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A NOTE ON LOCAL POLYNOMIAL FUNCTIONS ON COMMUTATIVE SEMIGROUPS

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Abstract

Given a universal algebra A, one can define for each positive integer n the set of functions on A which can be "interpolated" at any n elements of A by a polynomial function on A. These sets form a chain with respect to inclusion. It is known for several varieties that many of these sets coincide for all algebras A in the variety. We show here that, in contrast with these results, the coincident sets in the chain can to a large extent be specified arbitrarily by suitably choosing A from the variety of commutative semigroups.

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Let A be a universal algebra. Following Hule and Nöbauer [5], we define $F_1(A)$ to be the algebra of all functions from A to A, where the operations on $F_1(A)$ are the operations on A defined pointwise. The subalgebra of $F_1(A)$ generated by the identity function ξ and the constant functions is denoted by $P_1(A)$ and is called the *algebra of* 1-*place polynomial functions on* A. For each positive integer n, the algebra $L_nP_1(A) = \{g \in F_1(A)$: for any $a_1, \ldots, a_n \in A$ there exists $p \in P_1(A)$, $g(a_i) = p(a_i)$ for $1 \leqslant i \leqslant n\}$ is called the *algebra of* 1-*place n-local polynomial functions on* A, and p is said to *interpolate* g at $a_1, \ldots, a_n$. (For the definitions of $F_k(A)$, $P_k(A)$ and $L_nP_k(A)$ for any positive integer k, see [5].) The following chain of inclusions is evident:

$$L_1P_1(A) \supseteq L_2P_1(A) \supseteq L_3P_1(A) \supseteq \cdots \supseteq P_1(A).$$

Several recent papers [1, 2, 3, 4, 5, 6, 7] have investigated the conditions on A for equality to occur between members of this chain when A is an algebra in a

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particular variety. In most of these cases, many members of the chain coincide for all algebras A in the variety (although Lausch and Nöbauer [7] showed that the members can all be distinct in the variety of commutative rings with identity). In contrast with these earlier results, we shall show in this note that the chain from $L_3P_1(A)$ onwards is "badly-behaved" in the variety of commutative semigroups. More precisely, given any collection of pairs of adjacent members of the chain (from $L_3P_1(A)$ onwards), a commutative semigroup can be found for which the members in each pair are distinct and all other members of the chain coincide.

THEOREM. *For any $X \subseteq \{3,4,5,6,\dots\}$ there is a commutative semigroup S such that $L_nP_1(S) \neq L_{n+1}P_1(S)$ for all $n \in X$ and $L_nP_1(S) = L_{n+1}P_1(S)$ for all $n \in \{3,4,5,6,\dots\} \setminus X$.*

PROOF. Let $X \subseteq \{3,4,5,6,\dots\}$ be given. Define $S = \{0\} \cup X \cup \{(i, j) \in X \times Z^+ : j \leqslant i\}$. Define multiplication on S as follows:

$$xy = 0 \quad \text{if } x \in \{0\} \cup X \text{ or } y \in \{0\} \cup X,$$

$$(i, j)(k, l) = \begin{cases} i & \text{if } i = k \text{ and } j \neq l, \\ 0 & \text{otherwise.} \end{cases}$$

Multiplication is associative (since $(xy)z = 0$ and $x(yz) = 0$ for all $x, y, z \in S$) and commutative, so S is a commutative semigroup. Since $x^2 = 0$ for any $x \in S$, it is readily seen that every element of $P_1(S)$ is constant or of the form ξ or $(i, j)\xi$.

Let $n \in X$. Define $g: S \to S$, $g((n, j)) = n$ for all j, $g(x) = 0$ elsewhere. We shall show that $g \in L_nP_1(S) \setminus L_{n+1}P_1(S)$. To show $g \in L_nP_1(S)$, let $x_1, \dots, x_n \in S$. If $\{x_1, \dots, x_n\} \subseteq \{(n, 1), \dots, (n, n)\}$, then the constant function n interpolates g at these points. If not, then at most $n - 1$ of $x_1, \dots, x_n$ are in $\{(n, 1), \dots, (n, n)\}$, so there exists j such that $(n, j) \notin \{x_1, \dots, x_n\}$. Since $(n, j)x = g(x)$ for every $x \neq (n, j)$, $(n, j)\xi$ interpolates g at $x_1, \dots, x_n$. To establish $g \notin L_{n+1}P_1(S)$, we show that g cannot be interpolated by a polynomial at $0, (n, 1), \dots, (n, n)$. Clearly, neither a constant function nor ξ interpolates g at these points; and if $(i, j)\xi$ interpolates, then $n = g((n, j)) = (i, j)(n, j) = 0$, contradiction. Therefore $g \in L_nP_1(S) \setminus L_{n+1}P_1(S)$, so $L_nP_1(S) \neq L_{n+1}P_1(S)$.

Now suppose $n \in \{3,4,5,6,\dots\} \setminus X$. We shall show that $L_nP_1(S) = L_{n+1}P_1(S)$. Let $g \in L_nP_1(S)$. If $g(0) \neq 0$, then an interpolating polynomial at 0 and an arbitrary element x of S must be the constant function $g(0)$. Thus $g(x) = g(0)$, so g is constant and hence $g \in L_{n+1}P_1(S)$ in this case. Suppose $g(0) = 0$. By considering an interpolating polynomial at 0 and any $i \in X$, it is clear that $g(i) = 0$ or $g(i) = i$. If $g(i) = i$ for some $i \in X$, then an interpolating polynomial

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at 0, i and arbitrary $x \in S$ must be ξ, so $g(x) = x$ and hence $g = \xi \in L_{n+1}P_1(S)$. If not, then $g(i) = 0$ for every $i \in X$. In this case, consider any element of S of the form (i, j), and let $p \in P_1(S)$ interpolate g at $0, i$ and (i, j). Then either $p = 0$ or $p = (k, l)\xi$ for some k, l, so $g((i, j)) = 0$ or $g((i, j)) = i$. If $g((i, j)) = i$ and $g((i', j')) = i'$ for some distinct i and i' and for some j, j', let $q \in P_1(S)$ interpolate g at (i, j) and (i', j'). Clearly, q is neither constant nor ξ, so $q = (k, l)\xi$ for some k, l. Hence $(k, l)(i, j) = g((i, j)) = i$ and $(k, l)(i', j') = g((i', j')) = i'$, so $k = i$ and $k = i'$, contrary to assumption. Therefore there exists i_0 such that $g((i, j)) = 0$ for every $i \neq i_0$ and every j. Now suppose $g((i_0, j)) = g((i_0, j')) = 0$ for some distinct j and j'. Let $r \in P_1(S)$ interpolate g at $(i_0, j), (i_0, j')$ and (i_0, k), where k is arbitrary. Clearly $r \neq \xi$. If r is constant, then $g((i_0, k)) = 0$. If $r = (l, m)\xi$, then $(l, m)(i_0, j) = (l, m)(i_0, j') = 0$, which is possible only if $l \neq i_0$ (since $j \neq j'$). Thus $g((i_0, k)) = (l, m)(i_0, k) = 0$ (since $l \neq i_0$), so $g((i_0, k)) = 0$ in any case. Therefore $g(x) = 0$ for all $x \in S$, so $g = 0 \in L_{n+1}P_1(S)$. It remains to consider the case where $g((i_0, j)) = 0$ for at most one j. If $g((i_0, j_0)) = 0$, say, then $g((i_0, j)) = i_0$ whenever $j \neq j_0$, so $g = (i_0, j_0)\xi \in L_{n+1}P_1(S)$. If not, then $g((i_0, j)) = i_0$ for every j. Since $n \notin X$, $i_0 \neq n$. If $i_0 < n$, let $s \in P_1(S)$ interpolate g at $0, (i_0, 1), \ldots, (i_0, i_0)$. Clearly, s is neither constant nor ξ, while if $s = (k, l)\xi$ then $i_0 = g((i_0, l)) = (k, l)(i_0, l) = 0$, contradiction. Hence $i_0 > n$. Let $x_1, \ldots, x_{n+1} \in S$. If $\{x_1, \ldots, x_{n+1}\} \subseteq \{(i_0, 1), \ldots, (i_0, i_0)\}$, then the constant function i_0 interpolates g at $x_1, \ldots, x_{n+1}$. If not, then it follows from $i_0 > n$ that $(i_0, j) \notin \{x_1, \ldots, x_{n+1}\}$ for some j. In this case, $(i_0, j)\xi$ interpolates g at $x_1, \ldots, x_{n+1}$. Therefore $g \in L_{n+1}P_1(S)$, as required.

If $X \neq \varnothing$, then it can be shown that $L_1P_1(S) \neq L_2P_1(S) \neq L_3P_1(S)$ for the semigroup S defined in the proof of the theorem. It is not known whether the condition $n \geqslant 3$ in the theorem can be weakened.

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