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On the (In?)Stability of Spacetime Inextendibility

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Abstract

Leibnizian metaphysics underpins the universally held view that spacetime must be inextendible—that it must be "as large as it can be" in a sense. But here we demonstrate a surprising fact within the context of general relativity: the property of inextendibility turns out to be unstable when attention is restricted to certain collections of "physically reasonable" spacetimes.

1. Introduction

Within the context of general relativity, the "stability" of various spacetime properties has been one important focus of study. It has been argued that "in order to be physically significant, a property of space-time ought to have some form of stability, that is to say, it should be a property of 'nearby' space-times" (Hawking and Ellis 1973, 197). Questions concerning the stability of spacetime properties are often made precise using the so-called "C^k fine" topologies on any collection of spacetimes with the same underlying manifold. (The property of "stable causality" is often defined using the C^0 fine topology.) Here we review what is known concerning the (in) stability of spacetime properties within this framework. After considering some foundational results concerning causal properties (Hawking 1969; Geroch 1970a) and a fascinating drama concerning geodesic (in)completeness (Beem et al. 1996), we focus on the property of spacetime inextendibility, about which very little is known. Because inextendibility is defined relative to a background "possibility space" in the form of a standard collection of spacetimes, one can naturally consider variant definitions relative to other collections. (Some formulations of the "cosmic censorship" conjecture rely on such variant definitions of inextendibility.) We find that the stability of "inextendibility" can be highly sensitive to the choice of definition—even when attention is limited to definitions which are relative to "physically reasonable" collections of spacetimes. Indeed, it is not yet clear that there is a physically significant sense in which "inextendibility" is a stable property.

2. Preliminaries

Here we follow Wald (1984) and Malament (2012). An n-dimensional, general relativistic *spacetime* (for $n \geq 2$) is a pair of mathematical objects (M, g_{ab}) where M is a smooth, connected, n-dimensional, Hausdorff manifold and g_{ab} is a smooth metric of Lorentz signature $(+, -, \ldots, -)$ defined on M. In what follows, let $\mathcal{U}$ be the collection of all spacetimes. We say two spacetimes (M, g_{ab}) and (M', g'_{ab}) are *isometric* if there is a diffeomorphism $\varphi : M \to M'$ such that $\varphi^*(g'_{ab}) = g_{ab}$.

Fix a model (M, g_{ab}). For each point $p \in M$, the metric assigns a cone structure to the tangent space M_p. Any tangent vector ξ^a in M_p will be *timelike* if $g_{ab}\xi^a\xi^b > 0$, *null* if $g_{ab}\xi^a\xi^b = 0$, or *spacelike* if $g_{ab}\xi^a\xi^b < 0$. Null vectors create the cone structure; timelike vectors fall inside the cone while spacelike vectors fall outside. A *time orientable* model is one that has a continuous timelike vector field on M. In what follows, we assume that models are time orientable and that an orientation has been chosen.

For some connected interval $I \subset \mathbb{R}$, a smooth curve $\gamma : I \to M$ is *timelike* if its tangent vector ξ^a at each point in $\gamma[I]$ is timelike. Similarly, a curve is *null* if its tangent vector at each point is null. A curve is *causal* if its tangent vector at each point is either null or timelike. A causal curve is *future-directed* if its tangent vector at each point falls in or on the future lobe of the light cone. A causal curve $\gamma : I \to M$ is *closed* if the tangent vector is nowhere vanishing and there are distinct $s, s' \in I$ such that $\gamma(s) = \gamma(s')$. (M, g_{ab}) satisfies *chronology* if it does not contain a closed timelike curve; it satisfies *causality* if it does not contain a closed causal curve.

We write $p \ll q$ (respectively, $p < q$) if there exists a future-directed timelike (respectively, causal) curve from p to q. For any point $p \in M$, we define the *timelike future* of p as the set $I^+(p) = \{q : p \ll q\}$. Similarly, the *causal future* of p is the set $J^+(p) = \{q : p < q\}$. The timelike and causal pasts of p, denoted $I^-(p)$ and $J^-(p)$, are defined analogously. The spacetime (M, g_{ab}) satisfies *distinguishability* if there do not exist distinct points $p, q \in M$ such that $I^-(p) = I^-(q)$ or $I^+(p) = I^+(q)$. We say (M, g_{ab}) admits a *global time function* if there is a smooth function $t : M \to \mathbb{R}$ such that, for any distinct points $p, q \in M$, if $p \in J^+(q)$, then $t(p) > t(q)$. (M, g_{ab}) satisfies *global hyperbolicity* if it is causal and, for any points $p, q \in M$, the set $J^+(p) \cap J^-(q)$ is compact.

A curve $\gamma : I \to M$ is *maximal* if there is no curve $\gamma' : I' \to M$ such that I is a proper subset of I' and $\gamma(s) = \gamma'(s)$ for all $s \in I$. The curve $\gamma : I \to M$ is a *geodesic* if $\xi^a \nabla_a \xi^b = 0$, where ξ^a is its tangent vector and ∇_a is the unique derivative operator compatible with g_{ab}. A maximal geodesic $\gamma : I \to M$ is *incomplete* if $I \neq \mathbb{R}$. A spacetime is *geodesically incomplete* if it harbors an incomplete geodesic and *geodesically complete* otherwise; one can define *causal geodesic (in)completeness* in an analogous way.

Let the *energy–momentum tensor* T_{ab} for the spacetime (M, g_{ab}) be defined by Einstein's equation: $R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}$, where R_{ab} is the Ricci tensor and R the scalar curvature associated with g_{ab}. We say that (M, g_{ab}) is a *vacuum solution* if $T_{ab} = 0$. The *null energy condition* is satisfied if, for any null vector χ^a, we have $T_{ab}\chi^a\chi^b \geq 0$. The *weak energy condition* is satisfied if, for each timelike vector ξ^a, we have $T_{ab}\xi^a\xi^b \geq 0$. The *strong energy condition* is satisfied if, for any unit timelike vector ξ^a, we have $(T_{ab} - \frac{1}{2}Tg_{ab})\xi^a\xi^b \geq 0$. Finally, the *dominant energy condition* is satisfied if, for any future-directed unit timelike ξ^a, the vector $T^a{}_b\xi^b$ is causal and future-directed.

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3. Inextendibility

A spacetime (M', g'_{ab}) is an *extension* of the spacetime (M, g_{ab}) if there is a proper subset $N \subset M'$ such that the spacetimes (N, g'_{ab}) and $(M g_{ab})$ are isometric. A spacetime is *extendible* if it has an extension and *inextendible* otherwise. One can show that any extendible spacetime has an inextendible extension (Geroch 1970b). This fact helps to underpin the nearly universally held position that "any [physically] reasonable space-time should be inextendible" (Clarke 1993, 8). John Earman summarizes and responds to the usual line of argument (cf. Penrose 1969; Geroch 1970b):

> Metaphysical considerations suggest that to be a serious candidate for describing actuality, a spacetime should be [inextendible]. For example, for the Creative Force to actualize a proper subpart of a larger spacetime would seem to be a violation of Leibniz's principles of sufficient reason and plenitude. If one adopts the image of spacetime as being generated or built up as time passes then the dynamical version of the principle of sufficient reason would ask why the Creative Force would stop building if it is possible to continue. However, this image does not sit well with the four-dimensional way of thinking, and in any case it runs into trouble in its own terms: since extensions of spacetime are generally non-unique there may be many ways to continue building and the Creative Force may be stymied by a Buridan's ass choice. Some readers may be shocked by the introduction of metaphysical considerations in the hardest of the "hard sciences." But in fact leading workers in relativistic gravitation, though they don't invoke the name of Leibniz, are motivated by such principles. (Earman 1995, 32–33)

Setting aside the metaphysical issues outlined here, we see that the inextendibility condition also faces an important conceptual difficulty: the standard formulation is defined relative to the background "possibility space" $\mathcal{U}$ (the collection of all spacetimes) despite the fact that within $\mathcal{U}$ lurk "physically unreasonable" members (Manchak 2011, 2020). Shouldn't a spacetime which is extendible according to the standard definition count as "inextendible" if none of its extensions are "physically reasonable"? Even if we cannot pin down, once and for all, a single collection of "physically reasonable" spacetimes, one can still explore variant formulations of the inextendibility condition defined relative to subcollections of $\mathcal{U}$ (Geroch 1970b; Manchak 2016). For any collection $\mathcal{P} \subseteq \mathcal{U}$, consider the following: a $\mathcal{P}$*-spacetime* is a spacetime in $\mathcal{P}$; a $\mathcal{P}$-spacetime (M', g'_{ab}) is a $\mathcal{P}$*- extension* of a $\mathcal{P}$-spacetime (M, g_{ab}) if (M', g'_{ab}) is an extension of (M, g_{ab}); a $\mathcal{P}$-spacetime is $\mathcal{P}$*-extendible* if it has a $\mathcal{P}$-extension and is $\mathcal{P}$*-inextendible* otherwise. It is trivial that for any collection $\mathcal{P} \subset \mathcal{U}$ of inextendible spacetimes (e.g., the collection of geodesically complete spacetimes) a $\mathcal{P}$-inextendible spacetime must be inextendible. The general situation is quite different, however. For each $\mathcal{P} \subset \mathcal{U}$, consider the following statement:

(*) Any $\mathcal{P}$-inextendible spacetime must be inextendible.

Let $(V), (DEC), (SEC), (WEC), (NEC) \subset \mathcal{U}$ be, respectively, the collections of vacuum solutions and spacetimes satisfying the dominant, strong, weak, and null energy conditions; note that $(V) \subset (DEC) \subset (WEC) \subset (NEC)$ and $(V) \subset (SEC) \subset (NEC)$. Let $(GH), (TF), (Dist), (Caus), (Chron) \subset \mathcal{U}$ be, respectively, the collections of spacetimes which are globally hyperbolic, admit a global time function, are distinguishing, causal, and chronological. Of course, $(GH) \subset (TF) \subset (Dist) \subset (Caus) \subset (Chron)$. Let $(GI) \subset \mathcal{U}$ be the collection of geodesically incomplete spacetimes. We have the following proposition (Manchak 2017, 2021).

Proposition 1. (*) is false if: (i) $(DEC) \subseteq \mathcal{P} \subseteq (NEC)$; (ii) $(SEC) \subseteq \mathcal{P} \subseteq (NEC)$; (iii) $(GH) \subseteq \mathcal{P} \subseteq (Dist)$; (iv) $\mathcal{P} = (Caus)$; or (v) $\mathcal{P} = (GI)$.

The status of (*) is still unknown for $\mathcal{P} = (V)$ and $\mathcal{P} = (Chron)$ (cf. Krasnikov 2018). Indeed, the $\mathcal{P} = (Chron)$ case has been one focus of the "time travel" literature for some time but remains difficult to settle (cf. Krasnikov 2018). But in general, the proposition suggests that we should carefully attend to the differences between the standard definition of inextendibility and other variants. In formulating a version of the "cosmic censorship" conjecture, Wald (1984, 304–305) does just this when he appreciates that while some "maximal Cauchy developments ... are known to be extendible" it may be that all such extensions fail to be $\mathcal{P}$-extensions for some carefully chosen collection $\mathcal{P} \subset \mathcal{U}$ of "physically reasonable" spacetimes.

4. Stability

In their influential book *The Large Scale Structure of Space-Time*, Hawking and Ellis wrote:

> [I]n order to be physically significant, a property of space-time ought to have some form of stability, that is to say, it should be a property of "nearby" space-times. In order to give precise meaning to "nearby" one has to define a topology on the set of all space-times ... We shall leave the problem of uniting in one connected topological space manifolds of different topologies (this can be done); and shall just consider putting a topology on the set of all C^r Lorentz metrics ($r \geq 1$) on a given manifold. (Hawking and Ellis 1973, 197–198)

It is of some interest that despite the claim that a suitable topology can be put on the entire collection $\mathcal{U}$, no one has yet done this even after almost fifty years (Fletcher 2016). Instead, various topologies have been defined on each collection $\mathcal{L}(M) \subset \mathcal{U}$ of all spacetimes with underlying manifold M. The most commonly used are the "C^k fine" topologies (also called the "C^k open" topologies) for $k \geq 0$, which we shall consider here (cf. Geroch 1971; Hawking and Ellis 1973).

Let (M, g_{ab}) and (M, g'_{ab}) be spacetimes, let h^{ab} be any positive definite metric on M, and let ∇_a be the unique derivative operator compatible with h^{ab}. At each point in M,

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the *distance function* $d(g_{ab}, g'_{ab}, h^{ab}, k)$ between the kth partial derivatives (for $k \geq 0$) of the Lorentzian metrics g_{ab} and g'_{ab} on M relative to h^{ab} is given by

$$[h^{ac}h^{bd}(g_{ab} - g'_{ab})(g_{cd} - g'_{cd})]^{1/2} \text{ for } k = 0,$$

$$[h^{ac}h^{bd}h^{r_1 s_1} \cdots h^{r_k s_k}(\nabla_{r_1} \cdots \nabla_{r_k}(g_{ab} - g'_{ab}))(\nabla_{s_1} \cdots \nabla_{s_k}(g_{cd} - g'_{cd}))]^{1/2} \text{ for } k > 0 \ .$$

A C^k *fine neighborhood* of a spacetime (M, g_{ab}) is any collection $\mathcal{N} \subset \mathcal{L}(M)$ which includes all spacetimes (M, g'_{ab}) such that $\text{Sup}_M[d(g_{ab}, g'_{ab}, h^{ab}, j)] < \varepsilon$ for $j = 0, \ldots, k$, where h^{ab} is a positive definite metric on M and ε is a positive number. For all $\mathcal{Q} \subseteq \mathcal{P} \subseteq \mathcal{U}$, we say the property $\mathcal{Q}$ is C^k *stable relative to the collection* $\mathcal{P}$ if, for each $\mathcal{Q}$-spacetime (M, g_{ab}), there is a C^k fine neighborhood of (M, g_{ab}) such that every $\mathcal{P}$-spacetime in the neighborhood is a $\mathcal{Q}$-spacetime. Immediately we see that, for all $\mathcal{Q} \subseteq \mathcal{P} \subseteq \mathcal{U}$, if property $\mathcal{Q}$ is C^k stable relative to the collection $\mathcal{P}$, then $\mathcal{Q}$ is C^l stable relative to $\mathcal{P}$ for all $l \geq k$.

We know that even the coarsest of all of the C^k fine topologies is still quite fine: If (M, g_{ab}) is a spacetime and M is non-compact, then the collection $\{(M, \lambda g_{ab}) : \lambda \in (0, \infty)\}$ does not represent a C^0 fine continuous curve; in addition, the induced topology on the collection is discrete (Geroch 1971). It seems the C^k fine topologies have too many open sets to capture, once and for all, what it means for one spacetime to be "nearby" another. On the other hand, this means that instability results are all the more significant. Early results concerned two important causal properties (Hawking 1969; Geroch 1970a).

Proposition 2. *(TF) and (GH) are C^k stable relative to $\mathcal{U}$ for all $k \geq 0$.*

Consider a few remarks concerning Proposition 2. First, the (*TF*) case tells us that any spacetime (M, g_{ab}) with a global time function is "stably causal" in the sense that one can find a C^0 neighborhood of (M, g_{ab}) such that each spacetime in the neighborhood admits a global time function and is therefore causal. Second, there were significant gaps in the proof concerning the (*GH*) case which were filled in only recently (Navarro and Minguzzi 2011). Finally, a simple but physically significant corollary to the proposition ensures that the C^k stability of (*TF*) and (*GH*) with respect to $\mathcal{U}$ will "transfer down" to the C^k stability of (*TF*) $\cap \mathcal{P}$ and (*GH*) $\cap \mathcal{P}$ with respect to any "physically reasonable" collection $\mathcal{P} \subseteq \mathcal{U}$. In general, we have the following.

Proposition 3. *For all $\mathcal{Q} \subseteq \mathcal{P} \subseteq \mathcal{U}$ and for all $k \geq 0$, if the property $\mathcal{Q}$ is C^k stable relative to the collection $\mathcal{P}$, then, for any subcollection $\mathcal{P}' \subseteq \mathcal{P}$, the property $\mathcal{Q} \cap \mathcal{P}'$ is C^k stable relative to $\mathcal{P}'$.*

What about the stability of other important spacetime properties? Consider the collection (*GC*) $\subset \mathcal{U}$ of geodesic complete spacetimes. The following claim was made in the first edition of Beem and Ehrlich (1981).

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Claim 1. *(GC) is C^k stable relative to $\mathcal{U}$ for all $k \geq 2$.*

Soon after there was to be a dramatic turn of events. Ehrlich later recounted the following:

> That is how matters stood until 1985, when a copy of P. Williams' Ph.D. thesis, "Completeness and its stability on manifolds with connection," was received unexpectedly in the mail. This article revealed that there was a significant gap in the previous arguments for [the claim above] and that in fact neither geodesic completeness nor geodesic incompleteness was C^k-stable ... From a certain perspective, a good deal of research in global space-time geometry during the next decade can be viewed as trying to understand the more complicated geometry of the space of geodesics once it was realized that [the claim] failed to be valid. (Ehrlich 2006, 14)

From Williams (1984) we have the following result, which is all the more remarkable given how fine even the C^0 topologies have been shown to be.

Proposition 4. *(GC) and (GI) are not C^k stable relative to $\mathcal{U}$ for all $k \geq 0$.*

It is of some interest that this result fails within the Riemannian context where both geodesic completeness and geodesic incompleteness are C^k stable for all $k \geq 0$ (Beem and Ehrlich 1987). By the time the second edition of their book was published, Beem and Ehrlich had worked to salvage the stability of geodesic (in)completeness by restricting attention to special cases. Consider the following proposition, which is representative of this effort (Beem et al. 1996).

Proposition 5. *If (M, g) is a globally hyperbolic spacetime and causally geodesically complete (respectively, incomplete), then there is a C^1 fine neighborhood of (M, g) such that each spacetime in the neighborhood is causally geodesically complete (respectively, incomplete).*

Here, the physical significance is limited given that attention is restricted to globally hyperbolic spacetimes and the C^0 case is not considered. Are more general results available? Nothing so far—even today, we do not have a good understanding of the (in)stability properties of geodesic (in)completeness and closely related properties (cf. Manchak 2018; Doboszewski 2020).

5. Stability and Inextendibility

What is known concerning the (in)stability of the inextendibility properties? Very little. What we do have is due to Beem and Ehrlich (1987). Using their work concerning the stability of geodesic completeness, and drawing on on the fact that geodesic completeness implies inextendibility, they show the following.

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Proposition 6. *There is a C^1 fine neighborhood of Minkowski spacetime such that each spacetime in the neighborhood is inextendible.*

It is somewhat remarkable that, even after restricting attention to Minkowski spacetime, the C^0 case is unsettled. Are general results available? One would love to know the status of the following conjecture, for example.

Conjecture 1. *The collection of all inextendible spacetimes is C^k stable relative to $\mathcal{U}$ for all $k \geq 0$.*

Given the dramatic twists and turns so far concerning the (in)stability of geodesic (in)completeness—and the many surprises throughout the history of global Lorentzian geometry more generally—the status of the conjecture is anyone's guess. But even if it were true, there is a sense in which its physical significance would seem to be quite limited since there is no assurance here that the stability of inextendibility relative to $\mathcal{U}$ would "transfer down" to the stability of $\mathcal{P}$-inextendibility relative to some "physically reasonable" collection $\mathcal{P} \subset \mathcal{U}$. Indeed, consider the following.

Proposition 7. *There are collections $\mathcal{Q} \subset \mathcal{P} \subset \mathcal{U}$ such that $\mathcal{P}$-inextendibility is C^k stable relative to $\mathcal{P}$ for all $k \geq 0$ but $\mathcal{Q}$-inextendibility fails to be C^k stable relative to $\mathcal{Q}$ for all $k \geq 0$. Moreover, $\mathcal{P}$ can be chosen so that each member is a globally hyperbolic vacuum solution.*

Proof. We work in two dimensions to simplify the presentation, but one can generalize in the natural way. In first stage of the proof, we define the collections $\mathcal{Q}, \mathcal{P} \subset \mathcal{U}$. Consider the smooth bump function $u : [-2, 2] \to \mathbb{R}$ given by

$$u(t) = \begin{cases} \exp\left[1/(t^2-1)\right] & \text{if } \; -1 < t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each $n \in \mathbb{Z}^+$, we let $f_n, F_n : [-2, 2] \to \mathbb{R}$ be the functions $f_n(t) = [1 - u(t)/n]^{1/2}$ and $F_n(t) = \int_0^t f_n(x)\,dx$. The graphs of the functions $f_1(t)$ and $F_1(t)$ are given in Figure 1. Note that, for all n, $F_n(t)$ is strictly increasing and thus invertible. When no confusion arises, we will abuse notation and consider the functions $f_n(t)$ and $F_n(t)$ where the domain is restricted to $(-2, 2)$.

Let $M = \{(t, \varphi) \in \mathbb{R} \times S^1 : -2 < t < 2\}$. For each $n \in \mathbb{Z}^+$, let $(M, g_{ab}(n))$ be the spacetime defined by setting $g_{ab}(n) = f_n^2(t)\nabla_a t \nabla_b t - \nabla_a \varphi \nabla_b \varphi$. Let $(M, g_{ab}(\dagger))$ be the spacetime defined by setting $g_{ab}(\dagger) = \nabla_a t \nabla_b t - \nabla_a \varphi \nabla_b \varphi$. Finally, let $(M, g_{ab}(\ddagger))$ be the spacetime defined by setting $g_{ab}(\ddagger) = f_\ddagger^2(t)\nabla_a t \nabla_b t - \nabla_a \varphi \nabla_b \varphi$ where $f_\ddagger : (-2, 2) \to \mathbb{R}$ is given by $f_\ddagger(t) = \pi \sec^2(\pi t/4)/2$. Let $\mathcal{P} \subset \mathcal{U}$ be the collection $\{(M, g_{ab}(\dagger)), (M, g_{ab}(\ddagger))\} \cup \{(M, g_{ab}(n)) : n \in \mathbb{Z}^+\}$; let $\mathcal{Q} \subset \mathcal{P}$ be the collection $\mathcal{P} - \{(M, g_{ab}(\ddagger))\}$.

In the second stage of the proof, we establish the following facts: (i) for all n, $(M, g_{ab}(n))$ is $\mathcal{Q}$-extendible (and hence $\mathcal{P}$-extendible); (ii) $(M, g_{ab}(\dagger))$ is $\mathcal{Q}$-inextendible but $\mathcal{P}$-extendible; (iii) $(M, g_{ab}(\ddagger))$ is $\mathcal{P}$-inextendible; and (iv) each

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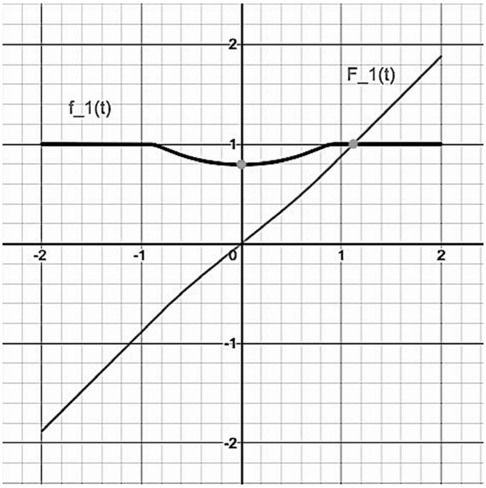

Figure 1. The functions $f_1(t)$ and $F_1(t)$.

member of $\mathcal{P}$ is a globally hyperbolic vacuum solution. All of these facts will follow easily once we define, for each spacetime in $\mathcal{P}$, an isometric variant.

First, consider $(M'(\ddagger), g'_{ab})$, where $M'(\ddagger) = \mathbb{R} \times S^1$ and $g'_{ab} = \nabla_a t' \nabla_b t' - \nabla_a \varphi' \nabla_b \varphi'$. We find that $(M, g_{ab}(\ddagger))$ is isometric to $(M'(\ddagger), g'_{ab})$; to see this, use the diffeomorphism $\Psi_\ddagger : M \to M'(\ddagger)$ defined by $\Psi_\ddagger((t, \varphi)) = (2\tan(\pi t/4), \varphi)$ and note that $\nabla_a(2\tan(\pi t/4)) = f_\ddagger(t)\nabla_a t$. Next, consider $(M'(\dagger), g'_{ab})$, where $M'(\dagger) = \{(t', \varphi') \in M'(\ddagger) : -2 < t' < 2\}$ and the domain of g'_{ab} is restricted in the natural way. We find that $(M, g_{ab}(\dagger))$ is isometric to $(M'(\dagger), g'_{ab})$; to see this, just use the identity map $\Psi_\dagger : M \to M'(\dagger)$. Finally, consider the spacetime $(M'(n), g'_{ab})$ for all $n \in \mathbb{Z}^+$, where $M'(n) = \{(t', \varphi') \in M'(\ddagger) : -F_n(2) < t' < F_n(2)\}$ and once again the domain of g'_{ab} is restricted in the natural way. One can verify that $F_1(2) \approx 1.88$ (see Figure 1) and, for all n, we have $F_1(2) < F_n(2) < 2$. Now, for all n, we find that $(M, g_{ab}(n))$ is isometric to $(M'(n), g'_{ab})$; to see this, use the diffeomorphism $\Psi_n : M \to M'(n)$ defined by $\Psi_n((t, \varphi)) = (F_n(t), \varphi)$ and note that $\nabla_a F_n(t) = f_n(t)\nabla_a t$.

It is immediate that, for all n, $M'(n)$. is a proper subset of $M(\dagger)$, which is, in turn, a proper subset of $M(\ddagger)$. So $(M'(\ddagger), g'_{ab})$ is an extension of $(M'(\dagger), g'_{ab})$, which is an extension of $(M'(n), g'_{ab})$ for all n. Moreover, these spacetimes are globally hyperbolic vacuum solutions since they are just portions of two-dimensional Minkowski spacetime which has been "rolled up" in the spacelike direction (see Figure 2). Using the isometries established above, we find that $(M, g_{ab}(\ddagger))$ is an extension of $(M, g_{ab}(\dagger))$, which is an extension of $(M, g_{ab}(n))$ for all n. Moreover, each of these $\mathcal{P}$-spacetimes must be a globally hyperbolic vacuum solution. So it follows that (i)–(iv) are true.

In the third stage of the proof, we show that the property of $\mathcal{P}$-inextendibility is C^0 stable (and hence C^k stable for all $k \geq 0$) relative to $\mathcal{P}$. Since $(M, g_{ab}(\ddagger))$ is the only $\mathcal{P}$-inextendible spacetime in $\mathcal{P}$, we are done if we can find a C^0 fine neighborhood $\mathcal{N} \subset \mathcal{U}$ of $(M, g_{ab}(\ddagger))$ such that none of the $\mathcal{P}$-extendible spacetimes can be found in $\mathcal{N}$. Let h^{ab} be the positive definite metric on M given by $h^{ab} = (\partial/\partial t)^a(\partial/\partial t)^b + (\partial/\partial\varphi)^a(\partial/\partial\varphi)^b$. Let $\mathcal{N} \subset \mathcal{L}(M)$ be the C^0 fine neighborhood

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Figure 2. The collections $\mathcal{P}$ and $\mathcal{Q}$.

of $(M, g_{ab}(\ddagger))$ defined as the collection of all spacetimes (M, g_{ab}) such that $\text{Sup}_M[d(g_{ab}(\ddagger), g_{ab}, h^{ab}, 0)] < 1$. We now show that each $\mathcal{P}$-extendible spacetime fails to make it into $\mathcal{N}$.

Consider the $\mathcal{P}$-extendible spacetime $(M, g_{ab}(n))$ for any n. We find that $g_{ab}(\ddagger) - g_{ab}(n)$ is just $\left(f_{\ddagger}^2(t) - f_n^2(t)\right)\nabla_a t \nabla_b t$. So $h^{ab}(g_{ab}(\ddagger) - g_{ab}(n)) = f_{\ddagger}^2(t) - f_n^2(t)$. But one can verify that, for all $t \in (-2, 2)$, we have $f_{\ddagger}^2(t) \geq f_{\ddagger}^2(0) = \pi^2/4 > 2$ and $f_n^2(t) \leq 1$. It follows that $\text{Sup}_M[d(g_{ab}(\ddagger), g_{ab}(n), h^{ab}, 0)] > 1$ and therefore $(M, g_{ab}(n))$ fails to be in $\mathcal{N}$ for all n. The argument for the remaining $\mathcal{P}$-extendible spacetime $(M, g_{ab}(\dagger))$ is analogous: We find that $g_{ab}(\ddagger) - g_{ab}(\dagger)$ is just $\left(f_{\ddagger}^2(t) - 1\right)\nabla_a t \nabla_b t$. So $h^{ab}(g_{ab}(\ddagger) - g_{ab}(\dagger)) = f_{\ddagger}^2(t) - 1$. Since $f_{\ddagger}^2(t) > 2$ for all $t \in (-2, 2)$, it follows that $\text{Sup}_M[d(g_{ab}(\ddagger), g_{ab}(\dagger), h^{ab}, 0)] > 1$ and therefore $(M, g_{ab}(\dagger))$ fails to be in $\mathcal{N}$. So, we have established that each $\mathcal{P}$-extendible spacetime fails to make it into $\mathcal{N}$ and therefore $\mathcal{P}$-inextendibility is C^0 stable (and hence C^k stable for all $k \geq 0$) relative to $\mathcal{P}$.

In the final stage of the proof, we show that the property of $\mathcal{Q}$-inextendibility is not C^k stable relative to $\mathcal{Q}$ for all $k \geq 0$. We restrict attention to the $k = 1$ case to simplify the presentation but one can generalize in the natural way. Since $(M, g_{ab}(\dagger))$ is $\mathcal{Q}$-inextendible, we are done if we can show that, for any C^1 fine neighborhood of $(M, g_{ab}(\dagger))$, there is some n such that the $\mathcal{Q}$-extendible spacetime $(M, g_{ab}(n))$ is in the neighborhood.

Let h^{ab} be any positive definite metric on M and ε any positive number. The smooth scalar fields $\alpha_0, \alpha_1 : M \to \mathbb{R}$ are defined by

$$\alpha_0(t, \varphi) = u(t) h^{ab} \nabla_a t \nabla_b t,$$

$$\alpha_1(t, \varphi) = [h^{ac} h^{bd} h^{rs} (\nabla_r (u(t) \nabla_a t \nabla_b t))(\nabla_s (u(t) \nabla_c t \nabla_d t))]^{1/2}.$$

The quantity $g_{ab}(\dagger) - g_{ab}(n)$ is just $(1 - f_n^2(t))\nabla_a t \nabla_b t = (u(t)/n)\nabla_a t \nabla_b t$ for all n. So $d(g_{ab}(\dagger), g_{ab}(n), h^{ab}, 0) = \alpha_0/n$ and $d(g_{ab}(\dagger), g_{ab}(n), h^{ab}, 1) = \alpha_1/n$. Let N be the compact region $\{(t, \varphi) \in M : -1 \leq t \leq 1\}$. By construction, α_0 and α_1 vanish on $M - N$. So $\text{Sup}_{M-N}[d(g_{ab}(\dagger), g_{ab}(n), h^{ab}, k)] = 0$ for $k = 0, 1$. Now consider N. Because this region is compact, we know that there is an $m \in \mathbb{R}$ such that $\alpha_0(p) < m$ and $\alpha_1(p) < m$ for all $p \in N$. So, for each n, we know $\text{Sup}_N[d(g_{ab}(\dagger), g_{ab}(n), h^{ab}, k)] < m/n$ for $k = 0, 1$. But $m/n < \varepsilon$ for large enough n. It follows that for $k = 0, 1$ we have

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$\mathrm{Sup}_M\left[d\left(g_{ab}(\dagger), g_{ab}(n), h^{ab}, k\right)\right] < \varepsilon$ for large enough n. So, for any C^1 fine neighborhood of $\left(M, g_{ab}(\dagger)\right)$, there is some n such that the $\mathcal{Q}$-extendible spacetime $\left(M, g_{ab}(n)\right)$ is in the neighborhood. So the property of $\mathcal{Q}$-inextendibility fails to be C^1 stable relative to $\mathcal{Q}$. QED

To highlight the physical significance of the proposition, suppose, for example, that it is true that "all physically reasonable spacetimes are globally hyperbolic" (Wald 1984, 304). And suppose that one were able to show that the property of (GH)-inextendibility is C^k stable relative to the collection (GH) for some $k \geq 0$. Because there remain "physically unreasonable" spacetimes lurking within (GH), one would also want assurance that the C^k stability of (GH)-inextendibility "transfers down" to the C^k stability of $\mathcal{P}$-inextendibility for any collection $\mathcal{P} \subset (GH)$. The proposition tells us that we do not have this assurance. Moreover, the predicament persists even if we further restrict attention to spacetimes which are well-behaved locally. Indeed, it is difficult to see how one might rule out as "physically unreasonable" a collection of globally hyperbolic vacuum solutions without invoking an inextendibility property of some kind.

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References

Beem, John K. and Ehrlich, Paul E. 1981. *Global Lorentzian Geometry*. 1st ed. New York: Marcel Dekker.

Beem, John, K. and Ehrlich, Paul E. 1987. "Geodesic Completeness and Stability." *Mathematical Proceedings of the Cambridge Philosophical Society* 102 (2):319–328.

Beem, John K., Ehrlich, Paul E., and Easley, K. 1996. *Global Lorentzian Geometry*. 2nd ed. New York: Marcel Dekker.

Clarke, Christopher J. S. 1993. *The Analysis of Space-Time Singularities*. Cambridge: Cambridge University Press.

Doboszewski, Juliusz. 2020. "Some Other 'No Hole' Spacetimes Properties Are Unstable Too." *Foundations of Physics* 50:379–384.

Earman, John. 1995. *Bangs, Crunches, Whimpers, and Shrieks: Singularities and Acausalities in Relativistic Spacetimes*. Oxford: Oxford University Press.

Ehrlich, Paul E. 2006. "A Personal Perspective on Global Lorentzian Geometry." In *Analytical and Numerical Approaches to Mathematical Relativity*, edited by Jörg Frauendiener, Domenico J. W. Giulini, and Volker Perlick, 3–32. Berlin: Springer.

Fletcher, Samuel C. 2016. "Similarity, Topology, and Physical Significance in Relativity Theory." *The British Journal for the Philosophy of Science* 67 (2):365–389.

Geroch, Robert. 1970a. "Domain of Dependence." *Journal of Mathematical Physics* 11:437–449.

Geroch, Robert. 1970b. "Singularities." In *Relativity*, edited by Moshe Carmeli, Stuart I. Fickler, and Louis Witten, 259–291. New York: Plenum Press.

Geroch, Robert. 1971. "Spacetime Structure from a Global Viewpoint." In *General Relativity and Cosmology*, edited by Rainer K. Sachs, 71–103. New York: Academic Press.

Hawking, Stephen. 1969. "The Existence of Cosmic Time Functions." *Proceedings of the Royal Society A* 308 (1494):433–435.

Hawking, Stephen and Ellis, George. 1973. *The Large Scale Structure of Space-Time*. Cambridge: Cambridge University Press.

Krasnikov, Serguei. 2018. *Back-in-Time and Faster-than-Light Travel in General Relativity*. New York: Springer.

Malament, David. 2012. *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory*. Chicago: University of Chicago Press.

Manchak, JB. 2011. "What is a Physically Reasonable Spacetime?" *Philosophy of Science* 78 (3):410–420.

Manchak, JB. 2016. "Is the Universe As Large as It Can Be?" *Erkenntnis* 81 (6):1341–1344.
Manchak, JB. 2017. "On the Inextendibility of Space-Time." *Philosophy of Science* 84 (5):1215–1225.
Manchak, JB. 2018. "Some 'No Hole' Spacetime Properties are Unstable." *Foundations of Physics* 48 (11): 1539–1545.
Manchak, JB. 2020. *Global Spacetime Structure*. Cambridge: Cambridge University Press.
Manchak, JB. 2021. "General Relativity As a Collection of Collections of Models." In *Hajnal Andréka and Istvań Németi on Unity of Science*, edited by Judit Madarász and Gergely Székely, 409–425. New York: Springer.
Navarro, J. and Minguzzi, E. 2011. "Global Hyperbolicity Is Stable in the Interval Topology," *Journal of Mathematical Physics* 52:112504.
Penrose, Roger. 1969. "Gravitational Collapse: The Role of General Relativity." *Revisita del Nuovo Cimento* 1:252–276.
Wald, Robert M. 1984. *General Relativity*. Chicago: University of Chicago Press.
Williams, P. M. 1984. "Completeness and Its Stability on Manifolds with Connection." Ph.D. thesis, University of Lancaster.

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