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HYPERGROUP ALGEBRAS AS TOPOLOGICAL ALGEBRAS

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Abstract

Let K be a locally compact hypergroup endowed with a left Haar measure and let $L^1(K)$ be the usual Lebesgue space of K with respect to the left Haar measure. We investigate some properties of $L^1(K)$ under a locally convex topology β^1. Among other things, the semireflexivity of $(L^1(K), \beta^1)$ and of sequentially β^1-continuous functionals is studied. We also show that $(L^1(K), \beta^1)$ with the convolution multiplication is always a complete semitopological algebra, whereas it is a topological algebra if and only if K is compact.

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1. Introduction and preliminaries

Throughout this paper, K will denote a locally compact hypergroup with Jewett's axioms with a fixed left Haar measure m; see [6]. For the sake of completeness and the convenience of the reader, let us recall the definition. A hypergroup K consists of a locally compact Hausdorff space K together with a bilinear, associative, weakly continuous convolution $*$ on the Banach space $M_b(K)$ of all bounded regular complex-valued Borel measures on K with the following properties.

(1) For all $x, y \in K$, the convolution of the point measures $\varepsilon_x * \varepsilon_y$ is a probability measure with compact support.

(2) The mapping $K \times K \to \mathcal{C}(K)$, $(x, y) \mapsto \text{supp}(\varepsilon_x * \varepsilon_y)$ is continuous with respect to the Michael topology on the space $\mathcal{C}(K)$ of all nonvoid compact subsets of K, where this topology is generated by the sets

$$U_{V,W} := \{L \in \mathcal{C}(K) : L \cap V \neq \emptyset, L \subset W\},$$

where V and W are open in K.

(3) There is an identity $e \in K$ with $\varepsilon_x * \varepsilon_e = \varepsilon_e * \varepsilon_x = \varepsilon_x$ for all $x \in K$.

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(4) There is a continuous involution $x \mapsto \overline{x}$ on K such that $(\varepsilon_x * \varepsilon_y)^- = \varepsilon_{\overline{y}} * \varepsilon_{\overline{x}}$ and $e \in \mathrm{supp}(\varepsilon_x * \varepsilon_y)$ if and only if $x = \overline{y}$ for $x, y \in K$. Here, for $\mu \in M_b(K)$, the measure μ^- is given by $\mu^-(A) = \mu(\overline{A})$ for Borel sets $A \subseteq K$.

A subset A is called symmetric if $A = \overline{A}$. For $A, B \subseteq K$, we define $A * B$ as the set $\cup\{\sup(\varepsilon_x * \varepsilon_y) : x \in A, y \in B\}$. Let us recall [6, Lemma 4.1 B]: for any sets A, B and C, $(A * B) \cap C = \emptyset$ if and only if $B \cap (\overline{A} * C) = \emptyset$. For more details, see also [1] and for other approaches to the notion of hypergroup, see [2, 10]. Locally compact groups and a wide class of locally compact semigroups are elementary examples of hypergroups.

For $p \geq 1$, let $L^p(K) := L^p(K, m)$ be the usual Lebesgue spaces with the norm $\|\cdot\|_p$ as defined in [5]. Recall that the dual of $L^1(K)$ can be identified with $L^\infty(K)$ via the pairing

$$\langle T(f), \varphi\rangle := \int_K f(x)\varphi(x)\, dm(x)$$

for all $\varphi \in L^1(K)$ and $f \in L^\infty(K)$. We denote by τ_n the topology generated by the norm $\|\cdot\|_1$.

We denote by $L_0^\infty(K)$ the subspace of $L^\infty(K)$ consisting of all functions $f \in L^\infty(K)$ that vanish at infinity; that is, those functions such that, for each $\varepsilon > 0$, there is a compact subset C of K for which $\|f\chi_{K\setminus C}\|_\infty < \varepsilon$, where $\chi_{K\setminus C}$ denotes the characteristic function of $K \setminus C$ on K.

We denote by $\mathcal{S}$ the set of increasing sequences of compact subsets of K and by $\mathcal{R}$ the set of increasing sequences (a_n) of real numbers in $(0, \infty)$ with $a_n \to \infty$. For any $(C_n) \in \mathcal{S}$ and $(a_n) \in \mathcal{R}$, set

$$U((C_n), (a_n)) = \{\varphi \in L^1(K) : \|\varphi\chi_{C_n}\|_1 \leq a_n, n \in \mathbb{N}\},$$

and note that $U((C_n), (a_n))$ is a convex balanced absorbing set in the space $L^1(K)$. It is easy to see that the family $\mathcal{U}$ of all sets $U((C_n), (a_n))$, for $(C_n) \in \mathcal{S}$ and $(a_n) \in \mathcal{R}$, is a base of neighbourhoods of zero for a locally convex topology on $L^1(K)$. This topology has been introduced and denoted by β^1 in [18] and [3] for locally compact groups and hypergroups, respectively. For a similar recent study in other contexts, see [11–13].

We shall also need the definition of strict topology in the general setting. Let $(V, \|\cdot\|)$ be a Banach left A-module, where A is a Banach algebra with a bounded approximate identity. The strict topology β on V induced by A is defined as the locally convex topology on V generated by the family of seminorms $\mathcal{P}_a(v) = \|a \cdot v\|$ $(a \in A, v \in V)$; for more details, see [15]. The strict topology for Banach modules has been studied extensively by Grosser [4]; see also [7–9] and [16, 17].

Recently, it was shown [11] that the β^1-topology can be viewed as a type of generalised strict topology in the sense of Sentilles and Taylor [15]. To this end, we consider the Banach space $L^1(K)$ as a Banach left $B_0(K)$-module, where the module action is the natural pointwise product of functions. Here $B_0(K)$ stands for the Banach algebra of all bounded Borel measurable functions on K vanishing at infinity under a pointwise product of functions. Then $L^1(K)$ is an essential $B_0(K)$-module (see [11] for more details).

Our contribution in this paper is to present some new properties of the topology β^1 for the more general setting of the Lebesgue space on a locally compact hypergroup. In particular, we show that $(L^1(K),\beta^1)$ with the convolution multiplication is always a semitopological algebra, whereas the convolution operator from $(L^1(K),\beta^1)\times(L^1(K),\beta^1)$ into $(L^1(K),\beta^1)$ is hypocontinuous if and only if K is compact. We also show that $(L^1(K),\beta^1)$ is a Mazur space.

2. Main results

Throughout this work, let K denote a locally compact hypergroup with a fixed left Haar measure m. We begin with the following result in which we collect some properties of the topology β^1 that we believe are interesting (for the proofs in a more general setting, we refer the reader to [11]).

Proposition 2.1. *Let K be a locally compact hypergroup. Then the following hold.*

- (i) *The topology β^1 on $L^1(K)$ is the strict topology β induced by $B_0(K)$ on the Banach left $B_0(K)$-module $L^1(K)$.*
- (ii) *The dual of $(L^1(K),\beta^1)$ under the strong topology can be identified with the Banach space $L_0^\infty(K)$.*
- (iii) *The space $(L^1(K),\beta^1)$ is metrisable if and only if K is compact.*
- (iv) *The space $(L^1(K),\beta^1)$ is complete.*
- (v) *A sequence (φ_n) in $L^1(K)$ is β^1-convergent to zero if and only if it is τ_n-bounded and $(\int_C \varphi_n\,dm)$ tends to zero for all compact subsets C.*

We shall need the following lemma for the forthcoming proposition.

Lemma 2.2. *Let K be a noncompact locally compact hypergroup. Then there exists a sequence (U_n) of mutually disjoint compact neighbourhoods such that for every compact subset K, $K\cap U_n=\emptyset$ for sufficiently large n.*

Proof. Let V be a compact neighbourhood of the identity element of K. Since K is not compact, we can choose a sequence (x_n) in K such that $x_n * V\cap x_m * V=\emptyset$ for $n\neq m$. Indeed, suppose that $x_1,\dots,x_{n-1}$ have been chosen as above. Now $\cup\{x_k * V * \overline{V} : 1\leq k\leq n-1\}$ is compact. Hence, there exists x_n with $x_n\notin\cup\{x_k * V * \overline{V} : 1\leq k\leq n-1\}$. It follows from [6, Lemma 4.1 B] that $x_n * V\cap(\bigcup_{k=1}^{n-1} x_k * V)=\emptyset$.

Now, if there is a compact subset C with $C\cap x_n * V\neq\emptyset$ for all n, another application of [6, Lemma 4.1 B] implies that $\bigcup_{n=1}^\infty x_n * V\subseteq\overline{V} * C * V$. This is a contradiction, because $m(\bigcup_{n=1}^\infty x_n * V)=\infty$, whereas $\overline{V} * C * V$ is a compact set and $m(\overline{V} * C * V)<\infty$. □

Proposition 2.3. *Let K be a locally compact hypergroup. Then β^1-convergence and τ_n-convergence coincide for sequences in $L^1(K)$ if and only if K is compact.*

Proof. Assume that K is not compact and, for $n\geq 1$, let $\varphi_n=\chi_{U_n}$, where (U_n) is the sequence from Lemma 2.2. Then, by Proposition 2.1(v), it is clear that $\varphi_n\to 0$ in

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the β^1-topology, while it does not converge in τ_n. The converse is clear, because if K is compact, then $\beta^1 = \tau_n$. □

Proposition 2.4. *Let K be a locally compact hypergroup. Then $(L^1(K), \beta^1)$ is a Mazur space; that is, every sequentially β^1-continuous linear functional on $L^1(K)$ is β^1-continuous.*

Proof. Let F be a sequentially β^1-continuous linear functional on $L^1(K)$. Hence, by Proposition 2.1(ii), there is a $g \in (L^1(K), \|\cdot\|_1)^* = L^\infty(K)$ such that $F(\varphi) = \int_K \varphi(x)g(x)\,dm(x)$ for all $\varphi \in L^1(K)$. Therefore, the proof would be complete once we show that $g \in L_0^\infty(K)$. Assume on the contrary that $g \notin L_0^\infty(K)$. Then there would exist a number $\epsilon_0 > 0$ and, by the regularity of m, an increasing sequence (K_n) of relatively compact open subsets of K and a number $\epsilon_0 > 0$ such that $\|g\chi_{K_n}\|_\infty > \epsilon_0$ for all n. It follows that there is a sequence (φ_n) in $L^1(K)$ such that, for all n,

$$\|\varphi_n\|_1 \le 1 \quad \text{and} \quad \left|\int_{K_n} \varphi_n(x)g(x)\,dm(x)\right| > \epsilon_0. \tag{2.1}$$

Define $\psi_n(x) = \varphi_n(x)\chi_{K_n}(x)$ for $n \ge 1$ and all $x \in K$. Now, by Proposition 2.1(v), the sequence (ψ_n) converges in the β^1-topology to zero. But this contradicts (2.1). It follows that $g \in L_0^\infty(K)$. □

We recall that a locally convex space (E, τ) is called *semireflexive* if $(E, \tau)^{**} = E$. In the next result, we deal with the semireflexivity of $(L^1(K), \beta^1)$.

Proposition 2.5. *Let K be a locally compact hypergroup. Then $(L^1(K), \beta^1)$ is semireflexive if and only if K is discrete.*

Proof. Suppose that K is discrete. By [6, Theorem 7.1 A], $m(\{x\}) = 1/(\varepsilon_{\bar{x}} * \varepsilon_x)(\{e\})$ for all $x \in K$ and hence $L^1(K)$ can be identified with the space $\ell^1(K)$ of all complex-valued functions on K such that $\sum_{x\in K} |\varphi(x)| < \infty$. In view of Proposition 2.1(ii), the dual of $(\ell^1(K), \beta^1)$ equipped with the strong topology can be identified with the Banach space $\ell_0^\infty(K)$ equipped with the $\|\cdot\|_\infty$-topology, where $\ell_0^\infty(K)$ denotes the space of all complex-valued functions f on K such that f is bounded and vanishing at infinity. Furthermore, $\ell_0^\infty(K)^*$ can be identified with $\ell^1(K)$. Thus, $(\ell^1(K), \beta^1)^{**} = \ell^1(K)$.

To prove the converse, note that ε_e, the Dirac measure at the identity of K, defines a bounded linear functional on $C_c(K)$, the space of continuous functions with compact support. Choose an extension u of ε_e to an element of $L_0^\infty(K)^*$ by the Hahn–Banach theorem. The assumption together with Proposition 2.1(ii) imply that $u = T^*(\varphi)$ for some $\varphi \in L^1(K)$, where T^* is the adjoint of the natural isometric isomorphism between $(L^1(K), \|\cdot\|_1)^*$ and the Banach space $L^\infty(K)$. In particular, for each $\psi \in C_c(K)$,

$$\langle \varepsilon_e, \psi\rangle = \langle u, \psi\rangle = \langle \varphi, T(\psi)\rangle = \int_K \varphi\psi\,dm.$$

It follows that ε_e is absolutely continuous with respect to m and hence $m(\{e\}) > 0$. By [1, Theorem 1.3.27], we conclude that K is discrete. □

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Let us recall some needed notation and definitions. For Borel functions φ and ψ, at least one of which is σ-finite, define the convolution $\varphi * \psi$ on K by

$$(\varphi * \psi)(x) = \int_K \varphi(y)\psi(\bar{y} * x)\, dm(y),$$

where

$$\varphi(x * y) = \int_K \varphi\, d(\varepsilon_x * \varepsilon_y).$$

Also recall that the Mackey topology $\mu_0 := \mu(L^1(K), L_0^\infty(K))$ on $L^1(K)$ is the topology of uniform convergence on absolutely convex weak* compact subsets of $L_0^\infty(K)$. We also denote the weak topology $\sigma(L^1(K), L_0^\infty(K))$ on $L^1(K)$ by σ_0.

Lemma 2.6. *Let K be a locally compact hypergroup. The following statements hold.*

(i) *For any $f \in L_0^\infty(K)$ and $\varphi \in L^1(K)$, $f\varphi \in L_0^\infty(K)$, where $f\varphi$ is defined on $L^1(K)$ by*

$$\langle f\varphi, \psi\rangle = \langle f, \varphi * \psi\rangle \quad (\psi \in L^1(K)).$$

(ii) *The convolution on $L^1(K)$ is separately continuous with respect to the weak topology σ_0 and the Mackey topology μ_0.*

Proof. (i) Let $\varphi \in L^1(K)$ and $f \in L_0^\infty(K)$. First, note that $f\varphi \in (L^1(K), \|\cdot\|_1)^*$ and thus $f\varphi \in L^\infty(K)$. Also,

$$f\varphi(x) = \int_K f(y * x)\varphi(y)\, dm(y)$$

for almost all $x \in K$; indeed, since

$$\begin{aligned}\int_K (f\varphi)(x)\psi(x)\, dm(x) = \langle f\varphi, \psi\rangle = \langle f, \varphi * \psi\rangle &\\ &= \int_K \int_K \varphi(y) f(x)\psi(\bar{y} * x)\, dm(x)\, dm(y)\\ &= \int_K \int_K \varphi(y) f(y * x)\psi(x)\, dm(y)\, dm(x).\end{aligned}$$

Now, for a given $\varepsilon > 0$, let D be a compact subset of K with $\int_{K\setminus D} |\varphi(x)|\, dm(x) < \varepsilon$ and also let C be a compact subset of K with $|f(t)| < \varepsilon$ for almost all $t \in K \setminus C$. Then, for each $x \in K \setminus \overline{D} * C$, we get $D * x \subseteq K \setminus C$. Observing that $\operatorname{supp}(\varepsilon_x * \varepsilon_y) \subseteq K \setminus C$ for $x \in K \setminus \overline{D} * C$ and $y \in D$, we therefore have

$$\begin{aligned}\left|\int_K f(y * x)\varphi(y)\, dm(y)\right| &\le \int_{K\setminus D} |f(y * x)\,\|\,\varphi(y)|\, dm(y) + \int_D |f(y * x)\,\|\,\varphi(y)|\, dm(y)\\ &\le \varepsilon\,(\|f\|_\infty + \|\varphi\|_1).\end{aligned}$$

That is, $|(f\varphi)(x)| < \varepsilon\,(\|f\|_\infty + \|\varphi\|_1)$ for almost all $x \in K \setminus \overline{D} * C$. Since $\overline{D} * C$ is compact, this means that $f\varphi$ vanishes at infinity and so $f\varphi \in L_0^\infty(K)$.

(ii) The σ_0-separate continuity of the convolution follows from (i). The μ_0-separate continuity of the convolution is an easy consequence of the σ_0-separate continuity; see for example [19, Corollary 26.15]. □

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In the next result, we prove that the convolution is also separately β^1-continuous. This generalises [12, Theorem 4.1] and gives for it a proof with a corrected base.

THEOREM 2.7. *Let K be a locally compact hypergroup. Then $(L^1(K), \beta^1)$ with the convolution product is a semitopological algebra.*

PROOF. Let us recall [14, Corollary 2.5] that a linear map from $(L^1(K), \beta^1)$ into a locally convex space is continuous if and only if its restriction to τ_n-bounded sets is continuous for β^1. So, we only need to show that the convolution on $L^1(K)$ is β^1-separately continuous on τ_n-bounded sets. It is well known that the Banach algebra $L^1(K)$ can be embedded isometrically isomorphically in $M_b(K)$ by means of the map $\varphi \mapsto \varphi m$, $\varphi \in L^1(K)$; see for example [6]. So, let (μ_α) be a norm-bounded net in $L^1(K) \subset M_b(K)$ convergent to zero in the β^1-topology. Let $\nu \in L^1(K) \subset M_b(K)$ with $\|\nu\| \neq 0$ and let $U((C_n), (a_n))$ be an arbitrary β^1-neighbourhood of zero.

Choose a compact set $C \subseteq K$ such that

$$|\nu|(K \setminus C) < \frac{a_1}{2M},$$

where $M > 0$ is such that $\|\mu_\alpha\| \leq M$ for all α. Now, if we put

$$F_n := C_n * \overline{C} \quad \text{and} \quad b_n := \frac{a_n}{2\|\nu\|},$$

then $((F_n), (b_n)) \in \mathcal{S} \times \mathcal{R}$ and so there is an α_0 such that $\mu_\alpha \in U((F_n), (b_n))$ for all $\alpha \geq \alpha_0$.

Now, using [1, Lemmas 1.2.21 and 1.2.14], we can write (for all $\alpha \geq \alpha_0$)

$$\begin{aligned}
|\mu_\alpha * \nu|(C_n) &\leq \int_K |(\mu_\alpha * \varepsilon_t)(C_n)|\, d|\nu|(t) \\
&\leq \int_K |\mu_\alpha|(C_n * \bar{t})\, d|\nu|(t) \\
&= \int_C |\mu_\alpha|(C_n * \bar{t})\, d|\nu|(t) + \int_{K\setminus C} |\mu_\alpha|(C_n * \bar{t})\, d|\nu|(t) \\
&\leq |\mu_\alpha|(F_n) \int_C d|\nu|(t) + M \int_{K\setminus C} d|\nu|(t) \\
&\leq b_n\|\nu\| + M\Big(\frac{a_1}{2M}\Big) \\
&\leq \frac{a_n}{2} + \frac{a_n}{2} = a_n.
\end{aligned}$$

Hence, $\mu_\alpha * \nu \longrightarrow 0$ in the β^1-topology. □

Recall from [6] that the modular function Δ is defined on K by the identity

$$m * \varepsilon_{\overline{x}} = \Delta(x) m.$$

It is known that Δ is a continuous homomorphism from K into the multiplicative group of positive real numbers.

For the next result, we need a slightly stronger version of [18, Remark 1(iv)].

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LEMMA 2.8. *Let K be a noncompact locally compact hypergroup, V a compact neighbourhood of the identity element and (C_n) an increasing sequence of compact subsets in K. Then there are a sequence (x_n) in K and an increasing sequence (K_n) of compact subsets such that $V * \overline{x_n} \subseteq K_n \setminus K_{n-1}$ and $C_n \subseteq K_n$, where $K_0 = \emptyset$. Moreover, if K is not unimodular, we can assume that $\Delta(x_n) > 1$ for all $n \geq 1$.*

PROOF. Let $K_0 = \emptyset$ and $K_1 = C_1 \cup V * \overline{x_1}$, where $x_1 \in K$ is arbitrary and, if K is nonunimodular, we choose x_1 with $\Delta(x_1) > 1$. Since K is not compact, by [6, Lemma 4.1 B], we can choose x_2 such that $V * \overline{x_2} \cap K_1 = \emptyset$ and, if K is nonunimodular, we choose x_2 with $\Delta(x_2) > 1$. Then let $K_2 = C_2 \cup V * \overline{x_1} \cup V * \overline{x_2}$. Proceeding by induction, when $x_1, \ldots, x_{n-1}$ and $K_0, \ldots, K_{n-1}$ have been defined, let x_n be any element such that $V * \overline{x_n} \cap K_{n-1} = \emptyset$ and, if K is nonunimodular, we can choose x_n with $\Delta(x_n) > 1$. Now put $K_n = C_n \cup V * \overline{x_1} \cdots \cup V * \overline{x_n}$; then clearly (x_n) and (K_n) have the desired properties. □

We conclude this paper with the following result.

THEOREM 2.9. *Let K be a locally compact hypergroup. Then the convolution operator from $(L^1(K), \beta^1) \times (L^1(K), \beta^1)$ into $(L^1(K), \sigma_0)$ is hypocontinuous if and only if K is compact. In particular, $(L^1(K), \beta^1)$ with the convolution product is a topological algebra if and only if K is compact.*

PROOF. The 'if' part is clear. For the converse, assume that K is not compact and consider any β^1-neighbourhood $U((C_n), (a_n))$ and a compact symmetric neighbourhood V of the identity element in K with $m(V) \leq 1$. For any $i \in \mathbb{N}$, we put $\varphi_i = a_i \Delta(x_i) \chi_{V * \overline{x_i}}$ and $\psi_i = \chi_{\overline{x_i} * V}$, where (x_i) is as in Lemma 2.8. Then each φ_i is in $U((K_n), (a_n)) \subset U((C_n), (a_n))$ and each ψ_i belongs to B, the closed unit ball of $L^1(K)$. But it is readily seen that $\|(\varphi_i * \psi_i)\chi_{V*V}\|_1 \geq a_i \Delta(x_i) m(V)^2$, so

$$U((C_n), (a_n)) * B \not\subseteq \left\{\phi \in L^1(K) : \left| \int_K \phi(x) \chi_{V*V}(x)\, dm(x) \right| < 1 \right\}.$$

This completes the proof. □

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