks D_1 of V_1 and q mutually disjoint meridian disks D_2 of V_2. If we smooth the intersections of ∂D_1 and ∂D_2 uniformly in T, then we can obtain a torus knot or link K of type (p, q). For each point of $\partial D_1 \cap \partial D_2$ we add two triangle regions along this smoothing to $D_1 \cup D_2$, and then we obtain a Seifert surface F_v for K. We remark that, by the construction, $\chi(F_v) = |D_1| + |D_2| - |\partial D_1 \cap \partial D_2| = p + q - pq$, and when K is a knot, $g(F_v) = (p-1)(q-1)/2 = g(K)$. We also have cabling annuli $F_h = T \cap E(K)$, where $E(K)$ denotes the exterior of K in S^3. Moreover, when K is a knot, we have $\Delta(\partial F_v, \partial F_h) = |\partial D_1 \cap \partial D_2| = pq$, where $\Delta(*, *)$ denotes the distance between two boundary slopes. We note that F_v is orientable and when K is a knot, F_h is connected.

1.3. From closed fake surfaces to dual surfaces. We define three subsets of $\mathbb{R}^3$:

(1) $\Sigma_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$;
(2) $\Sigma_2 = \{(x, y, z) \in \mathbb{R}^3 \mid y = 0, z \geq 0\}$;
(3) $\Sigma_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0, z \leq 0\}$.

A finite 2-polyhedron P is called a *closed fake surface* [4] if each of its points has a neighborhood homeomorphic to one of the following types (Figure 1):

- Type 1: Σ_1;
- Type 2: $\Sigma_1 \cup \Sigma_2$;
- Type 3: $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$.

We will refer to points in closed fake surfaces as points of Type 1, 2 and 3 depending on which of the above three neighborhoods they have. By P' we shall denote the set of points of Type 2 or 3. By P'', we denote the set of points of Type 3.Thus, P' can be

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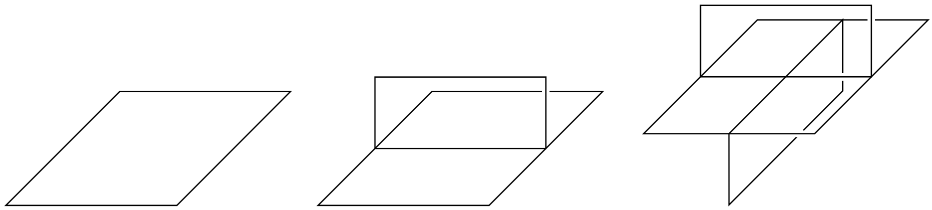

FIGURE 1. Local neighborhoods of a closed fake surface.

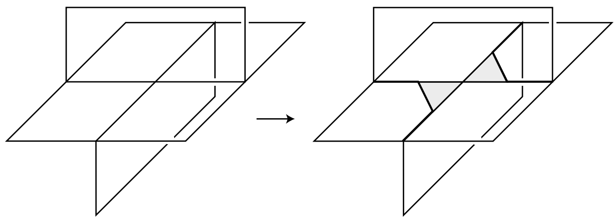

FIGURE 2. +-smoothing of a closed fake surface P.

regarded as a 4-valent graph with vertex set P''. A closed fake surface is *orientable* if each component of $P - P'$ is orientable.

We say that a closed fake surface P embedded in S^3 has a *vertical–horizontal decomposition* $P = P_v \cup P_h$ if P_h is a union of closed subsurfaces of P which corresponds to $(\Sigma_1, \mathbb{R}^3)$ at each neighborhood of points of Type 2 or 3 and P_v is a union of subsurfaces of P which corresponds to $(\Sigma_2, \mathbb{R}^3)$ or $(\Sigma_3, \mathbb{R}^3)$ at each neighborhood of points of Type 2 or 3. When $P' = \emptyset$, we define $P_h = P$ and $P_v = \emptyset$.

As in the procedure for constructing torus knots and links, for a closed fake surface P with the vertical–horizontal decomposition $P = P_v \cup P_h$, we obtain a knot or link K from P' and the vertical surfaces F_v and horizontal surfaces F_h from P_v and P_h, respectively, by smoothing P uniformly as follows. For each neighborhood of a point of Type 3, we add two triangle regions $\{(x, y, z) \in \mathbb{R}^3 \mid xy \geq 0, |x + y| \leq 1\}$ to P_v. Then we obtain surfaces from P_v, call them the *vertical surfaces* and denote them by F_v. We note that $\chi(F_v) = \chi(P_v) - |P''|$. The boundary of F_v consists of disjoint simple closed curves in P_h, namely, a knot or link, and we denote it by K. The *horizontal surfaces* F_h are the horizontal part P_h of P in $E(K)$. Then we say that F_v and F_h are obtained from P by the +-*smoothing* and that K is obtained from P' by the +-smoothing (Figure 2). The −-*smoothing* of P can be similarly defined, and the results for the +-smoothing also hold for the −-smoothing. We note that F_h is always orientable since P_h is a union of closed surfaces in S^3; however, F_v is nonorientable in almost all cases.

1.4. Definition of essential closed fake surfaces. Let P be a closed fake surface embedded in the 3-sphere S^3 with the vertical–horizontal decomposition $P = P_v \cup P_h$. A loop l properly embedded in $P - P'$ is *inessential* in P if l bounds a disk δ in $P - P'$,

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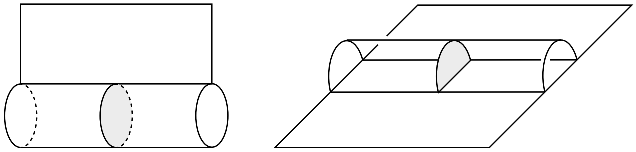

FIGURE 3. A monogon and bigon for $P = P_v \cup P_h$.

and l is *essential* if it is not inessential. Let $(\alpha, \partial\alpha)$ be an arc properly embedded in $(P_v, P' - P'')$ or $(P_h, P' - P'')$. An arc α is *inessential* in P if there is an arc β in $P' - P''$ such that $\alpha \cup \beta$ bounds a disk in P_v or P_h. Let D be a disk embedded in S^3 such that $D \cap P = \partial D \cap (P - P') = \partial D$. We call D a *compressing disk* for P if ∂D is essential in P. We call D a *monogon* if $\partial D \subset P_h - P''$ and $|\partial D \cap P'| = 1$. We call D a *bigon* if the boundary of D is decomposed into two arcs $\alpha \subset P_v$ and $\beta \subset P_h$ and at least one of α and β is an essential arc in P (Figure 3). A closed fake surface P embedded in S^3 is said to be *essential* if:

(1) $S^3 - P$ is irreducible;
(2) P has no compressing disk;
(3) P has no monogon;
(4) P has no bigon; and
(5) P_h has no 2-sphere component.

1.5. Definition of essential surfaces. Let K be a knot or link in S^3 and let $E(K)$ denote the exterior of K. Let F be a surface properly embedded in $E(K)$, possibly with boundary, except for the 2-sphere or disk, and let i denote the inclusion map $F \to E(K)$. We say that F is *algebraically incompressible* if the induced map $i_* : \pi_1(F) \to \pi_1(E(K))$ is injective and that F is *algebraically boundary incompressible* if the induced map $i_* : \pi_1(F, \partial F) \to \pi_1(E(K), \partial E(K))$ is injective for every choice of two base points in ∂F.

A disk D embedded in $E(K)$ is a *compressing disk* for F if $D \cap F = \partial D$ and ∂D is an essential loop in F. A disk D embedded in $E(K)$ is a *boundary compressing disk* for F if $D \cap F \subset \partial D$ is an essential arc in F and $D \cap \partial E(K) = \partial D - \mathrm{int}(D \cap F)$. We say that F is *geometrically incompressible* (respectively, *geometrically boundary incompressible*) if there exists no compressing disk (respectively, boundary compressing disk) for F.

In this paper, we say that surfaces F embedded in $E(K)$ are *geometrically essential* (respectively, *algebraically essential*) if each component of F is geometrically (respectively, algebraically) incompressible, geometrically (respectively, algebraically) boundary incompressible and not boundary parallel. In general, F is algebraically essential if and only if $\partial N(F) \cap E(K)$ is geometrically essential. If F is two-sided in $E(K)$, namely, orientable, then F is algebraically essential if and only if it is geometrically essential.

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Let K be a knot or link in S^3 and F be a union of closed surfaces embedded in S^3. We call F a union of *coiled surfaces* for K if $K \subset F$. We call a coiled surface F a *Neuwirth surface* if $F - C$ is connected for each component C of K and F is geometrically essential in the exterior $E(K)$. A union of surfaces S embedded in S^3 is a *spanning surface* for K if $\partial S = K$. The usual convention is that S has no closed components.

We remark that any nontrivial, nonsplittable knot or link has essential coiled surfaces since it bounds geometrically incompressible Seifert surfaces F, and $\partial N(F)$ gives coiled surfaces. Similarly, if a knot bounds an algebraically incompressible and boundary incompressible nonorientable spanning surface F, then $\partial N(F)$ gives a Neuwirth surface.

1.6. Main theorem.

Theorem 1.2. *Suppose that P is an essential orientable closed fake surface embedded in the 3-sphere S^3 with a vertical–horizontal decomposition $P = P_v \cup P_h$. Let F_v and F_h be the vertical and horizontal surfaces, respectively, obtained from P by the +-smoothing, and let K be the knot or link obtained from P' by the +-smoothing. Then F_v and F_h are algebraically essential in $E(K)$, and K is nonsplittable and prime. Moreover, when K is a knot, $\Delta(\partial F_v, \partial F_h) = |P''|$ and if F_v is orientable, then F_h is connected.*

We say that a knot or link K is *uniformly twisted* if it can be obtained from P' of an essential orientable closed fake surface P embedded in S^3 with a vertical–horizontal decomposition $P = P_v \cup P_h$ by the +-smoothing or −-smoothing.

In Theorem 1.2, if F_v is nonorientable, then K can be isotoped onto $\partial N(F_v)$ so that K is a nonseparating loop. Otherwise, F_h is a Neuwirth surface for K, by Theorem 1.2. Hence, we have the following corollary.

Corollary 1.3. *A uniformly twisted knot satisfies the Neuwirth conjecture.*

2. Proof

Proof of Theorem 1.2. First we isotope F_v near points of Type 3 so that F_v intersects F_h in arcs of the form $\{(x, y, z) : |x| \leq 1, y = z = 0\}$ in the neighborhoods of those points. Since F_h is orientable and F_v is possibly nonorientable, we need to show that F_h and the (twisted) ∂I-bundle $F_v \tilde{\times} \partial I$ are geometrically incompressible and boundary incompressible in $E(K)$. Then we may assume that in each neighborhood of points of Type 3, $F_h \cap (F_v \tilde{\times} \partial I)$ consists of two arcs.

Suppose that F_h or $F_v \tilde{\times} \partial I$ is compressible in $E(K)$ and let D be a compressing disk for it. Note that D is on the outside of $F_v \tilde{\times} \partial I$ since $F_v \tilde{\times} \partial I$ is incompressible in $F_v \tilde{\times} I$. We take D so that $|D \cap (F_v \cup F_h)|$ is minimal. If $D \cap (F_v \cup F_h) = \emptyset$, then D can be extended to a compressing disk for P. Otherwise, let α be an outermost arc in D and δ be the corresponding outermost disk of D. We extend δ so that $\partial\delta \subset F_v \cup F_h$. Then there are three possibilities; here we note that $\partial\alpha \subset F_v \cap F_h$:

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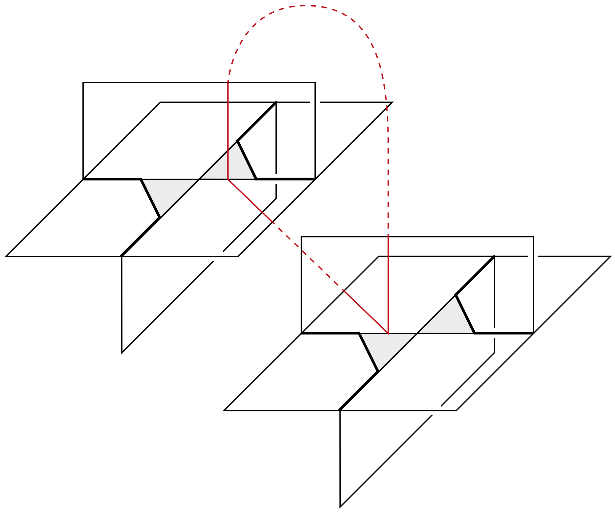

FIGURE 4. The boundary of an outermost disk (Case 1).

- Case 1: α connects two different arcs of $F_v \cap F_h$ which come from distinct points of Type 3.
- Case 2: α connects a single arc of $F_v \cap F_h$ which comes from a single point of Type 3, and δ lies on the same side of F_v near the point.
- Case 3: α connects a single arc of $F_v \cap F_h$ which comes from a single point of Type 3, and δ lies on both sides of F_v near the point.

In Case 1, δ cannot be trivial since there are two arcs of $P' - P''$ at the two points of $\partial\delta$ in δ (Figure 4). Here we remember that F_v and F_h are obtained by the +-smoothing. Hence, δ gives a bigon for P. In Case 2, δ is nontrivial since $|D \cap (F_v \cup F_h)|$ is taken to be minimal. Hence, δ again gives a bigon for P. Case 3 does not occur since $\partial\delta$ cannot run from one side of F_v to the other, since P_v is orientable. Hence, F_h and $F_v \tilde{\times} \partial I$ are incompressible in $E(K)$.

At this stage, we can show that K is nonsplittable and nontrivial as follows. Let S be an essential 2-sphere in $S^3 - K$. By the incompressibility of F_h, we may assume that S is disjoint from F_h. Moreover, since P_v is incompressible in the complements of F_h, we may assume that S is also disjoint from P_v. Then S bounds a 3-ball in $S^3 - K$ since $S^3 - P$ is irreducible. Hence, K is nonsplittable. Suppose that K is trivial. Then K is a trivial knot since K is nonsplittable. This shows that P_h consists of a single 2-sphere or a single torus since an orientable incompressible surface F_h in a solid torus $E(K)$ is a disk or annulus. In the former case, it contradicts that P_h has no 2-sphere component. In the latter case, F_h is an unknotted torus in S^3 which bounds a solid torus V, and K winds around V exactly once. Then $F_v \cap V$ consists of meridian disks or boundary parallel annuli. If $F_v \cap V$ consists of meridian disks, then Vcontains a monogon for P,

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and otherwise, V contains a bigon for P. In any case, we have a contradiction. Hence, K is nontrivial.

Next, suppose that F_h is boundary compressible in $E(K)$. Since it is well known that a geometrically incompressible but geometrically boundary compressible orientable surface in a link exterior is a boundary parallel annulus (cf. [7, Lemma 2]), there exists a solid torus V bounded by a component T of P_h such that the component of K contained in T winds around V exactly once longitudinally. Since $P_v \cap V$ is incompressible in V, it consists of meridian disks or boundary compressible annuli. If $P_v \cap V$ consists of meridian disks, then the remaining components of P_v having boundary in ∂V wind around V exactly once longitudinally. Therefore, there exists a monogon for P in V. Otherwise, there exists a bigon for P in V coming from a boundary compressing disk for $P_v \cap V$. Hence, F_h is incompressible and boundary incompressible in $E(K)$.

Suppose that $F_v \tilde{\times} \partial I$ is boundary compressible in $E(K)$. Since it is well known that an algebraically incompressible but algebraically boundary compressible nonorientable spanning surface for a link is a Möbius band whose boundary is the trivial knot (cf. [10, Lemma 2.2]), K is the trivial knot. This contradicts that K is nontrivial.

Now we know that both F_h and F_v are not boundary parallel in $E(K)$ since these surfaces are incompressible, boundary incompressible and have integral boundary slopes in $E(K)$. Hence, F_v and F_h are algebraically essential in $E(K)$.

Next we show that K is prime. Suppose that K is nonprime and let S be a decomposing sphere for K. We may assume that S intersects F_h in two arcs which form a loop l with two points p_1 and p_2 of $K \cap S$. Then l decomposes S into two disks, say, D_1 and D_2. By an isotopy, we may assume that ∂D_i does not run over neighborhoods of P'' for $i = 1, 2$. Since there is an arc of $D_i \cap P_v$ from a point p_j for $j = 1, 2$, there is an arc joining p_1 and p_2 in D_1 or D_2. Thus we may assume, without loss of generality, that D_1 intersects P_v in a single arc joining p_1 and p_2 and D_2 does not intersect P_v in its interior. Then we have a bigon for P as the subdisk of D_1. Hence, K is prime.

Hereafter, we assume that K is a knot. Then F_h consists of a single closed surface. It can be observed that the boundaries of F_v and F_h intersects in $|P''|$ points essentially on $\partial N(K)$ since F_v and F_h are obtained by the +-smoothing (Figure 5). Thus the distance $\Delta(\partial F_v, \partial F_h)$ is equal to $|P''|$.

Suppose that F_v is orientable. Then K can be oriented by the orientation of F_v. This shows that F_h is connected since F_v and F_h are obtained by the +-smoothing (Figure 6). □

3. Example

In this section, we observe that torus links and alternating links are uniformly twisted; these are typical examples for Theorem 1.2.

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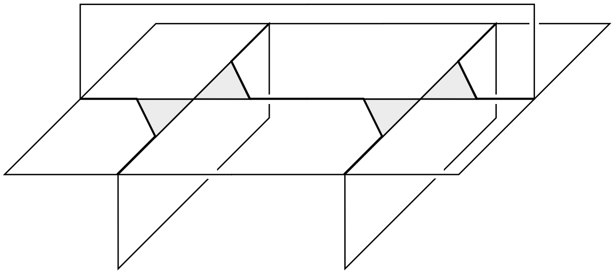

FIGURE 5. +-smoothing of a closed fake surface P.

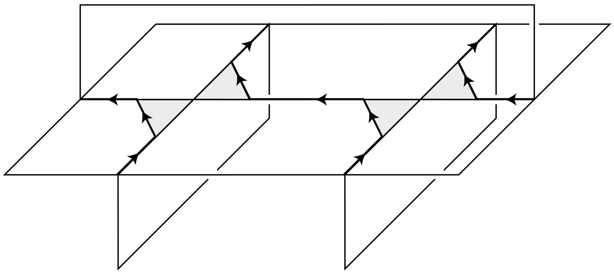

FIGURE 6. An orientation of K induced by F_v.

Let S^2 be a 2-sphere embedded in S^3 and G be a 2-connected graph embedded in S^2 with at least one edge. A graph G is said to be 2-*connected* if there does not exist a single vertex whose removal disconnects the graph, and we remark that this is required to have a prime knot or link K. Then the closed surface $P_h = \partial N(G)$ decomposes S^3 into two handlebodies V_1 and V_2, where V_1 contains G. For each edge of G, we take parallel copies of a meridian disk of V_1 which is dual to an edge of G, and for each region of $S^2 - \text{int}\, V_1$, we take parallel copies of a meridian disk of V_2 as the region. Let P_v be a union of these meridian disks. Then we obtain an essential orientable closed fake surface P with the vertical–horizontal decomposition $P = P_v \cup P_h$. Let F_v and F_h be the vertical and horizontal surfaces, respectively, obtained from P by the +-smoothing or −-smoothing, and let K be the knot or link obtained from P' by the +-smoothing or −-smoothing (Figure 7).

By the construction, torus knots and links are uniformly twisted. In the above construction, if we take exactly one copy of a meridian disk in V_1 or V_2, then we obtain a prime alternating knot or link as K. In this case, F_v is a checkerboard surface for the alternating diagram of K, and F_h is a boundary of a regular neighborhood of another checkerboard surface. Similarly, we note that generalized alternating knots and links [7] are also uniformly twisted. Moreover, we can take many parallel copies of each disk in P_v, and hence this construction gives a much larger class of knots that satisfy the Neuwirth conjecture.

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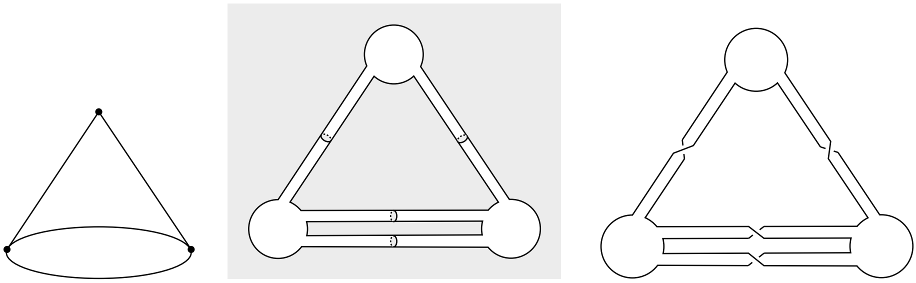

FIGURE 7. The procedure for obtaining a uniformly twisted knot.

4. Problem

We close this paper with some problems.

In the last section, we constructed an essential closed fake surface from a Heegaard surface in the 3-sphere. It is natural to ask how to get all essential closed fake surfaces obtained from a given closed surface as their horizontal surface.

PROBLEM 4.1. Find a construction of an essential closed fake surface from a given closed surface in the 3-sphere.

The class of uniformly twisted knots and links is somewhat wide, but it has a restriction, 'uniformly twisted'. We would like to know how wide is this class.

PROBLEM 4.2. Does there exist a knot or link which is not uniformly twisted?

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